J. Korean Math. Soc. 28(1991), No. 1, pp. 1–11

FINITELY GENERATED MODULES

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Introduction

In this paper, ***unless otherwise indicated, we shall not assume that our rings are commutative, but we shall always assume that every ring has an identity element.*** By a *module*, we shall always mean a unitary left module.

We provide a characterization of non-zero finitely generated Noetherian modules, some properties of finitely generated Noetherian and Artinian modules, and the localization E_S of a finitely generated module E over a commutative ring R with respect to a multiplicatively closed subset S of R not containing 0.

Finally, this paper deals with aspects of the identification of the maximal submodules of a finitely generated module over a commutative ring R. It shows an analogy between this set of submodules and the spectrum of R.

1. Finitely generated Noetherian and Artinian modules

The Cohen theorem [C50] says that if every prime ideal in a commutative ring R is finitely generated, then R is Noetherian. We first generalize this result.

Let E be an R-module. A submodule M of E is said to be a *maximal submodule* of E if (i) M is a proper submodule of E and (ii) there is no proper submodule of E strictly containing M.

It is well known [SV72] that every non-zero finitely generated R-module possesses a maximal submodule.

DEFINITION. Let E be an R-module. Then a submodule P of E is said to be a *prime submodule* of E if (a) P is proper and (b) whenever $re \in P$ $(r \in R,\ e \in E)$, then either $e \in P$ or $rE \subseteq P$.

Received December 29, 1989.

LEMMA 1. *Let R be a commutative ring and A a simple R-module. Then every zero-divisor on A is an annihilator of A.*

Proof. Let r be an arbitrary zero-divisor on A. Then there exists $e \in A$, $e \neq 0$ such that $re = 0$. Since A is a simple R-module, the submodule of A generated by e must be A itself. Hence

$$rA = r(Re) = (rR)e = (Rr)e = R(re) = 0$$

and so r is an annihilator of A.

PROPOSITION 2. *Let R be a commutative ring and A an R-module. Then every maximal submodule of A is prime.*

Proof. Let M be an arbitrary maximal submodule of A. Then M is proper. Replacing A by A/M, we can assume that A is a simple R-module and $M = 0$. It suffices to show that every zero-divisor on A is an annihilator of A. But, this follows from Lemma 1.

Note that for every R-module E, the annihilator, denoted by $\mathrm{Ann}_R E$, of E is a two-sided ideal of R. Let E be an R-module and P a submodule of E. Let $\mathfrak{p}$ denote $\mathrm{Ann}_R(E/P)$. Then if P is prime, then $\mathfrak{p}$ is a prime ideal of R by the definitions. Further, if P is maximal, then $\mathfrak{p}$ is a maximal ideal of R. In fact, E/P is a simple R-module and is R-isomorphic to $R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of R. This implies that

$$\mathfrak{p} = \mathrm{Ann}_R(E/P) = \mathrm{Ann}_R(R/\mathfrak{m}) = \mathfrak{m},$$

which becomes a maximal ideal of R. Hence we have the following result.

PROPOSITION 3. *Let E be an R-module and P a submodule of A. Let $\mathfrak{p}$ denote $\mathrm{Ann}_R(E/P)$. Then :*

(i) *P is prime if and only if the factor module E/P, as an $R/\mathfrak{p}$-module, is torsion-free ;*

(ii) *If P is maximal, then E/P, as an $R/\mathfrak{p}$-module, is divisible. In particular, when E is cyclic over a commutative ring R, P is maximal if and only if $R/\mathfrak{p}$ is a field.*

If N is an R-submodule of an R-module E and $\mathfrak{a}$ an ideal of R, we define $N :_E \mathfrak{a}$ to be the R-submodule of E consisting of all $x \in E$ such that $\mathfrak{a}x \subseteq N$.

LEMMA 4. *Let N be a submodule of an R-module E and r an element of R. If $N + rE$ and $N :_E rR$ are finitely generated, then N is also finitely generated.*

Proof. Adapt the proof of [N62, (3.3), p.8].

It is well known [N62] that every finitely generated module over a Noetherian ring is a Noetherian module. Note that any submodule of a finitely generated module is not necessarily finitely generated. The following result is a characterization of finitely generated Noetherian modules and is also a generalization of the Cohen theorem.

THEOREM 5. *A non-zero finitely generated R-module E is Noetherian if and only if every prime submodule of E is finitely generated.*

Proof. The *only if* part is a consequence of [N62, (3.1), p.7]. Use Zorn's lemma and Lemma 4 to prove the *if* part (cf. [N62, (3.4), p.8]).

The Formanek theorem [F73] says that if R is a commutative ring and $M = Rm_1 + \cdots + Rm_k$ is a faithful finitely generated R-module which satisfies the ascending chain condition (ACC) on "extended submodules" IM, where I is an ideal in R, then M is a Noetherian R-module and hence R is a Noetherian ring. In the remainder of this section we discuss under what conditions 'faithful' can be replaced.

LEMMA 6. *Let $M = Rm_1 + \cdots + Rm_k$ be a finitely generated module over a commutative ring R. Suppose that, for every ideal I in R and for each i, Im_i is equal to $Rm_i \cap IM$. Then M satisfies ACC (resp. DCC) on extended submodules if and only if each Rm_i, satisfies ACC (resp. DCC) on extended submodules.*

Proof. We consider only the ACC case since the proof of the DCC case is similar.

Assume that M satisfies ACC on extended submodules. We show only the case of $i = 1$ since the proof of the other case is similar. Consider the ascending chain

$$I_1 m_1 \subseteq I_2 m_1 \subseteq I_3 m_1 \subseteq \cdots$$

of extended submodules of Rm_1. This gives an ascending chain

$$\mathrm{Ann}_R(Rm_1/I_1m_1) \subseteq \mathrm{Ann}_R(Rm_1/I_2m_1) \subseteq \mathrm{Ann}_R(Rm_1/I_3m_1) \subseteq \cdots$$

of ideals of R. But, each $\mathrm{Ann}_R(Rm_1/I_im_1)$ is equal to the sum $I_i + \mathrm{Ann}_R(m_1)$. We get an ascending chain

$$(I_1 + \mathrm{Ann}_R m_1)M \subseteq (I_2 + \mathrm{Ann}_R m_1)M \subseteq (I_3 + \mathrm{Ann}_R m_1)M \subseteq \cdots$$

of extended submodules of M. By our assumption, there exists a positive integer s such that $(I_n + \mathrm{Ann}_R m_1)M = (I_s + \mathrm{Ann}_R m_1)M$ for all $n \geq s$. Thus,

$$\begin{aligned} I_n m_1 &= Rm_1 \cap I_n M \\ &\subseteq Rm_1 \cap (I_n + \mathrm{Ann}_R m_1)M \\ &= Rm_1 \cap (I_s + \mathrm{Ann}_R m_1)M \\ &= (I_s + \mathrm{Ann}_R m_1)m_1 \\ &= I_s m_1 \end{aligned}$$

for all $n \geq s$. Also, it is clear that $I_n m_1 \supseteq I_s m_1$ for all $n \geq s$. Therefore the given ascending chain terminates.

Conversely, assume that each Rm_i satisfies ACC on extended submodules. Consider the ascending chain

$$I_1 M \subseteq I_2 M \subseteq I_3 M \subseteq \cdots$$

of extended submodules of M. This gives ascending chains

$$Rm_i \cap I_1 M \subseteq Rm_i \cap I_2 M \subseteq Rm_i \cap I_3 M \subseteq \cdots, \ i = 1, \ldots, k.$$

But, $Rm_i \cap I_n M = I_n m_i$ for each $1 \leq i \leq k$ and for each $n \geq 1$. By our assumption, for each $1 \leq i \leq k$ there exists a positive integer s_i such that $I_n m_i = I_{s_i} m_i$ for all $n \geq s_i$. Take $s = \max\{s_1, s_2, \ldots, s_k\}$. Then $I_n m_i = I_s m_i$ for all $n \geq s$. Hence

$$I_n M = I_n m_1 + I_n m_2 + \cdots + I_n m_k = I_s m_1 + I_s m_2 + \cdots + I_s m_k = I_s M$$

for all $n \geq s$. Thus, the given ascending chain terminates.

THEOREM 7. *Let M be as in Lemma 6. If, for every ideal I in R and for each i, Im_i is equal to $Rm_i \cap IM$, and M satisfies ACC (resp. DCC) on extended submodules, then M is a Noetherian (resp. Artinian) R-module and hence $R/\mathrm{Ann}_R M$ is a Noetherian (resp. Artinian) ring.*

Proof. We consider only the Noetherian case, since the proof of the Artinian case is similar. By our hypothesis and Lemma 6 each Rm_i in M satisfies ACC on extended submodules. Since each Rm_i is R-isomorphic to $R/\mathrm{Ann}_R m_i$, it is Noetherian. By [SV72, Proposition 1.18, p.18] $Rm_1 \oplus \cdots \oplus Rm_k$ is Noetherian. Define a mapping $f : Rm_1 \oplus \cdots \oplus Rm_k \to Rm_1 + \cdots + Rm_k$ by $f(r_1m_1, \ldots, r_km_k) = r_1m_1 + \cdots + r_km_k$, where $r_i \in R$. Then f is an epimorphism. This gives an exact sequence

$$0 \to \mathrm{Ker} f \to Rm_1 \oplus \cdots \oplus Rm_k \to M \to 0$$

of R-modules. Hence M is Noetherian.

Now define a mapping $g : R \to Rm_1 \oplus \cdots \oplus Rm_k$ by $g(r) = (rm_1, \ldots, rm_k)$, where $r \in R$. Then g is an R-homomorphism and $\mathrm{Ker}\, g = \mathrm{Ann}_R M$. So, $R/\mathrm{Ann}_R M$ can be regarded as an R-submodule of $Rm_1 \oplus \cdots \oplus Rm_k$. Hence since $Rm_1 \oplus \cdots \oplus Rm_k$ is Noetherian, so is $R/\mathrm{Ann}_R M$.

PROPOSITION 8. *Let R be a commutative domain and M as in Lemma 6. If $M \neq 0$ is divisible, then M is faithful. Moreover, the converse holds if M is simple.*

Proof. M is faithful if and only if for each non-zero $r \in R$ there is at least one of $m_1, \ldots, m_k$ (depending on r) such that $rm_i \neq 0$. This latter property is inductive. The remainder of the proof is obvious.

It is well known [SV72, Proposition 2.6, p.33] that every injective module is divisible. Of course, every torsion-free divisible module over a commutative domain is injective [SV72, Proposition 2.7, p.34]. Hence the following proposition follows from Proposition 8 and the Formanek theorem.

PROPOSITION 9. *Let R be a commutative domain and M as in Lemma 6. If $M \neq 0$ is injective and satisfies ACC on extended submodules, then M is a Noetherian R-module and hence R is a Noetherian domain.*

2. Localization

In this section we discuss the localization E_S of a finitely generated module E over a commutative ring R with respect to a multiplicatively closed subset S of R not containing 0. Specifically, if P is a prime submodule of E, then we will take $S = R\backslash \mathrm{Ann}_R(E/P)$ and consider the corresponding localization of E.

PROPOSITION 10. *Let E be a non-zero finitely generated module over the commutative ring R, S a multiplicatively closed subset of R not containing 0, and P a prime R-submodule of E. Let $\mathfrak{p}$ denote $\mathrm{Ann}_R(E/P)$. Then:*

(i) *when $S \cap \mathfrak{p}$ is non-empty, then $P \otimes_R R_S = E \otimes_R R_S$;*

(ii) *when $S \cap \mathfrak{p}$ is empty, then $P \otimes_R R_S$ is a prime R_S-submodule of the finitely generated R_S-module $E \otimes_R R_S$.*

Hence there is a one-to-one order-preserving correspondence between the prime R_S-submodules of $E \otimes_R R_S$ and the prime R-submodules Q of E such that $S \cap \mathrm{Ann}_R(E/Q)$ is empty.

Proof. Note that $E \otimes_R R_S = E_S$ and $P \otimes_R R_S = P_S$. Let $E = Re_1 + \cdots + Re_n$. Then

(i) when $s \in S \cap \mathfrak{p}$, then, for $1 \leq i \leq n$, $e_i/1 = se_i/s \in P_S$; hence $P_S = E_S$.

(ii) Since E is finitely generated over R, E_S is finitely generated over R_S.

Assume $S \cap \mathfrak{p}$ is empty. Then E_S is non-zero. For, if not, then $e_i/1 = 0$ for $1 \leq i \leq n$; hence there exists σ in S such that $\sigma e_i = 0$, which belongs to P and so $\sigma \in \mathfrak{p}$, a contradiction. Clearly $P_S \neq E_S$.

Now let

$$(a/s)(e/t) \in P_S, \quad e/t \notin P_S,$$

where $a \in R, s, t \in S$, and $e \in E$. Then $ae/st = p/u$ for some $u \in S$ and $p \in P$; hence there is σ in S such that $\sigma((ua)e - (st)p) = 0$, which implies $(\sigma ua)e \in P$. Hence since $e \notin P$ and P is prime in E, we have $\sigma ua \in \mathfrak{p}$. Since $\sigma u \notin \mathfrak{p}$ we have $a \in \mathfrak{p}$. This implies $a/s \in \mathfrak{p}R_S$. It is well known [N76, p.41] that $\mathrm{Ann}_R(E/P)R_S = \mathrm{Ann}_{R_S}(E_S/P_S)$. Thus $a/s \in \mathrm{Ann}_{R_S}(E_S/P_S)$. Therefore every zero-divisor on E_S/P_S is an annihilator of E_S/P_S.

COROLLARY. *Let E be a non-zero finitely generated module over a commutative ring and P a prime R-submodule of E. Let $\mathfrak{p}$ denote* $\mathrm{Ann}_R(E/P)$. *Then the prime $R_{\mathfrak{p}}$-submodules of the non-zero finitely generated $R_{\mathfrak{p}}$-module $E \otimes_R R_{\mathfrak{p}}$ are in one-to-one order-preserving correspondence with the prime R-submodules Q of E such that* $\mathrm{Ann}_R(E/Q) \subseteq \mathfrak{p}$.

3. Spectra of finitely generated modules

This section deals with aspects of the identification of the maximal submodules of a finitely generated module over a commutative ring R. It shows an analogy between this set of submodules and the spectrum of R.

If E is an R-module, then the *radical* of E, denoted by $J(E)$, is defined to be the intersection of all maximal submodules of E, that is,

$$J(E) = \bigcap_{M \in \Omega_E} M,$$

where Ω_E is the collection of all maximal submodules of E. From now on we call Ω_E the *maximal spectrum* of E.

The following proposition is concerned with a relation between the radical $J(E)$ of a finitely generated R-module E and the Jacobson radical $J(R)$ of R.

PROPOSITION 11. *Let E be a non-zero finitely generated R-module and $\mathfrak{a}$ an ideal of R contained in the Jacobson radical $J(R)$ of R. Then* $\mathfrak{a} \subseteq \mathrm{Ann}_R(E/J(E))$. *In particular,* $J(R) \subseteq \mathrm{Ann}_R(E/J(E))$.

Proof. Let Ω_E denote the maximal spectrum of E. Then Ω_E is non-empty. For every M in Ω_E, $\mathrm{Ann}_R(E/M)$ is a maximal ideal of R. Hence by hypothesis

$$\mathfrak{a} \subseteq \bigcap_{M \in \Omega_E} \mathrm{Ann}_R(E/M). \tag{3.1}$$

Moreover, it is easy to show that

$$\bigcap_{M \in \Omega_E} \mathrm{Ann}_R(E/M) = \mathrm{Ann}_R(E/J(E)).$$

Therefore the proof is complete.

Note that (3.1) can also be proved by using Nakayama's lemma [AM69, Proposition 2.6, p.21].

Following the most general definition [CE56, p.147, M58, p.516, and SV72, p.63] we will call a ring R a *quasi-local ring* if the set of non-units of R forms a two-sided ideal, or equivalently, if R has only one maximal two-sided ideal.

DEFINITION. An R-module M is said to be a *quasi-local module* if M has the equivalent properties:
(a) M has a unique maximal submodule ;
(b) $M/J(M)$ is simple.

Let E be a finitely generated R-module. Then the *spectrum* of E, denoted by $\mathrm{Spec}_R(E)$, is defined to be the collection of all prime R-submodules of E. Thus, for any ring R, $\mathrm{Spec}_R(R)$ is the ordinary spectrum of R. Let $\bar{R}$ denote $R/\mathrm{Ann}_R E$ and define a mapping f : $\mathrm{Spec}_R(E) \to \mathrm{Spec}_R(\bar{R})$ by $f(P) = \overline{\mathrm{Ann}_R(E/P)}$, where $P \in \mathrm{Spec}_R(E)$. Then f is surjective. In fact, for any prime ideal $\mathfrak{p}$ of R containing $\mathrm{Ann}_R(E)$, $\mathfrak{p}E$ is a prime R-submodule of E and $\mathrm{Ann}_R(E/\mathfrak{p}E) = \mathfrak{p}$. The image of its restriction $f|_{\Omega_E} : \Omega_E \to \mathrm{Spec}_R(\bar{R})$ to the maximal spectrum Ω_E of E is $\Omega_{\bar{R}}$. The mapping f is not always injective since $f|_{\Omega_E}$ is not. The example is given as follows :

EXAMPLE. Note that the ring $\mathbf{Z}$ of integers is a faithful $\mathbf{Z}$-module. Consider the ring $\mathbf{Z}[i]$ of Gaussian integers, where $i = \sqrt{-1}$. Then since i is integral over $\mathbf{Z}$, $\mathbf{Z}[i]$ is a finitely generated $\mathbf{Z}$-module. It is trivial that, for any prime number p of $\mathbf{Z}$, $p\mathbf{Z} + i\mathbf{Z}$ and $\mathbf{Z} + i(p\mathbf{Z})$ are distinct maximal $\mathbf{Z}$-submodules of $\mathbf{Z}[i]$. Moreover,

$$\mathrm{Ann}_{\mathbf{Z}}(\mathbf{Z}[i]/(p\mathbf{Z} + i\mathbf{Z})) = \mathrm{Ann}_{\mathbf{Z}}(\mathbf{Z}[i]/(\mathbf{Z} + i(p\mathbf{Z}))) = p\mathbf{Z}.$$

Thus the mapping $f : \mathrm{Spec}_{\mathbf{Z}}(\mathbf{Z}[i]) \to \mathrm{Spec}_{\mathbf{Z}}(\mathbf{Z})$ is not injective.

If E is cyclic, then the mapping $f : \mathrm{Spec}_R(E) \to \mathrm{Spec}_R(\bar{R})$ is bijective. Hence we have the following lemma.

LEMMA 12. *Let E be a non-zero cyclic R-module. Then every prime R-submodule of E is of the form $\mathfrak{p}E$, where $\mathfrak{p}$ is a prime ideal of R containing $\mathrm{Ann}_R E$.*

THEOREM 13. *Let E be a non-zero cyclic R-module and P a prime R-submodule of E. Let $\mathfrak{p}$ denote* $\mathrm{Ann}_R(E/P)$. *Then the non-zero $R_{\mathfrak{p}}$-module $E \otimes_R R_{\mathfrak{p}}$ is a quasi-local $R_{\mathfrak{p}}$-module with unique maximal $R_{\mathfrak{p}}$-submodule $P \otimes_R R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}(E \otimes_R R_{\mathfrak{p}})$.*

Proof. Note that $E_{\mathfrak{p}} = E \otimes_R R_{\mathfrak{p}}$ and $P_{\mathfrak{p}} = P \otimes_R R_{\mathfrak{p}}$. Since E is cyclic, so is $E_{\mathfrak{p}}$. Hence $E_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-isomorphic to $R_{\mathfrak{p}}/\mathrm{Ann}_{R_{\mathfrak{p}}}(E_{\mathfrak{p}})$. Since $R_{\mathfrak{p}}$ is quasi-local with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, $E_{\mathfrak{p}}$ is quasi-local with unique maximal submodule $(\mathfrak{p}R_{\mathfrak{p}})E_{\mathfrak{p}}$ by Lemma 12.

This theorem can also be proved directly. In fact, since E is cyclic, $E \approx R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of R. It follows that $P \approx \mathfrak{q}/\mathfrak{a}$ for some prime ideal $\mathfrak{q}$ of R containing $\mathfrak{a}$. Hence

$$\mathfrak{p} = \mathrm{Ann}_R(E/P) = \mathrm{Ann}_R((R/\mathfrak{a})/(\mathfrak{q}/\mathfrak{a})) = \mathrm{Ann}_R(R/\mathfrak{q}) = \mathfrak{q}.$$

We need to show that $E_{\mathfrak{q}}$ is a quasi-local $R_{\mathfrak{q}}$-module.

$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ is exact so

$0 \to \mathfrak{a}R_{\mathfrak{q}} \to R_{\mathfrak{q}} \to (R/\mathfrak{a})_{\mathfrak{q}} \to 0$ is exact [AM69, Proposition 3.3].

Thus $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}} \approx (R/\mathfrak{a})_{\mathfrak{q}}$. In order to show that $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}$ is a quasi-local $R_{\mathfrak{q}}$-module it is sufficient to prove that the ring $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}$ is a quasi-local ring. But by using Proposition 3.1 of [AM69] it is easy to see that

$$R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}} \approx (R/\mathfrak{a})_{\mathfrak{q}/\mathfrak{a}}$$

which implies that $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}$ is a quasi-local ring [AM69, Example 1, p.38].

COROLLARY. *Let R be a commutative quasi-local ring with unique maximal ideal $\mathfrak{m}$. Let E be a non-zero finitely generated R-module. Then E is quasi-local if and only if E is cyclic.*

Proof. By Nakayama's lemma [AM69, Proposition 2.6, p.21], $\mathfrak{m}E \neq E$. This means that $E/\mathfrak{m}E$ is non-zero, or equivalently that $\mathrm{Ann}_R(E/\mathfrak{m}E) \neq R$. Also, $\mathfrak{m} \subseteq \mathrm{Ann}_R(E/\mathfrak{m}E)$. Hence $\mathfrak{m} = \mathrm{Ann}_R(E/\mathfrak{m}E)$. Note that $R_{\mathfrak{m}} = R$. Then if E is cyclic, then E is a quasi-local R-module with unique maximal submodule $\mathfrak{m}E$ (Theorem 13).

Conversely, assume that E is quasi-local with unique maximal submodule M. Take $e \in E \backslash M$. Then e generates E. For, otherwise, Re is a proper submodule of E. By [SV72, Proposition 1.6, p.7] $Re \subseteq M$, so $e \in M$, which contradicts.

Unless the finitely generated module E is cyclic Theorem 13 does not hold in general because the mapping $f : \mathrm{Spec}_R(E) \to \mathrm{Spec}_R(\bar{R})$ is not always injective.

Let R be a commutative quasi-local ring with unique maximal ideal $\mathfrak{m}$. Let E be a finitely generated R-module. $E/\mathfrak{m}E$ is annihilated by $\mathfrak{m}$, hence is naturally an $R/\mathfrak{m}$-module, i.e., a vector space over the field $R/\mathfrak{m}$, and as such is finite-dimensional. If $E/\mathfrak{m}E$ is zero-dimensional, then $\mathfrak{m}E = E$. This implies that E is zero by Nakayama's lemma [AM69, Proposition 2.6, p.21]. Therefore we have the following result.

PROPOSITION 14. *Let R be a commutative quasi-local ring with unique maximal ideal $\mathfrak{m}$. Let E be a non-zero finitely generated R-module. Then E has at least n distinct maximal R-submodules, where n is the dimension of the vector space $E/\mathfrak{m}E$ over the field $R/\mathfrak{m}$.*

Proof. Let $n = \dim_{R/\mathfrak{m}}(E/\mathfrak{m}E)$. As we have already observed, we have $1 \leq n < \infty$. Let e_i $(1 \leq i \leq n)$ be elements of E whose images $\overline{e}_i$ in $E/\mathfrak{m}E$ form a basis of this vector space. Then the e_i generate E [AM69, Proposition 2.8, p.22]. Now let $M_i = \mathfrak{m}e_i + \sum_{j \neq i} Re_j$, $1 \leq i \leq n$. Then we shall show that these are distinct maximal submodules of E.

Since $\{\overline{e}_1, \ldots, \overline{e}_n\}$ is a basis for the space $E/\mathfrak{m}E$ and $1 \notin \mathfrak{m}$ it follows that the M_i are distinct. In order to show that these are maximal, it is sufficient to prove that each E/M_i is a simple R-module. Again, to show this, it suffices to prove that E/M_i is a simple $R/\mathfrak{m}$-module.

$M_i = \sum_{j \neq i} Re_j + \mathfrak{m}E$, so each E/M_i is annihilated by $\mathfrak{m}$, hence is naturally an $R/\mathfrak{m}$-module, i.e., a vector space over the field $R/\mathfrak{m}$. Further, each E/M_i is $R/\mathfrak{m}$-isomorphic to $(E/\mathfrak{m}E)/(M_i/\mathfrak{m}E)$ [SV72, Proposition 1.9 Corollary 2, p.11]. Hence to show that each E/M_i is a simple $R/\mathfrak{m}$-module, it suffices to prove that each subspace $M_i/\mathfrak{m}E$ of the space $E/\mathfrak{m}E$ is a hyperspace in the space $E/\mathfrak{m}E$. But this follows immediately from the fact that each set $\{\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_n\}$ forms a basis for the subspace $M_i/\mathfrak{m}E$.

If E is a non-zero finitely generated module over a commutative quasi-local ring R with unique maximal ideal $\mathfrak{m}$, then the proposition implies that

$$\mathrm{Card}(\mathrm{Spec}_R(E)) \geq \mathrm{Card}(\Omega_E) \geq \dim_{R/\mathfrak{m}}(E/\mathfrak{m}E),$$

where Card A means the cardinality of a set A.

References

[AM69] Atiyah, M. F. and MacDonald, I. G., *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass., 1969.

[CE56] Cartan, H. and Eilenberg, S., *Homological algebra*, Princeton University Press, 1956.

[C50] Cohen, I. S., *Commutative rings with restricted minimum condition*, Duke Math. J. **17**(1950), 27–42.

[F73] Formanek, E., *Faithful Noetherian Modules*, Proc. Amer. Math. Soc. **41** (1973), 381–383.

[M58] Matlis, E., *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

[N62] Nagata, M., *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No.13, J. Wiley and Sons, 1962.

[N76] Northcott, D. G., *Finite Free Resolutions*, Cambridge Univ. Press, 1976.

[SV72] Sharpe, D. W. and Vámos, P., *Injective modules*, Cambridge, 1972.

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