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The Existence of an Upper Solution of $\frac{4(n-1)}{n-2}\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ on Compact Manifolds

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1. Introduction

In this paper, we prove some existence theorem for positive solutions of the elliptic nonlinear partial differential equation arising from conformal deformation of Riemannian metrics.

On compact manifolds of dimension $n \geq 3$ and with metric g_0, the problem of conformal deformation of metric is to find conditions on the function $K(x)$ so that $K(x)$ is the scalar curvature of a conformally related metric $g_1 = u^{n-2} g_0$, where u is a positive function on M.

If M admits $k = 0$ as the scalar curvature of g_0, then this is equivalent to the problem of solving the elliptic equation

$$\frac{4(n-1)}{n-2}\Delta u + Ku^{\frac{n+2}{n-2}} = 0, \quad u > 0, \tag{1}$$

where Δ is the Laplacian in the g_0 metric, (See [KW1], [KW2], [A] or [N]). Altough throuhout this paper we will assume that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs we need only assume that the curvature candidate K is Hölder continuous. In this case, the resulting metric with curvature K has Hölder continuous second derivatives.

In [KW1], J. L. Kazdan and F. W. Warner have studied the necessary conditions of the solvability of (1), i.e., K changes sign and $\bar{K} < 0$. In this paper, we shall prove the following, i.e., if K satisfies the necessary conditions, then there exists an upper (weak) solution of (1).

2. Main result

Let (M, g) be a complete connected manifold of dimension $n \geq 3$, which is not necessarily orientable. We denote the volume element of this metric by the gradient by ∇, and the mean value of a function f on M is written by $\bar{f}$, that is,

$$\bar{f} = \frac{1}{\mathrm{vol}(M)} \int_M f dV.$$

We let $Hs, p(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in $L_p(M)$.

It turns out that (1) is easier to analyze if we free it from geometry and consider instead

$$\Delta u + Hu^a = 0, \quad u > 0, \tag{2}$$

where H is an arbitrary function and $a > 1$ is a constant.

Lemma 1. *Let (M,g) be a compact Riemannian manifold. There exists a weak solution $w \in H_{1,2}(M)$ of $\Delta w = f$ if and only if $\bar{f} = 0$. The solution w is unique up to a constant. Moreover, if $f \in C^\infty(M)$, then $w \in C^\infty(M)$.*

Proof: See Theorem 4.7 in [A].

Lemma 2. *If a positive solution u of (2) exists and $H \neq 0$, then H must change sign and $\bar{H} < 0$.*

Proof: See Lemma 2.5 and Proposition 5.3 in [KW1].

Theorem (Existence of an upper (weak) solution). *Let $H(\neq 0)$ belong to $C^\infty(M)$ such that H changes sign and $\bar{H} < 0$. Then there exists an upper solution $u_+ > 0$ of (2), that is,*

$$\Delta u_+ + H u_+^a \leq 0.$$

Proof: Taking the change of variable $u_+ = e^v$,

$$\Delta u_+ + H u_+^a = e^v(\nabla v + |v|^2) + He^{av} \leq 0.$$

Hence it is sufficient to find v satisfying

$$\nabla v + |v|^2 + He^{cv} \leq 0,$$

where $c = a - 1 > 0$ is a constant.

But Lemma 1 implies that there exists a solution w of $\Delta w = \bar{H} - H$. We can pick $\alpha > 0$ so small that $|e^{c\alpha w} - 1| \leq \frac{-\bar{H}}{4\|H\|_\infty}$ and $\alpha|\nabla w|^2 \leq \frac{-\bar{H}}{4\|H\|_\infty}$. Let $e^{c\lambda} = \alpha$. Put $v = \alpha w + \lambda$. Then

$$\begin{aligned}
\Delta v + |\nabla v|^2 + He^{cv} &= \Delta(\alpha w + \lambda) + |\nabla(\alpha w + \lambda)|^2 + He^{c\alpha w + c\lambda} \\
&= \alpha \Delta w + \alpha^2 |\nabla w|^2 + \alpha H e^{c\alpha w} \\
&= \alpha \bar{H} - \alpha H + \alpha^2 |\nabla w|^2 + \alpha H e^{c\alpha w} \\
&= \alpha \bar{H} + \alpha^2 |\nabla w|^2 + \alpha H (e^{c\alpha w} - 1) \\
&\leq \alpha \bar{H} + \alpha^2 |\nabla w|^2 + \alpha \|H\|_\infty |e^{c\alpha w} - 1| \\
&\leq \alpha \bar{H} - \frac{\alpha \bar{H}}{4} - \frac{\alpha \bar{H}}{4} \\
&= \frac{\alpha \bar{H}}{2} < 0
\end{aligned}$$

Thus $u_+ = e^v = e^{\alpha w + \lambda}$ is an upper(weak) solution of (2).

Added in Proof: From the above theorem, if $\bar{H} < 0$, then we can always have an upper solution of (2). Hence in order to show that (2) has a solution, it suffices to find a lower (weak) solution u_- such that $0 < u_- < u_+$ and $\Delta u_- + H u_-^a \geq 0$. But as the author knows, this is still an open problem.

Remark: Many authers have studied the existence of a positive solution of (1) on the noncompact manifolds of dimension $n \geq 3$. (See [GS], [N] and [AM])

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