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Locally Convex Topologies of Vector Spaces[1)]

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1. Introduction

In the Hilbert space ℓ_2, a linear operator T can be represented by an infinite matrix $A = (a_{ij})$ using a basis (e. g. the standard basis) of ℓ_2. The known connexions between the continuity of T and the entries a_{ij} of A are very scarce [7].

In studying the connexion one usually considers the matrix A in an abstract vector space. However it will be more convenient to consider A in a topological vector space rather than in an abstract vector space.

Thus we are usually led to topologize the vector space of all infinite matrices with entries from a fixed field (e. g. the complex number field)

Let ℓ be the set of all sequences

$$x = (x_1, x_2, x_3, \dots)$$

of complex terms x_k with componentwise addition and componentwise scalar (complex) multiplication. Evidently ℓ forms a vector space and toplogizing the vector space of infinite matrices is evidently the same as topologizing the vector space ℓ. Hence topologizing the vector space ℓ is considered in § 3.

Locally convex topologies in a vector space are intimately connected with absorbing, balanced, convex subsets of the vector space [Theorem 3]. Hence some absorbing and non-absorbing subsets of vector spaces are considered in § 4.

Every vector space X has at least one admissible topology, namely the indiscrete topology (X, φ). Also it is evident that this indiscrete topology is the samllest admissible topology for the vector space X. Let

$$\dots\{A\}, \{B\}, \dots, \{C\}, \dots$$

be all admissible topologies of a vector space X. Then the familly $\psi = \{0 | 0 = \cup_\alpha 0\alpha,$ where each 0_α is a finite intersection $A \cap B \cap \cdots \cap C$ of open sets $A, B, \dots, C$ taken from the topologies $\{A\}, \{B\}, \dots, \{C\}\}$ defines a topology of X. That is, the finite intersections $A \cap B \cap \cdots \cap C$ constitute a basis for the topology ψ. Furthermore ψ is admissible to the vector space structure of X. For the continuity of the operation $x + y$, let 0 be an arbitrary neighborhood of $x + y$. Then there is a basis element $A \cap B \cap \cdots \cap C$ containing $x + y$ and which is inside 0. Since $\{A\}$ is an admissible topology for X, there are open subsets A' and A'' of the topology $\{A\}$ such that A' and A'' contain x and y respectively, and $A' + A'' \subset A$.

Likewise we can find $B', B''; \dots; C', C''$ from the topologies $\{B\}, \dots, \{C\}$ such that $B', \dots,$ C' contain x, $B'', \dots, C''$ contain y, and $B' + B'' \subset B, \dots, C' + C'' \subset C$.

Then $A' \cap B' \cap \cdots \cap C' + A'' \cap B'' \cap \cdots \cap C'' \subset A \cap B \cap \cdots \cap C$.

That is, the operation + is continuous. The continuity of scalar multiplication λx can also be proved in a similar fashion.

Thus ψ is the largest admissible topology for the vector space X. Hence we obtain the following Theorem.

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Theorem 1. *For an arbitrary vector space there always are a unique smallest admissible topology and a unique largest admissible topology.*

The indiscrete topology (X, φ) for a vector space X is clearly a locally convex topology. Therefore it is the unique smallest admissible locally convex topology for X.

Let $\ldots, \{A\}, \{B\}, \ldots, \{C\}, \ldots$

be all the admissible locally convex topologies of a vector space X. Then finite intersections $A \cap B \cap \cdots \cap C$ of open sets $A, B, \ldots, C$ taken from the topologies $\{A\}, \{B\}, \ldots, \{C\}$ constitute a basis of a topology ψ. That ψ is admissible to the vector space structure of X can be shown exactly as in Theorem 1. Furthermore it is easy to see that ψ is locally convex. Therefore ψ is the largest admissible locally convex topology for the vector space X. And we obtain the following theorem.

Theorem 2. *For an arbitrary vector space there always are a unique smallest admissible locally convex topology and a unique largest admissible locally convex topology.*

According to [2] this largest admissible locally convex topology can be described as follows.

Theorem 3. *Let X be an abstract vector space, and Γ the collection of all absorbing, balanced, convex subsets of X. Then the largest (finest) admissible locally convex topology of X can be obtained by letting Γ be the local base of neighborhoods of 0.*

Let X be an arbitrary abstract vector space and let $\{h_\lambda | \lambda \in \Lambda\}$ be a Hamel basis of X. Let α_λ be an arbitrary positive number, $\alpha_\lambda > 0$, for each $\lambda \in \Lambda$. Then $\|x\| = \sum_\lambda \alpha_\lambda |t_\lambda|$ for $x = \sum_\lambda t_\lambda h_\lambda \in X$ defines a norm in x. Therefore we obtain the following theorem.

Theorem 4. *Every abstract vector space can be made a normed vector space.*

2. The finest locally convex topology of the vector space $\mathbf{C}^n$

Theorem 5. *The finest admissible locally convex topology of the vector space $\mathbf{C}^n$ is the unique[2] (up to topological isomorphism) norm topology*

$$\|x\| = |x_1| + |x_2| + \cdots + |x_n|$$

where $x = (x_1, x_2, \ldots, x_n)$

Proof: Let Φ be the finest locally convex topology, and let ψ be the (uniform) topology induced by $\|x\|$. Since ψ also is a locally convex topology it is obvious that $\psi \subset \Phi$. So it is sufficient to show that $\Phi \subset \psi$. To this end it also is sufficient to show that the identity map

$$I : (\mathbf{C}^n, \psi) \to (\mathbf{C}^n, \Phi)$$

is continuous.

So what we have to show is that for any sequence $x^{(i)} \in \mathbf{C}^n$ $(i = 1, 2, 3, \ldots)$ with $\|x^{(i)}\| \to 0$ and Φ-nbhd 0 of the origin we can find an N such that $x^{(N)}, x^{(N+1)}, \ldots, \in O$.

[2] A. F. Taylor and D. C. Lay Introduction to Functional Analysis p.62

Now by the local convexity of Φ we can find a balanced convex neighborhood v of 0 such that v is contained in O. Since v is a Φ neighborhood of 0, it is an absorbing subset in C^n, and hence there is a positive number r such that $re_k \in O$ $(k = 1, 2, \ldots, n)$, where e_k are standard basis of the vector space C^n. Then by the absolute convexity (balanced and convex) of v the absolute convex hull of $re_1, re_2, \ldots, re_n$ is contained in v. That is

$$W = \{x \in C^n | x = x_1e_1 + \cdots + x_ne_n, |x_1| + \cdots + |x_n| \leq r\} \subset U$$

Since W is a ψ-neighborhood of 0, and since $\|x^{(i)}\| \to 0$ we can find an N such that

$$x^{(N)}, x^{(N+1)}, \ldots, \in W \subset v \subset O$$

proving the theorem.

Note: We know that admissible T_1-topology of C^n is unique [5]. Therefore if we know beforehand that the finest admissible locally convex topology of C^n is a T_1-topology, then it must coincide with the norm topology which is the unique admissible T_1-topology of C^n.

3. Topologies of the vector space ℓ

ℓ stands for the set of sequences $x = (x_1, x_2, x_3, \ldots)$ of complex entries x_k with componentwise addition and componentwise scalar multiplication.

For each n, $p_n(x) = |x_n|$ defines a semi-norm for the vector space ℓ. Hence the familly of all $p_n(x)$ defines a locally convex topology Φ in the vector space ℓ, and it is obvious that this topology is a Hausdorff one. Moreover it can be shown that the locally convex topology Φ indeed is a metric one.

Theorem 6. *Let*

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|/\{1 + |x_n - y_n|\} \tag{1}$$

for $x = (x_n)$ and $y = (y_n)$ of ℓ.

Then $d(x,y)$ is a metric of ℓ, and defines the semi-norms topology Φ

Proof: that $d(x,y)$ is a metric of ℓ is easy to see. An arbitrary neighborhood of 0 in the topology Φ of ℓ can be written

$$\begin{aligned} U &= \{x \in \ell \mid |x_1| < \varepsilon_1, |x_2| < \varepsilon_2, \ldots, |x_n| < \varepsilon_n\} \\ &= \{x \in \ell \mid 2^{-i}|x_i|/(1 + |x_i|) < \rho_i, i = 1, 2, \ldots, n\} \end{aligned}$$

where $\rho_i = 2^{-i}\varepsilon_i/(1 + \varepsilon_i)$

Let $\rho = \min(\rho_1, \rho_2, \ldots, \rho_n)$. Then $V = \{x \in \ell \mid d(x, 0) < P\} \subset U$ and V is a neighborhood of 0 in the metric topology induced by $d(x,y)$.

Conversely an arbitrary neighborhood of 0 in the metric topology of ℓ can be written

$$V = \{x \in \ell \mid d(x,0) < \varepsilon\}$$

Find an N such that $\sum_{k=N+1}^{\infty} 2^{-k} < \varepsilon/2$, Then

$$U = \{x \in \ell \mid 2^{-i}|x_i|/(1+|x_i|) < \varepsilon/(2N), i = 1,2,\ldots,n\} \subset V$$

and U is a neighborhood of 0 in the topology Φ.

Note: This topology Φ is not normable

Proof: The proof will be complete by [3] if we show that an arbitrary neighborhood $V(0)$ of O in the topology Φ is not bounded. Now we can find a neighborhood

$$W = \{(x_n) \in \ell \mid d((x_n), 0) < \varepsilon\}.$$

which is contained in $V(0)$. Find number p with $2^{-p} < \varepsilon$. And let $e_p = (0,\ldots,0,1,0,\ldots)$ whose only non-zero entry is the p-th entry 1. Let $\{\alpha_n\}$ be a null sequence of non-zero scalars, i. e. $\alpha_n \neq 0$ and $\alpha_n \to 0$. For example take α_n to be $1/n$.

Then $x_n = (1/\alpha_n)e_p \in W$ because

$$d(x_n, 0) = 2^{-p}|\frac{1}{\alpha_n}|/(1+|\frac{1}{\alpha_n}|) < 2^{-p} < \varepsilon.$$

That is, $x_n \in W$ $(n = 1,2,\ldots)$, $\alpha_n \to 0$, but $\alpha_n x_n = e_p$ does not tend to 0. Therefore by [4] W is not bounded. So $V(0)$ is not bounded.

The vector space ℓ can also be considered as the direct product

$$\ell = \prod_{n=1}^{\infty} Y_n \tag{2}$$

where each Y_n is the vector space $\mathbf{C}$ of complex numbers. Moreover each Y_n becomes a topological vector space $\mathbf{C}$ with the mertic

$$d_n(x_n, y_n) = 2^{-n}|x_n - y_n|/(1+|x_n - y_n|) \text{ for } x_n, y_n \in Y_n \tag{3}$$

And the cartesian product topology of (2) when each Y_n is topologized by the metric (3) can be defined by the metric

$$\rho(x,y) = \sup d_n(x_n, y_n) \text{ for } x = (x_n) \in \ell, y = (y_n) \in \ell \quad [1] \tag{4}$$

And it is easy to see that[3)] this metric (4) is equivalent to the metric $d(x,y)$ of (1). Hence we obtain the following theorem.

Theorem 6. *The semi-norms topology Φ of $\ell = \prod Y_n$ (each $Y_n = \mathbf{C}$) is the cartesian product topology of $\prod Y_n$ when to each Y_n is given the topology of the complex number plane.*

4. Absorbing and non-absorbing subsets in vector spaces

Let $r_k (k = 1,2,\ldots)$ be arbitrary positive numbers, and let

$$\begin{aligned} V(r_1, r_2, r_3, \ldots) &= V(r_1, r_2, r_3, \ldots; X) \\ &= \{x \in X \mid x = (x_k), |x_k| < r_k \text{ for } k = 1,2,3,\ldots\} \end{aligned} \tag{5}$$

be an (generalized) open box in the subspace X of ℓ. Then the following lemma holds.

[3)] Since $\rho(x,y) \leq d(x,y)$ every d-neighborhood of y, $\{x|d(x,y) < \varepsilon\}$, is contained in the ρ-neighborhood of y, $\{x|\rho(x,y) < \varepsilon\}$. Conversely let d-neighborhood of y, $\{x|d(x,y) < \varepsilon\}$, is given. Then there is an $n(\varepsilon)$ such that $\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots < \frac{\varepsilon}{2}$. Let $\delta = \frac{\varepsilon}{2n(\varepsilon)}$, then $\{x|\rho(x,y) < \delta\} \subset \{x|d(x,y) < \varepsilon\}$.

Lemma 1. *Let $1 \leq p < \infty$, $r_1 \geq r_2 \geq r_3 \geq \cdots \to 0$, and $\sum_{k=1}^{\infty} r_k^p < \infty$. Then the open box $V = V(r_1, r_2, r_3, \ldots; \ell_p)$ is not absorbing in the space ℓ_p.*

Proof: Let $\xi = (\xi_n)$ be such that

$$\xi_n = \begin{cases} (\frac{n}{2})^{1/p} r_n, & \text{when } n = 2^m, m = 1,2,3,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \|\xi\|_p^p &= \sum_{n=2^m} \{(n/2)^{1/p} r_n\}^p = \sum_n \frac{n}{2} r_m^p \\ &\leq \sum_n \{r_{2^{m-1}+1}^p + r_{2^{m-1}+2}^p + \cdots + r_{2^m}^p\} = \sum_k r_k^p < \infty \end{aligned}$$

Hence $\xi \in \ell_p$.

However no $s\xi \in V(r_1, r_2, r_3, \ldots; \ell_p)$ for $s > 0$, because $s\xi \in V$ implies that for all $n = 2^m$ $(m = 1,2,3,\ldots)$

$$s(\frac{n}{2})^{1/p} r_n < r_n.$$

and this is obviously impossible.

Lemma 2. *Let $1 \leq p < \infty$, and $r_1 \geq r_2 \geq r_3 \geq \cdots \to 0$. Then the open box $V = V(r_1, r_2, r_3, \ldots; \ell_p)$ is not absorbing in the sapce ℓ_p*

Proof: From the assumption we can find an increasing sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ such that

$$r_{n_1}^p + r_{n_2}^p + \cdots < \infty$$

Then

$$\ell_p \supset Y = \ell_p(n_1, n_2, \ldots) = \{x \in \ell_p | x = (x_k), x_k = 0 \text{ when } k \neq n_1, n_2, \ldots\} \cong \ell_p$$

That is, $\ell_p(n_1, n_2, \ldots)$ and ℓ_p are topologically isomorphic.

If $V(r_1, r_2, \ldots; \ell_p)$ were absorbing in the space ℓ_p, then

$$\{x \in Y | x = (x_k), |x_{n_1}| < r_{n_1} |x_{n_2}| < r_{n_2}, \ldots\}$$

would be absorbing in the space Y contradicting the lemma 1.

Theorem 7. *Let $1 \leq p < \infty$ and $r_k > 0 (k = 1,2,3,\ldots)$. If $\inf(r_1, r_2, r_3, \ldots) = 0$ then the open box $V = V(r_1, r_2, r_3, \ldots; \ell_p)$ is not absorbing in the space ℓ_p*

Proof: Put $n_1 = 1$. Then there is at least one n_2 such that

$$\frac{1}{2} r_{n_1} < r_{n_2}, n_1 < n_2. \tag{6}$$

Let n_2 be the least one satisfying (6). Next there is at least one n_3 such that

$$\frac{1}{2} r_{n_2} < r_{n_3}, n_2 < n_3. \tag{7}$$

Let n_3 be the least one satisfying (7).

Similarly we can choose $n_4 < n_5 < \dots$. And it is obvious that

$$r_{n_1} > r_{n_2} > r_{n_3} > \cdots \to 0.$$

Also it is obvious that

$$Y = \ell_p(n_1, n_2, \dots) = \{x \in \ell_p | x = (x_k), x_k = 0 \text{ when } k \neq n_1, n_2, \dots\} \cong \ell_p.$$

If V were absorbing in the space ℓ_p, then

$$\{x \in Y | x = (x_k), |x_{n_1}| < r_{n_1}, |x_{n_2}| < r_{n_2} \dots\}$$

would be absorbing in the space Y contradicting the lemma 2.

Let C_0, C and ℓ_∞ be the sub-vector spaces of ℓ such that

$$C_0 = \{x \in \ell | x = (x_k), x_k \text{ converges to o}\}$$
$$C = \{x \in \ell | x = (x_k), x_k \text{ converges }\}$$
$$\ell_\infty = \{x \in \ell | x = (x_k), x_k \text{ are bounded}\}$$

Then $\ell_p \subset C_0 \subset C \subset \ell_\infty \subset \ell$.

And in the following corollary let

$$V(r_1, r_2, r_3, \dots; X)$$

be the open box (5) with $X = C_0, C, \ell_\infty$ and ℓ respectively. Then we obtain the following corollary.

Corollary 1. *Let $r_k > 0$ $(k = 1, 2, 3,)$. If $\inf(r_1, r_2, r_3, \dots) = 0$ then the open box $V = V(r_1, r_2, r_3, \dots; X)$ is not absorbing in the vector spaces $X = C_0, C, \ell_\infty$ and ℓ.*

Proof: If V were absorbing in the space C_0, C, ℓ_∞ or ℓ, then V would be absorbing in the subspace ℓ_p contradicting the theorem 7

Theorem 8. *Let $r > 0$. Then the open box*

$$V = V(r, r, r, \dots; \ell_\infty)$$

is absorbing in the vector space 1_∞.

Proof: Let $\xi = (\xi_k) \in \ell_\infty$, then

$$|\xi_k| \leq M \quad \text{for } k = 1, 2, 3, \dots$$

Now we can find a number s such that $sM < r$. Then $s\xi \in V$.

Corollary 2. *Let $r > 0$. Then $V = V(r, r, r, \dots; X)$ is absorbing in the vector spaces $X = \ell_p, C_0$ and C.*

Proof: proof is immediate because ℓ_p, C_0 and C are sub-spaces of 1_∞.

Theorem 9. *Let $r_k > 0$ $(k = 1, 2, 3, \dots)$. Then the open box $V = V(r_1, r_2, r_3, \dots; \ell)$ is not absorbing in the vector space 1.*

Proof: Let $\xi = (r_1, 2r_2, 3r_3, \dots, nr_n, \dots)$. Then $\xi \in \ell$. And $\rho\xi \in V$ implies

$$|\rho n r_n| < r_n \quad \text{for } n = 1, 2, 3, \dots.$$

This is possible only when $\rho = 0$.

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