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On the Curvature Tensor in n-dimensional Semi-symmetric Einstein $^*g$-manifold

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I. n-dimensional $*g$-manifold X_n

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be semi-symmetric if its torsion tensor $S_{\lambda}\mu^{\nu}$ is of the form

$$S_{\lambda}\mu^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} \tag{1-1}$$

for an arbitrary vector X. A connection, which is semi-symmetric, and is satisfied by the system of equations

$$D_{\omega}{}^*g^{\lambda\mu} = -2S^{\mu}_{\omega\alpha}{}^*g^{\lambda\alpha}, \tag{1-2}$$

is called a SE-connection.

A generalized n-dimensional manifold X_n, on which the differential structure is imposed by the tensor $^*g^{\lambda\nu}$ by means of a SE-connection, is called an n-dimensional $^*g$-SE-manifold, denoted by $^*gSEX_n$.

In the next we state a theorem, proof of which is given by [1].

Theorem 1. *If there exists a SE-connection $\Gamma_{\lambda\mu}{}^{\nu}$, it must of the form*

$$\Gamma_{\lambda\mu}{}^{\nu} = *\{{}_{\lambda\mu}{}^{\nu}\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}_{\lambda\mu}, \tag{1-3}$$

where

$$U^{\nu}_{\lambda\mu} = - * h_{\lambda\mu}\overset{(1)}{X}{}^{\nu}. \tag{1-4}$$

II. The SE-curvature tensor in $*g - SEX_n$

Since we have found the SE-connection in the form (1-3), we may define the representation of the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$, defined by

$$R_{\omega\mu\lambda}{}^{\nu} \stackrel{\text{def}}{=} 2(\partial_{[\mu}\Gamma_{|\lambda|\omega]}{}^{\nu} + \Gamma_{\alpha[\mu}{}^{\nu}\Gamma_{|\lambda|\omega]}{}^{\alpha}), \tag{2-1}$$

as a function of X_{λ}, $^*g^{\lambda\nu}$ and their first two derivatives by substituting (1-3) into (2-1).

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Theorem 2. *In *g-SEX$_n$, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = *H^{\nu}_{\omega\mu\lambda} + \underset{1}{R}_{\omega\mu\lambda}{}^{\nu} + \underset{2}{R}_{\omega\mu\lambda}{}^{\nu}, \tag{2-2}$$

where

$$*H_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu} * \{{}^{\nu}_{\omega]\lambda}\} + *\{{}^{\nu}_{\alpha[\mu}\} * \{{}^{\alpha}_{\omega]\lambda}\}) \tag{2-3a}$$

$$\underset{1}{R}_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + \delta^{\nu}_{[\mu}\nabla_{\omega]}X_{\lambda} + \nabla_{[\mu}U^{\nu}_{\omega]\lambda}) \tag{2-3b}$$

$$\underset{2}{R}_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{[\mu}X_{\omega]}X_{\lambda} + *h_{\lambda[\omega}U_{\mu]}U^{\nu}) \tag{2-3c}$$

Proof: Substituting (1-3) into (2-1) and making use of (2-3), we have

$$\begin{aligned} R_{\omega\mu\lambda}{}^{\nu} &= 2\partial_{[\mu}({}^{*}\{{}_{\omega]\lambda}{}^{\nu}\} + X_{\omega]}\delta^{\nu}_{\lambda} - \delta^{\nu}_{\omega]}X_{\lambda} + U^{\nu}_{\omega]\lambda}) + 2({}^{*}\{{}_{\alpha[\mu}{}^{\nu}\} + \delta^{\nu}_{\alpha}X_{\mu} - X_{\alpha}\delta^{\nu}_{[\mu} \\ &\quad + U^{\nu}_{\alpha[\mu})({}^{*}\{{}^{\alpha]}_{\omega]\lambda}\} + X_{\omega]}\delta^{\alpha}_{\lambda} - \delta^{\alpha}_{\omega]}X_{\lambda} + U^{\alpha}_{\omega]\lambda}) \\ &= {}^{*}H_{\omega\mu\lambda}{}^{\nu} + 2\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + 2(\delta^{\nu}_{[\mu}\partial_{\omega]}X_{\lambda} - \delta^{\nu}_{[\mu}{}^{*}\{{}_{\omega]\lambda}{}^{\alpha}\}X_{\alpha}) \\ &\quad + 2(\partial_{[\mu}U^{\nu}_{\omega]\lambda} + {}^{*}\{{}^{\alpha}_{\lambda[\omega}\}U^{\nu}_{\mu]\alpha} + {}^{*}\{{}^{\nu}_{\alpha[\mu}\}U^{\alpha}_{\omega]\lambda}) \\ &\quad + 2(\delta^{\nu}_{[\mu}X_{\omega]}X_{\lambda} - X_{\alpha}\delta^{\nu}_{[\mu}U^{\alpha}_{\omega]\lambda} + U^{\nu}_{\alpha[\mu}U^{\alpha}_{\omega]\lambda]}). \end{aligned} \tag{2-4}$$

Crealy the sum of the second, third and fourth terms on the right hand side of (2-4) is equal to $\underset{1}{R}_{\omega\mu\lambda}{}^{\nu}$. Substituting (1-4) into the fifth term of (2-4), it is equal to $\underset{2}{R}_{\omega\mu\lambda}{}^{\nu}$. Hence our proof is completed.

Theorem 3. *In *g-SEX$_n$, we have the following identities:*

$$R_{\omega\mu\lambda}{}^{\nu} = R_{[\omega\mu]\lambda}{}^{\nu} \tag{2-5}$$

$$R_{[\omega\mu\lambda]}{}^{\nu} = 4\partial^{\nu}_{[\lambda}\delta_{\mu}X_{\omega]} \tag{2-6}$$

Proof: (2-5) follows immediately from (2-1). In order to prove (2-6), we use (2-2) to obtain

$$R_{\omega\mu\lambda]}{}^{\nu} = *H_{[\omega\mu\lambda]}{}^{\nu} + R_{[\omega\mu\lambda]}{}^{\nu} + \underset{2}{R}_{[\omega\mu\lambda]}{}^{\nu} \tag{2-7}$$

In virtue of (2-2), we have

$$*H^{\nu}_{[\omega\mu\lambda]} = \underset{2}{R}^{\nu}_{[\omega\mu\lambda]} = 0, \quad \underset{1}{R}^{\nu}_{[\omega\mu\lambda]} = 4\delta^{\nu}_{[\mu}\partial_{\omega}X_{\lambda]} \tag{2-8}$$

Our identity (2-6) follows by substitution of (2-8) into (2-7).

The following two theorems are direct consequences of Hlavaty's results [3 p. 129]

$$\begin{aligned} 2D_{[\omega}D_{\mu]}T^{\nu_1\ldots\nu_p}_{\lambda_1\ldots\lambda_q} &= -\sum_{\alpha=1}^{p} T_{\lambda_1\ldots\lambda_q}^{\nu_1\ldots\nu_p\xi\nu_{\alpha+1}\ldots\nu_p} R_{\omega\mu\xi}{}^{\nu_\alpha} \\ &\quad + \sum_{\beta=1}^{q} T^{\nu_1\ldots\nu_p}_{\lambda_1\ldots\lambda_{\beta-1}\xi\lambda_{\beta+1}\ldots\lambda_q} R_{\omega\mu\lambda_\beta}{}^{\xi} + 2S_{\omega\mu}{}^{\alpha}D_{\alpha}T^{\nu_1\ldots\nu_p}_{\lambda_1\ldots\lambda_q} \end{aligned} \tag{2-9}$$

$$D_{[\xi}R_{\omega\mu]\lambda}{}^{\nu} = -2S_{[\xi\omega}{}^{\beta}R_{\mu]\beta\lambda}{}^{\nu}, \tag{2-10}$$

which hold on a manifold to which an Einstein's connection is connected.

Theorem 4 (**Generalized Ricci's identity**). *The SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ in $*g$-SEX_n satisfies the following identities:*

$$\begin{aligned} 2D_{[\omega}D_{\mu]}T^{\nu_1\ldots\nu_p}_{\lambda_1\ldots\lambda_q} = &-\sum_{\alpha=1}^{p} T_{\lambda_1\ldots}{}^{\nu_1\ldots\nu_{\alpha-1}\xi\nu_{\alpha+1}\ldots\nu_p}{}_{\lambda_q} R^{\nu_\alpha}_{\omega\mu\xi} \\ &+\sum_{\beta=1}^{q} T^{\nu_1\ldots}{}_{\lambda_1\ldots\lambda_{\beta-1}\xi\lambda_{\beta+1}\ldots\lambda_q}{}^{\nu_p} R^{\xi}_{\omega\mu\lambda_\beta} - 4X_{[\omega}D_{\mu]}T^{\nu_1\ldots\nu_p}_{\lambda_1\ldots\lambda_q} \end{aligned} \tag{2-11}$$

Proof: Making use of (1-2), we have (2-11) as a direct consequence of (2-9).

Theorem 5 (**Generalized Bianchi's identity**). *In $^*g$-SEX_n, we have the following identities:*

$$D_{[\xi}R_{\omega\mu]\lambda}{}^{\nu} = -4X_{[\xi} * H_{\omega\mu]\lambda}{}^{\nu} + M_{[\xi\omega\nu]\lambda}{}^{\nu}, \tag{2-12}$$

where

$$\frac{1}{8}M_{\xi\omega\mu\lambda}{}^{\nu} = (\delta^{\nu}_{\lambda}X_{\xi}\partial_{\omega}X_{\mu} + X_{\xi}\delta^{\nu}_{\omega}\nabla_{\mu}X_{\lambda} + X_{\xi}\nabla_{\omega}U^{\nu}_{\mu\lambda}) + *h_{\lambda\omega}X_{\mu}U_{\xi}U^{\nu}. \tag{2-13}$$

Proof: In virtue of (1-2) and (2-2), (2-9) may be rewritten as

$$\begin{aligned} D_{[\xi}R_{\omega\mu]\lambda}{}^{\nu} &= -2S^{\beta}_{[\xi\omega} * H_{\mu]\beta\lambda}{}^{\nu} - 2S^{\beta}_{[\xi\omega}\underset{1}{R}_{\mu]\beta\lambda}{}^{\nu} - 2S^{\beta}_{\xi\omega}\underset{2}{R}_{\mu]\beta\lambda}{}^{\nu} \\ &= -4X_{[\omega} * H^{\nu}_{\mu\xi]\lambda} - 4X_{[\xi}\underset{1}{R}_{\omega\mu]\lambda}{}^{\nu} - 4X_{[\xi}\underset{2}{R}_{\omega\mu]\lambda}{}^{\nu}. \end{aligned} \tag{2-14}$$

In virtue of (2-3)b and (2-3)c, the last two terms on the right side of (2-14) may be written as

$$X_{[\xi}\underset{1}{R}_{\omega\mu]\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}X_{[\xi}\partial_{\mu}X_{\omega]} + X_{[\xi}\delta_{\mu}\nabla_{\omega]}X_{\lambda} + X_{[\xi}\nabla_{\mu}U^{\nu}_{\omega]\lambda}) \tag{2-15a}$$

$$X_{[\xi}\underset{2}{R}_{\omega\mu]\lambda}{}^{\nu} = -2X_{[\xi}U_{\mu}{}^{*}h_{\omega]\lambda}\overset{(1)}{X}{}^{\nu} = 2^{*}h_{\lambda[\omega}X_{\xi}U_{\mu]}U^{\nu} \tag{2-15b}$$

We substitute (2-15)a, b into (2-14) and make use of (2-13) to complete the proof of (2-12).

References

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