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Iterative Algorithm for Multiobjective Optimization with Set Functions*

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ABSTRACT. Multiobjective Optimization problem involving set functions is introduced. Then an iterative algorithm for these kinds of problems is suggested and its optimal process will be proved.

1. Introduction

In [9], Morris gave some examples for optimization problem involving set functions. In [7], the generalized Fenchel theorem on set functions is formulated and proved. In [12], Tanaka and Maruyama proved some properties for the multiobjective optimization problem of set functions. For a domination cone containing the nonnegative orthant, some results are known by Chou, Hsia and Lee [3]-[6] and by Lee [8].

Let $(X, \mathbb{U}, m)$ be a measure space and $F : \mathbb{U} \longrightarrow R^p$ be a p-dimensional vector valued set function, $G : \mathbb{U} \longrightarrow R^q$ be a q-dimensional vector valued set function. We wish to find $\Omega^* \in \mathbb{U}$ such that $F(\Omega^*)$ is a minimum subject to the constraint $G(\Omega) \leq 0$.

Problem (T):

$$\mathrm{Min}_{\substack{\Omega\in C\subseteq U \\ G(\Omega)\leq 0}} F(\Omega)$$

where $\mathbb{C}$ is a subset of the σ-algebra $\mathbb{U}$ that posseses convexity in some sense. Inequalities between vectors will be clear as we proceed.

In this paper, we concentrate on the investigation of an algorithm which finds Ω^* satisfying the minimality. In section 2, preliminaries concerning optimization with set functions will be given. And multiobjective programming problem with set functions will be defined. And some known results will be presented. In section 3, an iterative algorithm concerning multiobjective optimization with set functions will be introduced and its optimal process will be proved.

2. Multiobjective optimization with set functions

We first give a definition of inequalities between vectors.
Let $\overline{K}$ be the closure of a subset K of R^p and $\mathrm{int}(K)$ be the interior of a subset K of R^p. For two vectors $x = (x_1, x_2, \ldots, x_p)$, $y = (y_1, y_2, \ldots, y_p) \in R^p$, and a cone K of R^p,

(1) $x <_K y$ iff $y - x \in \mathrm{int}(K)$
(2) $x \leq_K y$ iff $y - x \in K - \{0\}$
(3) $x \leq y$ iff $y - x \in K$

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Let the nonnegative orthant be $R^p_+ = \{x \in R^p | x \geq 0\}$ and the nonpositive orthant $R^p_- = \{x \in R^p | x \leq 0\}$.
If K is the nonnegative orthant, K will be ommited in the inequalities. The inner product of the vectors x and y will be denoted by

$$< x, y >= \sum_{i=1}^{p} x_i y_i.$$

For a subset K in $\mathbf{R}^n$ is positive polar E^0 is defined by

$$K^0 = \{x^* \in \mathbf{R}^n | < x, x^* > \geq 0 \text{ for any } x \in E\}.$$

Lemma 2.1. *Let K be a pointed closed convex cone in R^p. If $0 \neq \lambda \in K^0$ and $x \in$ int(K), then $< \lambda, x >> 0$.*

Definition 2.1: Let K be a cone. A set A is said to be K-*convex* if $A + K = \{a + k | a \in A, k \in K\}$ is a convex subset in R^p.

In optimization problems, the set of efficient points of a feasible set gives a solution set.

Definition 2.2: A point x^* of E is called an *efficient point* of $E \subset R^p$ if $(E - x^*) \bigcap R^p_+ = \{0\}$, that is, for $x^* \in E$, there is no $x \in E$ such that $x \leq x^*$.

Another restricted solution concept, proper-efficiency, eliminates efficient points of certain types of abnormality.

Definition 2.3: A point $x^* = (x^*_1, x^*_2, \ldots, x^*_p) \in R^p$ is a *properly efficient point* of

$$E \subset R^p$$

if $(\overline{p(E + R^p_+ - x^*)} \bigcap R^p_-) = \{0\}$, where $p(S) = \{\alpha x | \alpha > 0, x \in S\}$ is the projecting cone for a set $S \subset R^p$.

Lemma 2.2 [8]. *Let E be a R^p_+-convex set of R^p. Then x^* is a properly efficient point if and only if there exists $\lambda^* \in$ int R^p_+ such that*

$$< \lambda^*, x^* > \leq < \lambda^*, x >,$$

for all $x \in E$.

Recently, multiobjective programming with set functions have been studied by Chou, Hsia, and Lee [3], [4]. The programming problem involves optimal selection of a measurable subset for a given measure space and set functions on it. The usual convexity theory can not be applied in this situation because of the pooly structured σ-algebra. The concept of a convex set functions was originally defined by Morris [mo] and then refined later by others.

For $\Omega \in U$, χ_Ω denotes the charateristic function of Ω. In this way, $\mathbf{U}$ can be viewed as a subspace of L_∞. We shall write L_p instead of $L_p(X, \mathbf{U}, m)$, for $0 \leq p < \infty$.

Proposition 2.3 [9]. *If a measure space $(X, \mathbb{U}, m)$ is finite atomless and L_1 separable, then for any $\Omega_1, \Omega_2 \in \mathbb{U}$ and $\lambda \in [0,1]$, there exists a sequence $\{\Gamma_n\} \subset \mathbb{U}$ such that $\chi_{\Gamma_n} \xrightarrow{w^*} \lambda\chi_{\Omega_1} + (1-\lambda)\chi_{\Omega_2}$, where $\xrightarrow{w^*}$ denotes the weak* convergence of elements in L_∞.*

This $\{\Gamma_n\} \subset \mathbb{U}$ is called Morris-sequence associated with $(\lambda, \Omega_1, \Omega_2)$.

Throughout this paper, let $(X, \mathbb{U}, m)$ be a finite atomless measure space with $L_1(X, \mathbb{U}, m)$ separable. Then, by the above proposition, we could replace convex combinations by a Morris sequence.

Definition 2.4: [4]. A subfamily $\mathbb{C}$ of $\mathbb{U}$ is called *convex* if, given $(\alpha, \Omega_1, \Omega_2) \in [0,1] \times \mathbb{C} \times \mathbb{C}$ and a Morris sequence $\{\Gamma_{n_k}\}$ in $\mathbb{U}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathbb{C}$.

A convex set function is now generalized to a convex subfamily $\mathbb{C}$ of $\mathbb{U}$.

Definition 2.5: [5]. Let $\mathbb{C}$ be a convex subfamily of $\mathbb{U}$ and K be a convex cone in R^p. A multi-valued set function $H : \mathbb{C} \longrightarrow R^p$ is *K-convex* if, given $(\alpha, \Omega_1, \Omega_2)$ in $[0,1] \times \mathbb{C} \times \mathbb{C}$ and any Morris-sequence $\{\Gamma_n\}$ in $\mathbb{U}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathbb{C}$ such that

$$\limsup_{k \to \infty} \mathrm{H}(\Gamma_{n_k}) \leq_k \alpha \mathbb{H}(\Omega_1) + (1-\alpha)\mathrm{H}(\Omega_2),$$

where lim sup is taken over each component.

Proposition 2.4 [4]. *If $\mathbb{C}$ is convex, then $\overline{\mathbb{C}}$, the w^*-closure of $\mathbb{C}$ in L_∞, is the w^*-closed convex hull of $\mathbb{C}$ and $\overline{\mathbb{U}} = \{f \in L_\infty | 0 \leq f \leq 1\}$.*

Using this ideas now we able to introduce a version of continuity on a convex subfamily $\mathbb{C}$.

Definition 2.6: A vector-valued set function $H = (H_1, H_2, \ldots, H_p) : \mathbb{C} \longrightarrow R^p$ is called *w^*-continuous* on $\mathbb{C}$ if for each $f \in \overline{\mathbb{C}}$ and for each $j = 1, 2, \ldots, p$, $\{H_j(\Omega_m)\}$ converges to the same limit for all $\{\Omega_m\}$ with $\chi_{\Omega_m} \xrightarrow{w^*} f$.

Given a cone E of R^p and a set function H on convex subfamily $\mathbb{C}$, a restricted result comparable to that of Proposition 2.2.7 [sa] obtains:

Lemma 2.5 [5]. *Let K be a closed convex cone in R^p. If $H : \mathbb{C} \longrightarrow R^p$ is a w^*-continuos, K-convex set function, then the closure of $H(\mathbb{C})$ in R^p, $\overline{H(\mathbb{C})}$, is K-convex.*

For a p-dimensional function

$$f = (f^1, f^2, \ldots, f^p)^t, \quad f^i \in L_1(X, \mathbb{U}, m), \quad i = 1, 2, \ldots, p$$

and characteristic function $\chi_\Omega \in L_\infty(X, \mathbb{U}, m)$, we denote

$$\ll f, \chi_\Omega \gg = (< f^1, \chi_\Omega >, < f^2, \chi_\Omega >, \ldots, < f^p, \chi_\Omega >)^t,$$

where $< f^i, \chi_\Omega > = \int_\Omega f^i dm$, $i = 1, \ldots, p$, is the inner product of f^i and χ_Ω.

A pseudometric ρ on the σ-algebra $\mathbb{U}$ will be defined by the following way

$$\rho(\Omega_1, \Omega_2) = m(\Omega_1 \Delta \Omega_2), \quad \Omega_1, \Omega_2 \in \mathbb{U},$$

where $\Omega_1 \Delta \Omega_2$ denotes the symmetric difference for Ω_1 and Ω_2.

Definition 2.7: A p-dimensional vector valued set function $F : \mathbf{U} \longrightarrow \mathbf{R}^p$ is said to be *differentiable* at Ω_0 if for $F = (F_1, \dots, F_p)$ with each element set function $F_i : \mathbf{U} \longrightarrow \mathbf{R}$, $i = 1, 2, \dots, p$, there exists

$$f_{\Omega_0} = (f^1_{\Omega_0}, \dots, f^p_{\Omega_0}), \quad f^i_{\Omega_0} \in L_1, \quad i = 1, 2, \dots, p,$$

the derivative at Ω_0 such that

$$F(\Omega) = F(\Omega_0) + \ll f^i_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0} \gg + E_F(\Omega, \Omega_0)$$

where $E_F(\Omega_0, \Omega) = (E_{F_1}(\Omega_0, \Omega), \dots, E_{F_p}(\Omega_0, \Omega))$ and each $E_{F_i}(\Omega, \Omega_0)$'s are $o(\rho(\Omega, \Omega_0))$, i.e.,

$$\lim_{\rho(\Omega,\Omega_0)\to 0} (E_{F_i}(\Omega, \Omega_0)/\rho(\Omega, \Omega_0)) = 0,$$

for $i = 1, \dots, p$.

By [9], the derivative f_Ω is unique.

We assume, in optimization problem (T), that F, G are w^*-continuous and differentiable at each $\Omega \in \mathbf{U}$.

Let us consider a common regularity condition on the feasible set:

$$G(\Omega_0) < 0,$$

for some $\Omega_0 \in \mathbf{C}$. This is only a straightforward generalization of well-known Slater's constraint qualification.

Lemma 2.6. *Assume Slater's constraint qualification for the problem* (T). *Then* $\overline{F(\mathbf{C}')}$ *is a convex subset in* R^p, *where the sets* $\mathbf{C}' = \{\Omega \in \mathbf{C} | G(\Omega) \leq 0\}$ *and* $F(\mathbf{C}') = \{F(\Omega) | \Omega \in \mathbf{C}'\}$.

Remark:: Note that the feasible family $\mathbf{C}' = \{\Omega \in \mathbf{C} | G(\Omega) \leq 0\}$ is not convex in general (see Example 3.1 of [5]). However,in the proof of lemma, $\overline{F(\mathbf{C}')} = \overline{F(\mathbf{C}'')}$ is convex by the fact that $\mathbf{C}'' = \{\Omega \in \mathbf{C} | G(\Omega) < 0\}$ is convex under Slater's constraint qualfication. Since F is w^*-continuous and $\overline{\mathbf{C}'}$ is w^*-compact in L_∞,we have that $\overline{F(\mathbf{C}')} = \overline{F}(\mathbf{C}')$ is compact, where $\overline{F}$ is an extention of F on L_∞.

Next we have the Lagrange multiplier theorem for vector-valued programming with set functions. The set of all $p \times q$ matrices is denoted by $R^{p\times q}$. The set $\{M \in R^{p\times q} | < M, R^q_+ > \subset R^p_+\}$ is denoted by L.

Theorem 2.7 [8]. *Let* Ω^* *be a properly efficient solution to the problem* (T). *If there is* $\Omega_0 \in \mathbf{C}$ *such that* $G(\Omega_0) < 0$, *then there exists* $M^* \in L$ *such that*

(i) $F(\Omega^*) \in \mathrm{Min}\{L(\Omega, M^*) | \Omega \in \mathbf{C}\}$

(ii) $< M^*, G(\Omega^*) >= 0$,

where $L(\Omega, M) = F(\Omega) + < M, G(\Omega) >$, *for* $\Omega \in \mathbf{C}$ *and* $M \in R^{p\times q}$.

Corollary 2.8 [8]. *Suppose that* Ω^* *is a properly efficient solution to the proplem* (T). *If there is a* $\Omega_0 \in \mathbf{C}$ *such that* $G(\Omega_0) < 0$, *then there exists a vector* $\lambda^* \in R^q_+$ *such that*

(i) $F(\Omega^*) \in \mathrm{Min}\,\overline{\{F(\Omega) + \ll \lambda^*, G(\Omega) \gg | \Omega \in \mathbf{C}\}}$
$\qquad \bigcap \mathrm{Min}\{F(\Omega) + \ll \lambda^*, G(\Omega) \gg | \Omega \in \mathbf{C}\}$,

and

(ii) $\ll \lambda^*, G(\Omega) \gg = 0$, *where* $\ll \lambda, x \gg$ *is a vector with each component* $< \lambda, x >$.

3. Iterative Algorithm

The constrained problem (T) can be converted into unconstrained problem using the correct Lagrange multipliers . Hence we will concentrate on unconstrained optimization problem. It will be assumed that $(X, \mathbb{U}, m)$ is the Lebesque measure space over $X \subseteq R^p$, $m(X) < \infty$ and that all set funtions are differentiable.

Definition 3.1: $\Omega_0 \in \mathbb{U}$ is *local mimimum* in (T) if there exists $\varepsilon > 0$ such that for Ω satisfying that $\rho(\Omega_0, \Omega) < \varepsilon, G_i(\Omega) \leq 0$, $i = 1, 2, \ldots, q$, it follows that $F(\Omega_0) \leq F(\Omega)$.

Definition 3.2: Given a set function $f = (f^1, \ldots, f^p) : X \longrightarrow R^p, f^j \in L_1$, $j = 1, 2 \ldots, p$, we say that f *separates* Ω_0 if

$$0 \leq \ll f, \chi_\Omega - \chi_{\Omega_0} \gg \quad \text{for all } \Omega \in \mathbb{U}$$

or equivalently $f \leq 0$ a.e. on Ω_0, $f \geq 0$ a.e. on Ω_0^c.

Note that a seprating set satisfies the necessary condition for minimality.

A numerical approximation to an optimal set is to represent it in terms of a finite number of elementary sets. Our approach is to partition X into a finite union of disjoint elementary sets or finite elements as follows :
Let $\{\Lambda_i^h\}_{i=1}^N$ be a family of disjoint measurable sets with

$$\rho(X, \bigcup_{i=1}^{N} \Lambda_i^h) = 0$$

and

$$m(\Lambda_i^h) = h, \qquad i = 1, 2, \ldots, N.$$

Denote by $\mathbb{U}_{\Lambda^h}$ the power set $P[\{\Lambda_i^h\}_{i=1}^N]$.

An iterative algorithm to find a separating element of $\mathbb{U}_{\Lambda^h}$ will be stated now. This element approximately satisfies the necessary condition for the problem (T). It is an extention of an algorithm from [9].

Algorithm 3.3:

Let $M > 0$, $\Omega_0 \in \mathbb{U}_{\Lambda^h}$.
(1) $\Omega \leftarrow \Omega_0$.
(2) For each $j = 1, 2, \ldots, p$,
(i) select $\Lambda_j \in \{\Lambda_i^h\}_{i=1}^N$, $\Lambda_j \subseteq \Omega$ with $< f_\Omega^j, \chi_{\Lambda_j} > > Mh^2$
or
(ii) select $\Lambda_j' \in \{\Lambda_i^h\}_{i=1}^N$, $\Lambda_j' \subseteq \Omega^c$ with $< f_\Omega^j, \chi_{\Lambda_j'} > < -Mh^2$
If neither (i) nor (ii) is satisfiable, then stop.
(3) $\Omega_{\text{Next}} \leftarrow [\Omega \backslash \bigcap_{j \in K} \Lambda_j] \bigcup [\bigcup_{j \in K'} \Lambda_j']$
where $K = \{j|$ index $1 \leq j \leq p$ for which (i) in step (2) holds$\}$
and $K' = \{j||$ index $1 \leq j \leq p$ for which (ii) in step (2) holds$\}$
(4) $\Omega \leftarrow \Omega_{\text{Next}}$
go to (2)

Theorem 3.1. *Let $F = (F_j) : \mathbf{U} \longrightarrow \mathbf{R}^p$ be differentiable with derivative*

$$f_\Omega = (f_\Omega^1, f_\Omega^2, \dots, f_\Omega^p)$$

and $E_F(\Omega_1, \Omega_2) \leq (M/p)[\rho(\Omega_1, \Omega_2)]^2$ for all $j = 1, 2, \dots, p$ and Ω_1, Ω_2 in $\mathbf{U}$. Then the algorithm (3.3) stops at Ω satisfying, $\forall j = 1, \cdots, p$,

$$< f_\Omega^j, \Lambda > \leq Mh^2, \quad \Lambda \in \{\Lambda_i^h\}_{i=1}^N, \quad \Lambda \subseteq \Omega$$

and

$$< f_\Omega^j, \Lambda > \geq -Mh^2, \quad \Lambda \in \{\Lambda_i^h\}_{i=1}^N, \quad \Lambda \subseteq \Omega^c$$

Remark: The conclusion holds for all $\Lambda \subseteq \mathbf{U}_{\Lambda^h}$, since U_{Λ^h} is the σ-algebra of disjoint sets Λ_i^h.

Proof: Assume that at step (2) j is in K. Let us denote $\Lambda_1 = \bigcap_{j \in K} \Lambda_j$ and $\Lambda_2 = \bigcup_{j \in K'} \Lambda_j'$. Then

$$\begin{aligned}
F_j(\Omega_{\text{Next}}) &= F_j(\Omega) + < f_\Omega^j, \chi_{\Omega_{\text{Next}}} - \chi_\Omega > + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&= F_j(\Omega) + < f_\Omega^j, \chi_{\Omega_{\text{Next}}} > - < f_\Omega^j, \chi_\Omega > + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&= F_j(\Omega) + < f_\Omega^j, \chi_\Omega - \chi_{\Lambda_1} + \chi_{\Lambda_2} > - < f_\Omega^j, \chi_\Omega > + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&= F_j(\Omega) - < f_\Omega^j, \chi_{\Lambda_1} > + < f_\Omega^j, \chi_{\Lambda_2} > + E_{F_j}(\Omega_{\text{Next}}, \Omega).
\end{aligned}$$

Since $\Lambda = \Lambda_j$ or ϕ and $< f_\Omega^j, \chi_{\Lambda_2} >< -Mh^2 < 0$,

$$\begin{aligned}
F_j(\Omega_{\text{Next}}) &\leq F_j(\Omega) - < f_\Omega^j, \chi_{\Lambda_1} > + 0 + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&\leq F_j(\Omega) - < f_\Omega^j, \chi_{\Lambda_1} > + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&\leq F_j(\Omega) - Mh^2 + Mh^2 = F_j(\Omega).
\end{aligned}$$

In the case that at step (2) $j \in K'$, similarly we have that

$$\begin{aligned}
F_j(\Omega_{\text{Next}}) &= F_j(\Omega) + < f_\Omega^j, \chi_{\Lambda_2} > - < f_\Omega^j, \chi_{\Lambda_1} > + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&\leq F_j(\Omega) + < f_\Omega^j, \chi_{\Lambda_2} > - 0 + E_{F_j}(\Omega_{\text{Next}}, \Omega) \\
&\leq F_j(\Omega) - Mh^2 + Mh^2 = F_j(\Omega).
\end{aligned}$$

Thus if the algorithm does not stop at step (2), then $F_j(\Omega_{\text{Next}})$ decreases at each cycle. Since N is finite, such a decrease occurs only a finite number of times and the algorithm stops. At that point, i.e.

$$< f_\Omega^j, \Lambda > \leq \; Mh^2, \quad \Lambda \in \{\Lambda_i^h\}_{i=1}^N, \quad \Lambda \subseteq \Omega$$

and

$$< f_\Omega^j, \Lambda > \geq \; -Mh^2, \quad \Lambda \in \{\Lambda_i^h\}_{i=1}^N, \quad \Lambda \subseteq \Omega^c$$

Therefore, the proof is complete.

Since algorithm (3.3) solves an approximated version of the problem it is important now to enquire whether as $\mathbf{U}_{\Lambda^h}$ is given a finer structure. The repeated application of algorithm (3.3) will produce a sequence whose accumulation points satisfy the necessary conditions for optimality. To show this we define a sequence of discretization schemes $\{\Lambda_i^h\}_{i=1}^{N(h)}$ where $h \in H = \{h_1, h_2, \dots\}$ and the sequence H converges strictly monotonically to zero. Again define $\mathbf{U}_{\Lambda^h} = P[\{\Lambda_i^h\}_{i=1}^{N(h)}]$. The discretization scheme is required to possess the following properties :

(1) for each $h \in H$, $\{\Lambda_i^h\}_{i=1}^{N(h)}$ is a family of disjoint measurable sets satisfying $m(\Lambda_i^h) = h$, $i = 1, \dots, N(h)$ and $\rho(X, \bigcup_{i=1}^{N(h)} \Lambda_i^h) = 0$,

(2) given $\Omega \in \mathbf{U}$ and $\varepsilon > 0$, there exists N such that $n \geq N$ implies there exists $A \in \mathbf{U}_{\Lambda^{h_n}}$ with $\rho(A, \Omega) < \varepsilon$.

Note that property (2) provides that the discretization scheme does not deteriorate in its ability to approximate a given set as $n \to \infty$, and moreover becomes arbitrarily fine. The following theorem gives a separating set that satisfies the necessary condition for minimality.

Theorem 3.2. *Assume*

(i) $F = (F_j) : \mathbf{U} \longrightarrow R^p$ *is differentiable with derivative* $f_\Omega = (f_\Omega^j)$

(ii) $E_{F_j}(\Omega_1, \Omega_2) \leq (M/p)[\rho(\Omega_1, \Omega_2)]^2$, *for* $j = 1, \dots, p$

(iii) $\rho(\Omega, \Omega_k) \to 0 \Longrightarrow \|f_\Omega^j - f_{\Omega_k}^j\|_{L_1} \to 0$, *for* $j = 1, \dots, p$

Suppose a discretization scheme satisfying (1) and (2) above is employed, Algoritm (3.3) is executed successively with $h = h_1, h = h_2, \dots$, and the resulting sets are denoted $\Omega_1, \Omega_2, \dots,$. Then if $\{\Omega_k\}$ has a ρ-accumulating point Ω it follows that f_Ω separates Ω.

Proof: Rename the ρ-convergence subsequence of $\{\Omega_k\}_{k=1}^\infty$ as $\{\Omega_k\}_{k=1}^\infty$. Fix $\varepsilon > 0$ and $A \subseteq \Omega$, $A \in \mathbf{U}$. Now since $f_\Omega^j \in L_1(X, \mathbf{U}, m)$, there exists $\delta > 0$ such that

$$m\Lambda < \delta \Longrightarrow \int_\Lambda |f_\Omega^j| dm < \varepsilon/5 \tag{1}$$

Choose k so large that (2)-(5) hold

$$\|f_\Omega^j - f_{\Omega_k}^j\|_{L_1} < \varepsilon/5 \tag{2}$$

There exists $A_k \in \mathbf{U}_{\Lambda^{h_k}}$ with

$$\rho(A_k, A) \leq \delta/2 \tag{3}$$

$$\rho(\Omega, \Omega_k) \leq \delta/2 \tag{4}$$

$$M h_k m(X) \leq \varepsilon/5 \tag{5}$$

Then

$$\begin{aligned}
\int_A f_\Omega^j dm &\leq \left| \int_A f_\Omega^j dm - \int_{A_k} f_\Omega^j dm \right| \\
&\quad + \left| \int_{A_k} f_\Omega^j dm - \int_{A_k} f_{\Omega_k}^j dm \right| + \int_{A_k - \Omega_k} f_{\Omega_k}^j dm + \int_{A_k \bigcap \Omega_k} f_{\Omega_k}^j dm \\
&\leq \varepsilon/5 + \varepsilon/5 + \left| \int_{A_k - \Omega_k} f_{\Omega_k}^j - f_\Omega^j dm \right| + \int_{A_k - \Omega_k} f_\Omega^j dm \\
&\quad + \int_{A_k \bigcap \Omega_k} f_{\Omega_k}^j dm \leq 2\varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \int_{A_k \bigcap \Omega_k} f_{\Omega_k}^j dm
\end{aligned}$$

Now, $A_k \bigcap \Omega_k \in \mathbf{U}_{\Lambda^{h_k}}$ and $A_k \bigcap \Omega_k \subseteq \Omega_k$.
Thus by Theorem 4.2,

$$\int_{A_k \bigcap \Omega_k} f^j_{\Omega_k} dm \le M h_k^2 \le M h_k m(X) \le \varepsilon/5.$$

Thus

$$\int_A f^j_\Omega dm \le \varepsilon, \quad \forall A \subseteq \Omega, \quad A \in \mathbf{U}, \quad \text{for all} \quad j = 1, \ldots, p.$$

Therefore

$$f^j_\Omega \le 0 \quad \text{a.e. on} \quad \Omega, \quad \text{for all} \quad j = 1, \ldots, p.$$

Similarly $f^j_\Omega \ge 0$ a.e. on Ω^c, for all $j = 1, \ldots, p$.
Therefore $f_\Omega = (f^j_\Omega)$ separates Ω. The proof is complete.

4. Conclusion

In this paper, We have attempted numerical method which approximately converges to the element satisfying the necessary condition for the unconstrained problem involving multiobjective set functions. An existing computational approach can be analysed further and a new numerical method can be proposed for unconstrained set function optimization. Constrained problem can be converted into unconstrained case by the result of section 2. But this will require more effort in choosing appropriate multipliers. And it will be a new area of research. The algorithm obtained should be further tested experimentally to gain experience on such matters as reliability, rate of convergence.

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