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Further Properties of a Model for a System Subject to Continuous Wear

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ABSTRACT

A generalization of an earlier diffusion model for system subject to continuous wear is presented. It is assumed that the state of the system is modelled by Brownian motion with negative drift and an absorbing barrier at the origin. A repairman arrives according to a stationary renewal process and increases the state of the system by a random amount if the state does not exceed a threshold. Various properties of this model are investigated including the distribution of the state of the system at time t, the first passage time to state 0 and the probability that the state of the system exceeds a certain level throughout a specified interval.

1. Introduction

Baxter and Lee(1987) introduced a diffusion model for a system whose state changes continuously with time. It was assumed that the state of the system is initially x_0 and thereafter follows Brownian motion with negative drift and an absorbing barrier at the origin. It was further assumed that the state of the system is enhanced by a repairman whose sequence of arrivals comprises a Poisson process. If the state of the system when the repairman arrives exceeds a threshold $\alpha(0\leq\alpha<x_0)$, no action is taken, otherwise the repairman instantaneously increases the state of the system by a random amount Y where $Y\geq\alpha$ almost surely. Baxter and Lee(1987) deduced an integro-differential equation for $f(x, t)$, the density of $X(t)$, the state of the system at time t, and solved it to yield an expression for the Laplace transform of $f(x, t)$ with respect to x. The stationary case was examined in detail and the moments of the stationary distribution were calculated. Further, an expression for the distribution funtion $F(x, t)$ of $X(t)$, $0\leq x\leq\alpha$, was derived by a purely probabilistic argument which did not make use of the integro-differential equation. Applications of the model to the control of a production process, to the fuelling of a power station, to inventory theory and to reliability theory were discussed. See Baxter and

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Lee(1987) for full details.

In this paper we continue our analysis by relaxing the assumption that the sequence of arrivals of the repairman comprises a Poisson process; instead, we assume that the arrivals comprise a stationary renewal process. For this more general model, we generalize our previous analysis, study the first passage time to the state 0 and deduce an expression for the probability that the state of the system exceeds a certain level throughout a given interval.

2. A Formula for F(x, t), $0 \leq x \leq \alpha$

In this section, we present a formula for $F(x, t)=P\{X(t)\leq x\}$, the distribution function of $X(t)$, for $0\leq x\leq\alpha$.

Let Q be the commom distribution function of the times between successive arrivals of the repairman and let $v=\int_0^\infty t dQ(t)$ denote the expected inter-arrival time. We assume that $0<v<\infty$. Then the distribution function of the time to the first arrival is $P(x)=\int_0^x \bar{Q}(u)du/v$ where $\bar{Q}=1-Q$.

Consider the sequence of points at which the state of the system crosses the threshold α for the first time after a visit by the repairman. Since Brownian motion has independent increments and since the distribution function of the forward recurrence time to the next visit of the repairman is always P, this sequence comprises an embedded renewal process. Hence we can use an argument similar to that of Baxter and Lee(1987) to deduce an expression for $F(x, t)$, $0\leq x\leq\alpha$.

Let μ and $\sigma^2(\mu<0,\ \sigma>0)$ denote the parameters of the Brownian motion and let $W_{a,b}$ denote the distribution function of S_{a-b}, the first passage time from a to b in Brownian motion with parameters $\mu<0$ and $\sigma^2(a>b)$. Then, from Karlin and Taylor(1975),

$$W_{a,b}(t)=\int_0^t \frac{a-b}{\sigma\sqrt{2\pi x^3}}\ exp[-(a-b+\mu x)^2)/2\sigma^2 x]dx. \tag{2.1}$$

Further, let $B(x, t)$ denote the distribution function of $Z(t)$ where $\{Z(t),\ t\geq 0\}$ denotes Brownian motion with parameters μ and σ^2, an absorbing barrier at the origin and initial condition $Z(0)=\alpha$. From Cox and Miller(1965),

$$B(x,\ t)=1-\int_x^\infty \frac{1}{\sigma\sqrt{2\pi t}}[exp\{-\frac{(z-\alpha-\mu t)^2}{2\sigma^2 t}\} -exp\{-\frac{2\mu\alpha}{\sigma^2}-\frac{(z-\alpha-\mu t)^2}{2\sigma^2 t}\}]dz \qquad (x\geq 0). \tag{2.2}$$

Then, by an argument similar to that of Baxter and Lee (1987), it can be shown that

$$F(x,\ t)=\int_0^t B(x,\ t-u)\bar{P}(t-u)dH(u), \tag{2.3}$$

where

$$H(t)=W_{x_0,\alpha}(t)+\sum_{n=1}^{\infty} W_{x_0,\alpha} * K^{(n)}(t), \tag{2.4}$$

and where the asterisk denotes Stieltjes convolution, the superscript denotes n-fold recursive convolution and

$$K(t)=\int_0^t\int_\alpha^\infty W_{u,\alpha}(t-s)dV(u)dP(s), \tag{2.5}$$

where

$$V(x)=\int_0^\infty\int_\alpha^x B(x-y)\wedge\alpha,\ t)dG(y)dP(t)+\int_0^\infty [B(x,t)-B(\alpha,t)]dP(t), \tag{2.6}$$

G being the distribution function of Y.

3. The Laplace-Stieltjes Transform of F(x, t)

Since the renewal density of the sequence of arrivals of the repairman is constant, we can make use of an argument similar to that which was used to derive equation (A1.1) of Baxter and Lee(1987) in order to deduce the following integro-differential equation:

$$\frac{\partial}{\partial t}F(x,t)=\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t)-\mu\frac{\partial}{\partial x}F(x,t)-\frac{1}{v}F(x\wedge\alpha,t)$$
$$+\frac{1}{v}\int_\alpha^x F((x-z)\wedge\alpha,t)dG(z). \tag{3.1}$$

This equation is vallid for $x\geq 0$. Since the origin is an absorbing state, we have the boundary condition $\frac{\partial}{\partial x}F(x,t)\mid_{x=0}=0$ for $t\geq 0$. See Cox and Miller(1965).

We now dreive an expression for the Laplace transform of (3.1) with respect to x, we see that

$$\frac{\partial}{\partial t}F^\circ(x,t)=\frac{1}{2}\sigma^2[s^2F^\circ(s,t)-sF(0,t)]-\mu[sF^\circ(s,t)-F(0,t)]-\int_0^\infty e^{-sx}F(x\wedge\alpha,t)dx/v$$
$$+\int_0^\infty e^{-sx}\int_\alpha^x F((x-z)\wedge\alpha,t)dG(z)dx/v, \tag{3.2}$$

where $F^\circ(s,t)=\int_0^\infty e^{-sx}F(x,t)dx$. Now

$$\int_0^\infty e^{-sx}F(x\wedge\alpha,t)dx=\int_0^\alpha e^{-sx}F(x,t)dx+e^{-s\alpha}F(\alpha,t)/s,$$

and

$$\int_0^\infty e^{-sx}\int_\alpha^\infty F((x-z)\wedge\alpha,t)dG(z)dx=\int_\alpha^\infty e^{-sz}\int_0^\infty e^{-sy}F(y\wedge\alpha,t)dydG(z)$$
$$=G^*(s)[\int_0^\alpha e^{-sx}F(x,t)dx+e^{-s\alpha}F(\alpha,t)/s]$$

where $G^*(s)=\int_\alpha^\infty e^{-sy}dG(y)$. Thus (3.2) becomes

$$\frac{\partial}{\partial t}F^\circ(s,t)=(\frac{1}{2}\sigma^2s^2-\mu s)F^\circ(s,t)-(\frac{1}{2}\sigma^2s-\mu)F(0,t)$$
$$-[1-G^*(s)][F^\alpha(s,t)+e^{-s\alpha}F(\alpha,t)/s]/v, \tag{3.3}$$

where $F^\alpha(s,t)=\int_0^\alpha e^{-sx}F(x,t)dx$. Solving (3.3) with the boundary condition $F^\circ(s,0)=e^{-sx_0}/s$

and recalling that the Laplace-Stieltijes transform of $F(x,t)$ with respect to x, $F^*(s,t)$ say, is given by $F^*(s,t)=sF^\circ(s,t)$, it follows that

$$F^*(s,t)=exp[(\frac{1}{2}\sigma^2 s^2-\mu s)t][e^{-sx_0}-\int_0^t exp[(\mu s-\frac{1}{2}\sigma^2 s^2)u]$$
$$\{(\frac{1}{2}\sigma^2 s^2-\mu s)F(0,u)+\frac{1}{v}[1-G^*(s)][sF^\alpha(s,u)+e^{-s\alpha}F(\alpha,u)]\}du], \qquad (3.4)$$

where, from(2.3),

$$F(0,t)=\int_0^t B(0,t-u)\bar{P}(t-u)dH(u),$$
$$F(\alpha,t)=\int_0^t B(\alpha,t-u)\bar{P}(t-u)dH(u),$$

and

$$F^\alpha(s,t)=\int_0^\alpha e^{-sx}\int_0^t B(x,t-u)\bar{P}(t-u)dH(u)dx.$$

The Laplace-Stieltjes transform of the disrtibution function of $X(t)$ for the case where the process $\{X(t),t\geq 0\}$ is stationary is considered in Section 6 below.

4. The First Passage Time to State 0

Define $T_0=inf\{t \mid X(t)=0\}$, the first passage time to state 0. In this section, we deduce expressions for the distribution of T_0 and for $E(T_0)$ and we derive the limiting distribution of T_0 as $\alpha\to\infty$.

Let p be the probability that when a repair is performed, the system is at state 0. Clearly,

$$p=\int_0^\infty B(0,t)dP(t)/\int_0^\infty B(\alpha,t)dP(t). \qquad (4.1)$$

Observe that T_0 satisfies the following relation:

$$T_0\overset{D}{=}s_{x_0}+\sum_{i=1}^{n} Y_i\overset{D}{=}S_{x_0}+\sum_{i=1}^{n} S_{Y_i} \text{ with probability } p(1-p)^n, \qquad (4.2)$$

for $n=1,2,3,\cdots$, where $Y_i\overset{D}{=}Y$ for all i and where $\overset{D}{=}$ denotes equality in distribution. It is readily seen that

$$P\{S_Y\leq t\}=\int_0^\infty W_{y,0}(t)dG(y)=D(t), \text{ say,}$$

and hence the distribution function of T_0, $L(t)$ say, is given by

$$L(t)=\sum_{n=0}^{\infty} W_{x_0,0}*D^{(n)}(t)p(1-p)^n, \qquad (4.3)$$

where $D^{(0)}(t)$ is the Heaviside function.

It follows from (4.2) that

$$E(T_0)=\sum_{n=0}^{\infty}[E(S_{x_0})+nE(S_Y)]p(1-p)^n. \tag{4.4}$$

Since $E(S_{a-b})=(b-a)/\mu$ (this follows (5.5) of Karlin and Taylor(1975)), and hence

$$E(S_Y)=\int_0^{\infty}E(S_y)dG(y)=-m/\mu,$$

it is easily seen that (4.4) simplifies to yield the following formula:

$$E(T_0)=-[px_0+m(1-p)]/p\mu, \tag{4.5}$$

where $m=E(Y)$.

We now investigate the limiting behavior of T_0 as $\alpha\to\infty$.

Theorem 4.1 As $\alpha\to\infty$, $T_0/E(T_0)$ converges in distribution to a unit exponential variate.

Proof A straightforward calculation shows that the Laplace-Stieltjes transform of $L(t)$, $L^*(s)$ say, is given by

$$L^*(s)=pW^*_{x_0,0}(s)/[1-(1-p)D^*(s)],$$

where $W^*_{x_0,0}(s)$ and $D^*(s)$ are the Laplace-Stieltjes transfors of $W_{x_0,0}(t)$ and $D(t)$, respectively. Hence the Laplace-Stieltjes transform of the distribution function of $T_0/E(T_0)$, $L_1^*(s)$ say, is given by

$$L_1^*(s)=L^*(s/E(T_0))=\frac{pW^*_{x_0,0}(s/E(T_0))}{1-(1-p)D^*(s/E(T_0))}.$$

From (4.5), it follows that

$$L_1^*(s)=\frac{pW^*_{x_0,0}(s\mu p/[(m-x_0)p-m])}{1-(1-p)D^*(s\mu p/[(m-x_0)p-m])}. \tag{4.6}$$

Since $p\to0$ as $\alpha\to\infty$, the numerator and denominator of (4.6) both tend to 0 as $\alpha\to\infty$ and hence we apply l'Hôpital's rule. A straightforward calculation shows that $\lim_{\alpha\to\infty}L_1^*(s)=1/(1+s)$. Since this is the Laplace-Stieltjes transform of a unit exponential variate, the result follows from the uniqueness of Laplace-Stieltjes transforms. ■

5. The Probability that the System Exceeds a Given Level

In this sectin, we deduce an expression for $\Pi_x(t_1,t_2)=P\{X(t)>x$ for all $t\in[t_1,t_2]\}$. Observe that $X(t)>x$ for all $t\in[t_1,t_2]$ if and only if $X(t_1)>x$ and the first passage time from $X(t_1)$ to x is greater than t_2-t_1.

Consider, firstly, the case $x\geq\alpha$. The first passage time from $X(t_1)$ to x is equal in disrtibution to $S_{x(t_1)-x}$. Hence

$$\Pi_x(t_1, t_2) = P\{X(t_1) > x,\ S_{x(t_1)-x} > t_2 - t_1\}$$
$$= \int_x^\infty P\{S_{x(t_1)-x} > t_2 - t_1 \mid X(t_1) = y\} d_y F(y, t_1),$$

so that

$$\Pi_x(t_1, t_2) = \int_x^\infty \bar{W}_{y,x}(t_2 - t_1) d_y F(y, t_1) \quad (x \geq \alpha). \tag{5.1}$$

Consider, now, the case $x < \alpha$. Let $\{Z_x(t), t \geq 0\}$ denote Brownian motion with parameters μ and σ^2, an absorbing barrier at x and initial condition $Z_x(0) = \alpha$. Let $B_x(y, t) = P\{Z_x(t) \leq y\}$ denote the distribution function of $Z_x(t)$. By an argument similar to that of Cox and Miller(1965), it can be shown that

$$B_x(y, t) = 1 - \int_y^\infty \frac{1}{\sigma\sqrt{2\pi t}} \{exp[-\frac{(z-\alpha-t)^2}{2\sigma^2 t}] - exp[-\frac{2\mu(\alpha - x)}{\sigma^2} - \frac{(z - 2x + \alpha - \mu t)^2}{2\sigma^2 t}]\} dz \quad (y \geq x). \tag{5.2}$$

Let $T_{y,x}$ be the first passage time from y to x in the process $\{X(t), t \geq 0\}$. By an argument similar to that of Section 4, it can be shown that the distribution function of $T_{y,x}$, $L_{y,x}(t)$ say, is given by

$$L_{y,x}(t) = \sum_{n=0}^{\infty} W_{y,x} * D^{(n)}(t) p_x (1 - p_x)^n, \tag{5.3}$$

where $p_x = \int_0^\infty B_x(x, t) dP(t) / \int_0^\infty B_x(\alpha, t) dP(t)$.
Hence, by an argument similar to the distribution of (5.1),

$$\Pi_x(t_1, t_2) = \int_x^\infty \bar{L}_{y,x}(t_2 - t_1) d_y F(y, t_1) \quad (x < \alpha). \tag{5.4}$$

6. The Stationary Case

Suppose, now, that distribution of $X(t)$ does not depend on t; this corresponds to the stationary distribution, i.e. when $\partial F(x,t)/\partial t = 0$, and to the equilibrium disrtibution $\lim_{t\to\infty} F(x,t)$. The corresponding distribution function is denoted $F(x)$. As would be expected, in this case a more tractable expression for the Laplace-Stieltjes transform of F, $F^*(s)$ say, is obtained.
Form (3.3), it follows that

$$F^*(s) = \frac{(\sigma^2 s^2/2 - \mu s)F(0) + [1 - G^*(s)][sF^\alpha(s) + e^{-s\alpha}F(\alpha)]/v}{\sigma^2 s^2/2 - \mu s}, \tag{6.1}$$

where $F^\alpha(s) = \int_0^\alpha e^{-sx} F(x) dx$. We now show how $F(0)$, $F(\alpha)$ and $F^\alpha(s)$ may be evaluated.
To evaluate $F(0)$, consider the indicator variable $J(t) = I_{\{X(t)>0\}}$. Observe that $\{J(t), t \geq 0\}$ is an embedded alternating renewal process: all sojourn times are mutually independent, the sojourn times in state 0 are identically distributed with common distribution function P and the sojourn times in state 1 are, with the exception of the first, identically distributed with distribution function $\sum_{n=1}^{\infty} D^{(n)}(t) p (1-p)^{n-1}$ (this can be proved by the argument of Section 4). Futher, since the sequence of arrivals of the repairman comprises a stationary process, the process $\{J(t), t \geq 0\}$

is also stationary. Hence $F(0)=E(D^*)/[E(U^*)+E(D^*)]$ where D^* and U^* are generic random variables denoting the durations of sojourns in state 0 and 1 respectively. It is easily seen that $E(U^*)=-m/\mu p$ and that $E(D^*)=v_2/2v$ where $v_2=\int_0^\infty x^2dQ(x)$ and hence

$$F(0)=\mu p v_2/(\mu p v_2-2vm). \tag{6.2}$$

To determine $F(\alpha)$ and $F^\alpha(s)$, observe that in the stationary case, (3.1) reduces to

$$\frac{1}{2}\sigma^2\frac{d^2}{dx^2}F(x)-\mu\frac{d}{dx}F(x)-F(x)/v=0, \tag{6.3}$$

for all $0\leq x\leq\alpha$. Recall that if

$$a\frac{d^2}{dx^2}y(x)+b\frac{d}{dx}y(x)+cy(x)=0$$

and $b^2>4ac$, then $y(x)=a_1e^{\eta_1x}+a_2e^{\eta_2x}$ where η_1 and η_2 are the solutions to the equation $a\eta^2+b\eta+c=0$. Hence $F(x)=a_1e^{\eta_1x}+a_2e^{\eta_2x}$ where $\eta_1=(\mu+\sqrt{\mu^2+2\sigma^2/v})/\sigma^2$ and $\eta_2=(\mu-\sqrt{\mu^2+2\sigma^2/v})/\sigma^2$. To determine a_1 and a_2, observe that, since $\frac{d}{dx}F(x)\mid_{x=0}=0$ and $F(0)=\mu p v_2/(\mu p v_2-2vm)$, we have the following two constraints:

$$a_1\eta_1+a_2\eta_2=0 \text{ and } a_1+a_2=\mu p v_2/(\mu p v_2-2vm).$$

Hence

$$a_1=\eta_2\mu p v_2/(\eta_2-\eta_1)(\mu p v_2-2vm)$$

and

$$a_2=-\eta_1\mu p v_2/(\eta_2-\eta_1)(\mu p v_2-2vm).$$

It thus follows that

$$F(\alpha)=a_1e^{\eta_1\alpha}+a_2e^{\eta_2\alpha} \tag{6.4}$$

and that

$$F^\alpha(s)=\frac{a_1[e^{(\eta_1-s)\alpha}-1]}{\eta_1-s}+\frac{a_2[e^{(\eta_2-s)\alpha}-1]}{\eta_2-s} \tag{6.5}$$

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