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Asymptotic Density of Quadratic Forms

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ABSTRACT

The theory of the asymptotic behavior of Toeplitz forms is applicable to some problems concerning the local limit theorem.

1. Introduction

This paper is concerned with the problem of obtaining satisfactory approximation for the determination of the density function of quadratic forms in correlated normal variates. Such quadratic forms are of importance in many applications of the theory of stochastic processes. The quadratic forms can be transformed to weighted sums of squares of independent identically distributed normal variates. In many applications, these weights are or approximate the eigenvalues of a Toeplitz matrix. This paper is intended to show that the theory of the asymptotic behavior of Toeplitz forms is applicable to some problems concerning the local limit theorem. That summary is given in the followings. In Section 2, we study the asymptotic distribution of the eigenvalues of Toeplitz forms. The remainder of the paper uses the results of this section. Section 3 is devoted to the problem of finding the asymptotic density of quadratic forms.

2. Asymptotic Distribution of the Eigenvalues of Toeplitz Forms

In certain applications of the theory of stochastic processes, it happens that the weights $\lambda_{n\nu}$ of the diagonalized quadratic form

$$Q_0=\sum_{\nu=1}^{n} \lambda_{n\nu}\xi_\nu^2$$

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are the eigenvalues(or approximate eigenvalues) of a Toeplitz matrix. The following approximation for the distribution of Q_0 is based upon the theory of these matrices. This method is an application of a general theorem of Szego on the asymptotic distribution of the eigenvalues of Toeplitz forms, which we are going to state(See Grenander and Szego(1958)).

Consider a Toeplitz matrix

$$T_n=\{c_{\nu-\mu}\,;\ \nu,\ \mu=1,\ 2,\ \cdots,\ n\}$$

where

$$c_\nu=(1/2\pi)\int_{-\pi}^{\pi} e^{i\nu x}\, g(x)dx, \qquad \nu=0,\ \pm 1,\ \pm 2,\ \cdots.$$

and g(x) is a measurable bounded function, say

$$|g(x)|\leq M<\infty,\quad \forall x.$$

Denote the eigenvlues by $\lambda_{n1},\ \lambda_{n2},\ \cdots,\ \lambda_{nn}$. The quadratic form

$$Q=x^*T_n\,x=\sum_{\nu,\mu=1}^{n} c_{\nu-\mu}\, x_\nu \bar{x}_\mu$$
$$=(1/2\pi)\int_{-\pi}^{\pi}\Big|\sum_{\nu=1}^{n} x_\nu\, e^{i\nu x}\Big|^2 g(x)dx$$

is Hermitian so that the λ's are real. Moreover,

$$|x^*T_n\,x|\leq(M/2\pi)\int_{-\pi}^{\pi}\Big|\sum_{\nu=1}^{n} x_\nu e^{i\nu x}\Big|^2dx$$
$$\leq\sum_{\nu=1}^{n}|x_\nu|^2$$

for all $x=(x_1,\ \cdots,\ x_n)$, so

$$|\lambda_{n\nu}|\leq M,\quad \forall n,\ \nu.$$

For any n we can consider the distribution of the eigenvalues in the interval $(-M,\ M)$. Let

$$s_{np}=(1/n)\sum_{\nu=1}^{n}\lambda^p_{n\nu}$$

denote the moments of these distributions. Then

$$\lim_{n\to\infty} s_{np}=(1/2\pi)\int_{-\pi}^{\pi} g^p(x)dx,\ \ p=1,\ 2,\ \cdots,\ ;$$

Since the moments converge, the same is true of the distributions. The limiting moments belong to the stochastic variable $g(U)$ where U is uniformly distributed in $(-\pi,\ \pi)$. Hence the eigenvalues of a Toeplitz form behave asymptotically like the ordinates of the function $g(U)$ with equidistributed U. More precisely,

$$\lim_{n\to\infty}\frac{\text{number of eigenvalues}\leq t}{n}=(1/2\pi)\ \text{Lebesgue measure of}\ [x\,|\,g(x)\leq t].$$

3. Asymptotic Density of Quadratic Forms

As an example, consider the Hermitian matrix $\mathbf{B}_n$ of the form

$$B_n(j,k)=\begin{cases}\rho^{|k-j|-1}, & \text{if } j\neq k\,;\\ 0, & \text{if } j=k.\end{cases}$$

where $\rho\in(-1,1)$. In this case, $\mathbf{B}_n$ is a Toeplitz matrix with

$$g(x)=\sum_{k\neq 0}\rho^{|k|-1}e^{ikx}$$
$$=2(\cos x-\rho)/(1-2\rho\cos x+\rho^2)$$

and

$$|g(x)|\leq 2/(1-\rho)=\mathrm{M},\quad \forall x.$$

By Szego's theorem the distribution of the eigenvalues converges to the distribution of $g(U)$ where U is uniformly distributed over $(-\pi,\pi)$. Let μ_n and σ_n^2 denote the mean and variance of the n^{th} distribution. Then

$$\mu_n=(1/n)\sum_{\nu=1}^{n}\lambda_{n\nu}=(1/n)\,\mathrm{tr}(\mathbf{B}_n)=0$$

and

$$\lim_{n\to\infty}\sigma_n^2=(1/2\pi)\int_{-\pi}^{\pi}g^2(x)dx$$
$$=(1/2\pi)\int_{-\pi}^{\pi}\left(\frac{2(\cos(x)-\rho)}{1-2\rho\cos(x)+\rho^2}\right)^2dx$$
$$=(2/(1-\rho^2)^3)\,[3\rho^3-3\rho^2+1].$$

Using the distribution of the eigenvalues, it is possible to study the density of certain quadratic forms. First, we are concerned with the distribution of the random variable

$$Q_n=\mathbf{b}'\begin{pmatrix}\varepsilon_1\\ \vdots\\ \varepsilon_n\end{pmatrix}+(\varepsilon_1,\cdots,\varepsilon_n)\mathbf{B}_n\begin{pmatrix}\varepsilon_1\\ \vdots\\ \varepsilon_n\end{pmatrix}$$

where $\varepsilon_1,\varepsilon_2,\cdots$ are i.i.d. standard normal random variables, ρ and $\mathbf{B}_n$ are defined in Section 2 and $\mathbf{b}=\mathbf{b}_n$ are vectors for which

$$\|\mathbf{b}_n\|^2=O(n).$$

Lemma 3.1 *If Z has the standard normal distribution, then*

$$Q(s,t)=E[\exp(isZ+(1/2)\,itZ^2)]$$
$$=(1-it)^{-1/2}\exp\{-s^2/(2(1-it))\}$$

Proof.

$$Q(s,t)= E[\exp(isZ+(1/2)\ itZ^2)]$$
$$= (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \exp[isZ+ (1/2)\ itZ^2- (1/2)\ Z^2]$$
$$= \exp[-s^2/(2(1-it))] \int_{-\infty}^{\infty} (1/\sqrt{2\pi}) \exp[-(1/2)\ (1-it)\ (z-(is/(1-it))^2]dz.$$

This integral is easily evaluated and obtains

$$Q(s,t)=(1-it)^{-1/2} \exp\{-s^2/(2(1-it))\}.$$

Lemma 3.2 *The characteristic function of* Q_n *is given by*

$$\varphi_n(t)=E(e^{itQ_n})$$
$$= |\mathbf{I}-it\mathbf{B}_n|^{-1/2} \exp\{-(1/2)t^2\mathbf{b}'(\mathbf{I}-it\mathbf{B}_n)^{-1}\mathbf{b}\}.$$

Proof. We introduce an orthogonal matrix $\mathbf{C}$ for which $\mathbf{CB}_n\mathbf{C}'=\mathbf{D}$ is a diagonal matrix. Then $\mathbf{B}_n=\mathbf{C'DC}$ and

$$|\mathbf{I}-it\mathbf{B}_n| = |\mathbf{I}-it\mathbf{D}|$$
$$=(1-it\lambda_{n1}) \cdots (1-it\lambda_{nn})$$

where the $\lambda_{n1}, \cdots, \lambda_{nn}$ are the eigenvalues of the matrix $\mathbf{B}_n$. Let $\mathbf{Z}=\mathbf{C}\varepsilon$ and $\beta=\mathbf{Cb}$. Then it is easily seen that $Z_1, \cdots Z_n$ are independent unit normal random variables. Moreover,

$$\mathbf{b}'(\mathbf{I}-it\mathbf{B}_n)^{-1}\mathbf{b}=\mathbf{b}'[\mathbf{C}'(\mathbf{I}-it\mathbf{D})^{-1}\mathbf{C}]\mathbf{b}$$
$$=\beta'(\mathbf{I}-it\mathbf{D})^{-1}\beta.$$

so that

$$\mathbf{b}'(\mathbf{I}-it\mathbf{B}_n)^{-1}\mathbf{b}= \sum_{k=1}^{n} (\beta_k^2/(1-it\lambda_{nk})) .$$

Thus

$$Q_n= \sum_{k=1}^{n} [\beta_k Z_k+(1/2)\ \lambda_{nk}Z_k^2]$$

and

$$E\{e^{itQ_n}\}= E\{\exp \sum_{k=1}^{n} [it\beta_k Z_k+(1/2)\ it\lambda_{nk}Z_k^2]\}$$
$$= \Pi^n_{k=1} E\{\exp[it\beta_k Z_k+(1/2)\ it\lambda_{nk}Z_k^2]\}.$$

Hence

$$\varphi_n(t)= \left[\Pi^n_{k=1}(1-it\lambda_{nk})\right]^{-1/2}\exp\{-(1/2)\ t^2 \sum_{k=1}^{n} (\beta_k^2/(1-it\lambda_{nk}))\}$$
$$= |\mathbf{I}-it\mathbf{B}_n|^{-1/2} \exp\{-(1/2)\ t^2\mathbf{b}'(\mathbf{I}-it\mathbf{B}_n)^{-1}\mathbf{b}\}.$$

This is not too convenient an expression, partly because the λ_{nk} are, in general, difficult to compute and partly because it is difficult to carry out the Fourier inversion ledading to the frequency function for Q_n. We find readily the mean value μ_n and variance σ_n^2 are

$$\mu_n = (1/2) \sum_{k=1}^{n} \lambda_{nk} = (1/2)\ \mathrm{tr}(\mathbf{B}_n) = 0$$

$$\sigma_n^2 = \sum_{k=1}^{n} \beta_k^2 + (1/2) \sum_{k=1}^{n} \lambda_{nk}^2.$$

Now let $f_n(\cdot\ ; \rho, \mathbf{b})$ denote the density of Q_n.

Theorem 3.1 *Let*

$$Q_n^* = Q_n/\sigma_n\ ,\qquad n \geq 1.$$

Then Q_n^* *has a density* f_n^* *for all* $n \geq 1$ *and*

$$\lim_{n\to\infty} \sup_z | f_n^*(z) - \phi(z) | = 0$$

where

$$\phi(z) = (1/\sqrt{2\pi})\ e^{-1/2z^2},\qquad -\infty < z < \infty.$$

Proof. There is no loss of generality in supposing that $M \geq 1$, that $\|\mathbf{b}\|^2/\sigma_n^2 \leq M$, and that $n/\sigma_n^2 \leq M$, for all n. Let $\psi_n(t)$ denote the characteristics function of Q_n^* and let $\delta = 1/8M^4 \leq 1/8M$. If $|t| \leq \delta\sigma_n$, then

$$\begin{aligned}
\log \psi_n(t) &= -(1/2) \sum_{k=1}^{n} \log\left(1 - it\lambda_{nk}/\sigma_n\right) \\
&\quad - (1/2)\ (t^2/\sigma_n^2) \sum_{k=1}^{n} \left\{\beta_k^2/(1 - (it\ \lambda_{nk}/\sigma_n))\right\} \\
&= (1/2) \sum_{p=2}^{\infty} (1/p) \left\{\sum_{k=1}^{n} (\lambda_{nk}/\sigma_n)^p\right\}(it)^p \\
&\quad - (1/2)\ (t^2/\sigma_n^2) \sum_{p=0}^{\infty} \left\{\sum_{k=1}^{n} \beta_k^2(\lambda_{nk}/\sigma_n)^p\right\}(it)^p \\
&= -(1/2)\ t^2 + R_n(t),\qquad -\infty < t < \infty,
\end{aligned}$$

where

$$\begin{aligned}
|R_n(t)| \leq{}& (1/2) \sum_{p=3}^{\infty} (nM^p/\sigma_n^p)\ |t|^p \\
&+ (1/2)\ (t^2/\sigma_n^2)\ \|\mathbf{b}\|^2 \sum_{p=1}^{\infty} |\ (M_t/\sigma_n)\ |^p,
\end{aligned}$$

for all $|t| \leq \delta\sigma_n$ and all $n \geq 1$, by Taylor's Theorem applied to the logarithm. Here R_n tends to zero as n tends to infinity, for all t, since

$$|R_n| \leq (1/2)\ (n|Mt|^3/\sigma_n^3)\ \sum_{p=3}^{\infty} |(Mt/\sigma_n)|^{p-3}$$

$$+ (1/2)\ (M|t|^3/\sigma_n^3)\ \|\mathbf{b}\|^2\ \sum_{p=1}^{\infty} |(Mt/\sigma_n)|^{p-1}$$

$$\leq (2/3)\ (n|Mt|^3/\sigma_n^3) + (2/3)\ (M|t|^3/\sigma_n^3)\ \|\mathbf{b}\|^2.$$

Hence

$$\lim_{n\to\infty} \psi_n(t) = e^{-1/2t^2},\quad \forall t;$$

and, therefore the distribution function of Q_n^* converges to the standard normal.

It is clear that Q_n^* has a density f_n^* for all $n \geq 1$ and that ψ_n is integrable with respect to Lebesgue measure for all $n \geq 3$. So,

$$f_n^*(z) = (1/2\pi) \int_{-\infty}^{\infty} e^{-itz}\,\psi_n(t)dt$$

and

$$|f_n^*(z) - \phi(z)| = |(1/2\pi) \int_{-\infty}^{\infty} e^{-itz}[\psi_n(t) - e^{-1/2t^2}]dt|$$

$$\leq (1/2\pi) \int_{-\infty}^{\infty} |\psi_n(t) - e^{-1/2t^2}|\,dt$$

for all $-\infty < z < \infty$ and all $n \geq 3$.

If n is sufficiently large and $|t| \leq \delta\sigma_n$, then

$$|R_n(t)| \leq (1/3)\ (nM^3t^2/\sigma_n^3)\ \delta\sigma_n + (1/3)\ (Mt^2/\sigma_n^3)\|\mathbf{b}\|^2\delta\sigma_n$$

$$\leq (1/4)\ t^2$$

and, since we have $|\psi_n(t) - e^{-1/2t^2}| \leq 2$ and

$$|\psi_n(t)| \leq e^{-1/4t^2}.$$

So

$$\int_{-\delta\sigma_n}^{\delta\sigma_n} |\psi_n(t) - e^{-1/2t^2}|\,dt \to 0$$

by the dominated convergence theorem. It remains to show that

$$\lim_{n\to\infty} \int_{|t|>\delta\sigma_n} |\psi_n(t)|\,dt \to 0.$$

as $n\to\infty$. Now

$$|\psi_n(t)| = \Pi_{k=1}^{n} |1 - it\ (\lambda_{nk}/\sigma_n)|^{-1/2} |\exp\Big(-(1/2)\ (t^2/\sigma_n^2)) \sum_{k=1}^{n}(\beta_k^2/(1-\lambda t\ \lambda_{nk}/\sigma_n))\Big)|$$

$$\leq \Pi_{k=1}^{n} |1 + t^2\ (\lambda_{nk}^2/\sigma_n^2)|^{-1/4}$$

$$\leq \Big(1/(1+t^2/(4\sigma_n^2))\Big)^{1/4N_n}$$

where

$$N_n = \#\{k : \lambda_{nk}^2 \geq (1/4)\}$$

and

$$\lim_{n \to \infty} (1/n)\ N_n = (1/2\pi)\ \text{meas}\left[x \mid \grave{g}(x) \geq (1/2)\right] > 0.$$

So

$$\begin{aligned}
\int_{|t| \geq \delta\sigma_n} |\psi_n(t)|\, dt &\leq \int_{|t| \geq \delta\sigma_n} \left(1/(t^2/4\sigma_n^2)\right)^{1/4N_n} dt \\
&= \int_{|t| \geq \delta\sigma^n} \left((1/(1+\delta^2/4)\right)^{1/4N_n} dt \\
&= \int \left(1/(1+\delta^2/4)\right)^{1/4N_n} I_{[|t| > \delta\sigma]_n}\, dt \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

The theorem follows easily.

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이차형식의 점근밀도함수

최 기 헌*

요 약

본 논문에서는 여러 분야에서 중요한 역할을 하는 이차형식의 정확한 밀도함수를 구하는 것이 어려운 경우에 극소극한정리를 포함하는 문제들에 토에플리츠(Toelitz) 형식의 고유값에 대한 점근 이론을 갖고서 이차형식의 점근 밀도함수를 구하였다.

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