A Mixed Variational Principle of Fully Anisotropic Linear Elasticity

異方性弾性問題의 混合形變分原理

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Abstract

In this paper, a mixed variational principle applicable to the linear elasticity of inhomogeneous anisotropic materials is presented. For derivation of the general variational principle, a systematic procedure for the variational formulation of linear coupled boundary value problems developed by Sandhu et al. is employed. Consistency condition of the field operators with the boundary operators results in explicit inclusion of boundary conditions in the governing functional. Extensions of admissible state function spaces and specialization to a certain relation in the general governing functional lead to the desired mixed variational principle. In the physical sense, the present variational principle is analogous to the Reissner's recent formulation obtained by applying Lagrange multiplier technique followed by partial Legendre transform to the classical minimum potential energy principle. However, the present one is more advantageous for the application to the general anisotropic materials since Reissner's principle contains an implicit function which is not easily converted to an explicit form.

요 약

본고에서는 Sandhu 등에 의해 개발된 다변수경계치문제의 변분모델화 방법을 이용하여 범함수의 독립변수로써 변위와 응력을 동시에 포함하는 이방성탄성문제의 혼합형변분원리(Mixed Variational Principle)를 유도한다. 탄성방정식을 內積空間에서 self-adjoint한 미분연산자매트릭스 방정식으로 표시한 후 다변수경계치문제의 변분이론을 적용하므로써 일반적 범함수가 구해지며, 이때에 지배방정식의 미분연산자와 경계조건식의 연산자의 일관성(Consistency)을 유지하므로써 경계조건도 체계적으로 범함수내에 포함시킬 수 있다. 이 일반적 범함수에서 미분연산자의 self-adjointness 성질을 이용하여 응력함수의 도함수를 제거하고 탄성방정식중 특정식이 항상, 정확히 만족된다고 가정하므로써 원하는 혼합형변분원리의 범함수를 유도할 수 있다. 여기에서 유도된 변분원리는 최근 Reissner에 의해 개발된 변분원리와 유사한 물리적 의미를 가지나 유도방법이 다를 뿐 아니라 일반적 이방성탄성체에 적용할 때 보다 편리한 형태로 된다. 이 혼합형변분원리는 다양하게 응용될 수 있으나, 복합재료적층판과 같은 이질성, 이방성 평판이론, 또는 쉘이론의 유도에 유용하게 사용할 수 있다.

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INTRODUCTION

Due to growing importance of variational principle in the study of theoretical mechanics as well as in the development of direct approximate solution procedures for the boundary value problem, a great research interest has been attracted to the development of systematic procedure for derivation of variational principles governing linear and certain nonlinear boundary value problems. As representatives of a fairly large body of literatures on this subject, we cite the monographs by Mikhlin[1], Vainberg[2], Tonti [3] and Oden and Reddy[4], wherein references to other work may be found.

Mikhlin[1] stated the basic variational theorem for self−adjoint linear operators on an inner product space, in which the inner product was used as the nondegenerate bilinear mapping. Gurtin[5, 6] used convolution product for writing variational principles governing initial value problems and explicitly included nonhomogeneous initial and boundary conditions in the formulation. Sandhu and Pister[7, 8] extended this to the linear coupled problem. In the context of application of finite element method to the plate problem, Prager[9] included, in the variational formulation, jump discontinuities which may exist across interelement boundaries. Sandhu and Salaam[10] examined the general case of linear operators with nonhomogeneous boundary conditions and internal jump discontinuities based on the extension of Mikhlin's theorem. By introducing the concept of boundary operators consistent with field operators, a systematic procedure to obtain variational principles for linear coupled problem was developed.

For elasticity problem, Reissner[11] developed a mixed vaiational principle that contains displacement and stress components as the state variables and used it for the derivation of approximate theory for the homogeneous isotropic plate, while potential and complementary energy principles had been widely used for most applications. Obviously, its development was based upon his physical intuition. Recently, Reissner [12] derived another type of mixed variational principle for the development of anisotropic plate theory by applying Lagrange multiplier technique and partial Legendre transform to the potential energy functional. He later presented similar version of the principle[13], which was derived form the original mixed variational principle[11], and discussed possibility of its application to the anisotropic nonhomogeneous plate like composite laminates. However, explicit form of the governing functional for the fully anisotropic material was not given.

In this paper, we present a mixed variational principle analogous to Reissner's[12, 13] following Sandhu's framework for the variational formulation of coupled boundary value problems. For the derivation of the governing principle, elasticity equations are first splitted and recast in a self−adjoint form with respect to inner product. Extension of the space of admissible state functions in the general governing functional and specialization to a certain relation lead to the desired variational principle. Physical meaning and possible application of the derived functional are discussed.

VARIATIONAL PRINCIPLE OF COUPLED B.V.P

Consider the boundary value problem

$$A(u)=f \text{ on } R \tag{1}$$

$$C(u)=g \text{ on } \partial R \tag{2}$$

where R is an open connected region in a

Euclidean space E^3 and ∂R is its boundary ; A, C are linear, bounded differential operators. We also assume that the field operator A is self-adjoint, i.e.,

$$\langle Au,\ v\rangle_R = \langle u,\ Av\rangle_R + D_{\partial R}(u,\ v) \tag{3}$$

where $\langle\ ,\ \rangle_R$ is a nondegenerate bilinear mapping and $D_{\partial R}(u,\ v)$ is the quantities associated with the boundary ∂R. Sandhu and Salaam[10] generalized Mikhlin's basic theorem of variational principle to show that the functional governing the problem (1) and (2) is given by

$$\Omega(u) = \langle Au,\ u\rangle_R - 2\langle u,\ f\rangle_R + \langle Cu,\ u\rangle_{\partial R} - 2\langle u,\ g\rangle_{\partial R} \tag{4}$$

In other words, the Gateaux differential of (4) along arbitrary path v

$$\Delta_v\Omega(u) = \frac{d}{d\lambda}\ \Omega(u+\lambda v)\,|_{\lambda=0} \tag{5}$$

vanishes if and only if u satisfies (1) and (2), and the field operator A is consistent with the boundary operator C, i.e.,

$$D_{\partial R}(u,\ v) = \langle v,\ Cu\rangle_{\partial R} - \langle u,\ Cv\rangle_{\partial R} \tag{6}$$

The framework of variational formulation stated above can further be extended to the coupled boundary value problem with multiple field variables. With nonhomogeneous boundary conditions, it is written as

$$\sum_{j=1}^{n} A_{ij}u_j = f_i \text{ on } R \tag{7}$$

$$\sum_{j=1}^{n} C_{ij}u_j = g_i \text{ on } \partial R_i,\ i=1,\ 2,\ \cdots,\ n \tag{8}$$

in which ∂R_i, denote segments of ∂R such that

$$\partial R = \cup_{i=1}^{n}\partial R_i \tag{9}$$

and n is the number of independent field variables. Operators A_{ij} are regarded as the transformations that correspond the elements M_{ij} onto P_{ij}

$$A_{ij} : M_{ij} \rightarrow P_{ij} \tag{10}$$

such that

$$u_j \in W_j = \cap_{i=1}^{n} M_{ij} \tag{11}$$

$$f_i \in V_i = \cup_{j=1}^{n} P_{ij} \tag{12}$$

where W_j and V_j are the linear vector spaces. Let V be a linear vector space defined as the direct sum

$$V = V_1 + V_2 + \cdots\cdots + V_n \tag{13}$$

and an element $u \in V$ be the ordered set $u = \{u_1,\ u_2,\ \cdots,\ u_n\}$ such that $u_i \in V_i$. Then, the bilinear mapping on V is defined as

$$\langle u,\ v\rangle_R = \langle u_i,\ v_i\rangle_R + \cdots\cdots + \langle u_n,\ v_n\rangle_R \tag{14}$$

In line with (3), the set of operators A_{ij} is said to be self-adjoint with respect to this bilinear mapping if

$$\sum_{j=1}^{n}\langle u_j,\ A_{ji}u_i\rangle_R = \langle u_i,\ \sum_{j=1}^{n} A_{ij}u_j\rangle_R + D_{\partial R}(u_i,\ u_j) \tag{15}$$

where $D_{\partial R}(u_i,\ u_j)$ denote quantities associated with ∂R. If the set of operators A_{ij} is self-adjoint, as a generalization of (4), the governing functional of (7) and (8) is defined as

$$\Omega(u_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}\{\langle u_i,\ A_{ij}u_j - 2f_i\rangle_R + \langle u_i, C_{ij}u_j - 2g_i\rangle_{\partial R}\} \tag{16}$$

For vanishing of the Gateaux differential of this functional to imply (7) and (8), the boundary operators C_{ij} must be consistent with the field operators A_{ij}. Sandhu[14] stated the consistent condition as

$$D_{\partial R}(u_i,\ u_j) = \langle u_i,\ \sum_{j=1}^{n} C_{ij}u_j\rangle_{\partial R} - \sum_{j=1}^{n}\langle u_j,\ C_{ji}u_i\rangle_{\partial R} \tag{17}$$

In other words, for (16) to be a governing functional in variational formulation of the problem given by (7) and (8), the boundary operators must satisfy (17).

SELF-ADJOINT FORM OF ELASTICITY EQUATIONS

We summarize below the field equations of elasticity and rewrite them in the self-adjoint form which is necessary to derive variational principle. Throughout, all functions are defined on the domain $\bar{R}$, closure of the open connected spatial region of interest R. Standard index notation is used, in which Latin indices take on values of 1, 2, 3 and Greek indices take on value of 1, 2. Summation on repeated indices is implied and a comma indicates differentiation with respect to the spatial coordinate denoted by following index.

For a linear elastic body, the governing differential equations can be written by

$$\text{Equilibrium}: \sigma_{ij,j}+f_i=0 \tag{18}$$

$$\text{Kinematics}: e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{19}$$

$$\text{Constitutive Eq.}: \sigma_{ij}=E_{ijkl}e_{kl} \text{ or } e_{ij}=C_{ijkl}\sigma_{kl} \tag{20}$$

in which σ_{ij}, f_i are the components of Cauchy stress tensor and body force vectors; u_i, e_{ij} are the components of displacement vector and symmetric linear strain tensor; E_{ijkl} and C_{ijkl} are the components of elasticity and compliance tensors, respectively. Decomposing these equations into the components associated with x_α and x_3 coordinates and rewriting in matrix form, we have

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}\frac{\partial}{\partial x_3}\delta_{\alpha\gamma} & L_2 \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{1}{2}\frac{\partial}{\partial x_\gamma} & 0 \\ 0 & -\frac{\partial}{\partial x_3} & C_{3333} & C_{33\gamma3} & C_{33\gamma\delta} \\ -\frac{1}{2}\frac{\partial}{\partial x_3}\delta_{\alpha\gamma} & -\frac{1}{2}\frac{\partial}{\partial x_\alpha} & C_{\alpha333} & C_{\alpha3\gamma3} & C_{\alpha3\gamma\delta} \\ -L_1 & 0 & C_{\alpha\beta33} & C_{\alpha\beta\gamma3} & C_{\alpha\beta\gamma\delta} \end{bmatrix} \begin{bmatrix} u_\gamma \\ u_3 \\ \sigma_{33} \\ 2\sigma_{\gamma3} \\ \sigma_{\gamma\delta} \end{bmatrix} = \begin{bmatrix} -f_\alpha \\ -f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

in which $\delta_{\alpha\beta}$ is the identity tensor and

$$L_1=\frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\beta}+\delta_{\beta\gamma}\frac{\partial}{\partial x_\alpha}\right) \tag{22}$$

$$L_2=\frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial x_\delta}+\delta_{\alpha\delta}\frac{\partial}{\partial x_\gamma}\right) \tag{23}$$

Elements of the operator matrix in (21) satisfy self-adjointness in the sense of (15) if the bilinear mapping is defined as the inner product

$$\langle f, g\rangle_R = \int_R fg\,dR \tag{24}$$

Note that this is nondegenerate. The operators on the diagonal are symmetric and the off-diagonal operators constitute adjoint pairs. It can be shown that consistent boundary conditions associated with the field equations (21), in the sense of (17), are

$$u_\alpha\eta_\beta=\hat{u}_\alpha\eta_\beta \quad u_\alpha\eta_3=\hat{u}_\alpha\eta_3 \quad \text{on } S_1 \tag{25}$$

$$u_3\eta_3=\hat{u}_3\eta_3 \quad u_3\eta_\beta=\hat{u}_3\eta_\beta \quad \text{on } S_1 \tag{26}$$

$$-(\sigma_{\alpha\beta}\eta_\beta+\sigma_{\alpha3}\eta_3)=-\hat{t}_\alpha \quad \text{on } S_2 \tag{27}$$

$$-(\sigma_{33}\eta_3+\sigma_{\alpha3}\eta_\alpha)=-\hat{t}_3 \quad \text{on } S_2 \tag{28}$$

where a superposed circumflex denotes the value of the prescribed quantity over the boundary surface; $\hat{t}_i$ and η_i are the components of the prescribed traction vector and of the unit outward normal to the boundary. S_1 and S_2 are complementary subsets of ∂R(Fig.1).

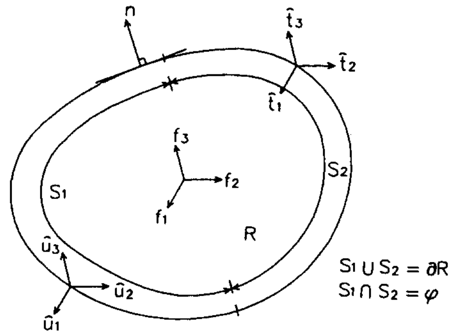

Fig. 1 Force system of an elastic body embedded in E^3 space

Referred to (15) and (17), off-diagonal elements of the self-adjoint matrix in (21) satisfy the following relations:

$$\langle u_\alpha, \sigma_{\alpha3,3}\rangle_R = -\langle \sigma_{\alpha3}, u_{\alpha,3}\rangle_R + \langle u_\alpha, \sigma_{\alpha3}\eta_3\rangle_{S_2} + \langle \sigma_{\alpha3}, u_\alpha\eta_3\rangle_{S_1} \tag{29}$$

$$\langle u_3, \sigma_{33,3}\rangle_R = -\langle \sigma_{33}, u_{3,3}\rangle_R + \langle u_3, \sigma_{33}\eta_3\rangle_{S_2} + \langle \sigma_{33}, u_3\eta_3\rangle_{S_1} \tag{30}$$

$$\langle u_\alpha, \sigma_{\alpha\beta,\beta}\rangle_R = -\langle \sigma_{\alpha\beta}, u_{\alpha,\beta}\rangle_R + \langle u_\alpha, \sigma_{\alpha\beta}\eta_\beta\rangle_{S_2} + \langle \sigma_{\alpha\beta}, u_\alpha\eta_\beta\rangle_{S_1} \tag{31}$$

$$\langle u_3, \sigma_{\alpha3,\alpha}\rangle_R = -\langle \sigma_{\alpha3}, u_{3,\alpha}\rangle_R + \langle u_3, \sigma_{\alpha3}\eta_\alpha\rangle_{S_2} + \langle \sigma_{\alpha3}, u_3\eta_\alpha\rangle_{S_1} \tag{32}$$

The relations (29)-(32) play a very important role in deriving various alternative forms of the general variational principle.

GENERAL VARIATIONAL PRINCIPLE

Using the definition (16), the governing functional for the field equations (21) and associated consistent boundary conditions (25)-(28) can be written as

$$\begin{aligned}\Omega = {} & \langle u_\alpha, \sigma_{\alpha3,3}\rangle_R + \langle u_\alpha, \sigma_{\alpha\beta,\beta}\rangle_R + \langle u_3, \sigma_{33,3}\rangle_R \\ & + \langle u_3, \sigma_{\alpha3,\alpha}\rangle_R + 2\langle u_\alpha, f_\alpha\rangle_R + 2\langle u_3, f_3\rangle_R \\ & - \langle \sigma_{33}, u_{3,3}\rangle_R - \langle \sigma_{\alpha3}, (u_{\alpha,3}+u_{3,\alpha})\rangle_R \\ & - \langle \sigma_{\alpha\beta}, u_{\alpha,\beta}\rangle_R \\ & + \langle \sigma_{33}, C_{3333}\sigma_{33} + 2C_{33\alpha3}\sigma_{\alpha3} + C_{33\alpha\beta}\sigma_{\alpha\beta}\rangle_R \\ & + \langle 2\sigma_{\alpha3}, C_{\alpha333}\sigma_{33} + 2C_{\alpha3\gamma3}\sigma_{\gamma3} + C_{\alpha3\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\ & + \langle \sigma_{\alpha\beta}, C_{\alpha\beta33}\sigma_{33} + 2C_{\alpha\beta\gamma3}\sigma_{\gamma3} + C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\ & + \langle \sigma_{\alpha\beta}, (u_\alpha - 2\hat{u}_\alpha)\eta_b\rangle_{S_1} + \langle \sigma_{\alpha3}, (u_3 - 2\hat{u}_3)\eta_\alpha\rangle_{S_1} \\ & + \langle \sigma_{\alpha3}, (u_\alpha - 2\hat{u}_\alpha)\eta_3\rangle_{S_1} + \langle \sigma_{\alpha3}, (u_3 - 2\hat{u}_3)\eta_\alpha\rangle_{S_1} \\ & - \langle u_\alpha, (\sigma_{\alpha\beta}\eta_\beta + \sigma_{\alpha3}\eta_3) - 2\hat{t}_\alpha\rangle_{S_2} \\ & - \langle u_3, \sigma_{33}\eta_3 - 2\hat{t}_3\rangle_{S_2}\end{aligned} \tag{33}$$

Let $\{\tilde{v}\} = \{\tilde{u}_\alpha, \tilde{u}_3, \tilde{\sigma}_{\alpha\beta}, \tilde{\sigma}_{\alpha3}, \tilde{\sigma}_{33}\}$ be an admissible state corresponding to the set of field variables $\{v\} = \{u_\alpha, u_3, \sigma_{\alpha\beta}, \sigma_{\alpha3}, \sigma_{33}\}$. Assuming that the $\{v\} + \lambda\{\tilde{v}\}$, λ a scalar, is an admissible state for all λ, Gateaux differential (5) of the functional (33) gives

$$\begin{aligned}\Delta_v\Omega = {} & -2\langle \tilde{\sigma}_{\alpha3}, (u_{\alpha,3} + u_{3,\alpha}) - 2C_{33\alpha3}\sigma_{33} \\ & \quad - 4C_{\alpha3\gamma3}\sigma_{\gamma3} - 2C_{\gamma\delta\alpha3}\sigma_{\gamma\delta}\rangle_R \\ & -2\langle \tilde{\sigma}_{\alpha\beta}, u_{\alpha,\beta} - C_{33\alpha\beta}\sigma_{33} - 2C_{\gamma3\alpha\beta}\sigma_{\gamma3} \\ & - C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\ & - \langle \tilde{\sigma}_{33}, u_{33} - C_{3333}\sigma_{33} - 2C_{\alpha333}\sigma_{\alpha3} - C_{\alpha\beta33}\sigma_{\alpha\beta}\rangle_R \\ & + 2\langle \tilde{u}_\alpha, \sigma_{\alpha3,3} + \sigma_{\alpha\beta,\beta} + f_\alpha\rangle_R \\ & + 2\langle \tilde{u}_3, \sigma_{33,3} + \sigma_{\alpha3,\alpha} + f_3\rangle_R \\ & + 2\langle \tilde{\sigma}_{\alpha\beta}, (u_\alpha - \hat{u}_\alpha)\eta_\beta\rangle_{S_1} + 2\langle \tilde{\sigma}_{33}, (u_3, \hat{u}_3)\eta_3\rangle_{S_1} \\ & + 2\langle \tilde{\sigma}_{\alpha3}, (u_3 - \hat{u}_3)\eta_\alpha\rangle_{S_1} + 2\langle \tilde{\sigma}_{\alpha3}, (u_\alpha - \hat{u}_\alpha)\eta_3\rangle_{S_1} \\ & - 2\langle \tilde{u}_\alpha, (\sigma_{\alpha\beta}\eta_\beta + \sigma_{\alpha3}\eta_3) - \hat{t}_\alpha\rangle_{S_2} \\ & - 2\langle \tilde{u}_3, (\sigma_{33}\eta_3 + \sigma_{\alpha3}\eta_\alpha) - \hat{t}_3\rangle_{S_2}\end{aligned} \tag{34}$$

The Gateaux differential (34) vanishes if and only if all the field equations and boundary conditions are satisfied due to linearity and nondegeneracy of the product. In other words, vanishing of $\Delta_v\Omega$ for all $\{\tilde{v}\}$ implies satisfaction of (21) and the boundary conditions (25)-(28)

DERIVATION OF A MIXED VARIATIONAL PRINCIPLE

Eqs. (29)-(32) relate pairs of off-diagonal operators in the operator matrix of (21) and may be used to eliminate either of elements in each pair from the functional Ω. Elimination of an operator A_{ij} implies that state variable u_j needs not to be in the domain M_{ij} of A_{ij}. This results in relaxing the requirement of differentiability of u_j, thereby extending the space of admissible states.

Through this procedure, numerous alternative forms of the functional Ω are possible, even though all of them are not enumerated herein. For the present purpose, however, we use Eqs. (29)−(32) simultaneously to eliminate $\sigma_{\alpha\beta,\beta}$, $\sigma_{\alpha 3,\alpha}$, $\sigma_{\alpha 3,3}$ and $\sigma_{33,3}$ from Ω, giving

$$
\begin{aligned}
\Omega_1 = & -2\langle \sigma_{\alpha\beta},\ u_{\alpha,\beta}\rangle_R - 2\langle \sigma_{\alpha 3},\ (u_{\alpha,3}+u_{3,\alpha})\rangle_R - 2 \\
& \langle \sigma_{33},\ u_{3,3}\rangle_R \\
& + 2\langle u_\alpha,\ f_\alpha\rangle_R + 2\langle u_3,\ f_3\rangle_R \\
& + \langle \sigma_{33},\ C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}\rangle_R \\
& + \langle 2\sigma_{\alpha 3},\ C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\
& + \langle \sigma_{\alpha\beta},\ C_{\alpha\beta 33}\sigma_{33} + 2C_{\alpha\beta\gamma 3}\sigma_{\gamma 3} + C_{\alpha\beta\gamma\delta}\rangle_R \\
& + 2\langle \sigma_{\alpha\beta},\ (u_\alpha - \hat{u}_\alpha)\eta_\beta\rangle_{S_1} + 2\langle \sigma_{33},\ (u_3 - \hat{u}_3)\eta_3\rangle_{S_1} \\
& + 2\langle \sigma_{\alpha 3},\ (u_\alpha - \hat{u}_\alpha)\eta_3\rangle_{S_1} + 2\langle \sigma_{\alpha 3},\ (u_3 - \hat{u}_3)\eta_\alpha\rangle_{S_1} \\
& + 2\langle u_\alpha,\ \hat{t}_\alpha\rangle_{S_2} + 2\langle u_3,\ \hat{t}_3\rangle_{S_2} \qquad (35)
\end{aligned}
$$

This is equivalent to Hellinger−Reissner variational principle. For this functional, certain specializations are possible by constraining the admissible state to satisfy some field equations. Assuming that the kinematic relations of x_1-x_2 plane, i.e. the fifth eq. of (21) is identically satisfied, Ω_1 reduces to

$$
\begin{aligned}
\Omega_2 = & -\langle \sigma_{\alpha\beta},\ u_{\alpha,\beta}\rangle_R - 2\langle \sigma_{\alpha 3},\ (u_{\alpha,3}+u_{3,\alpha})\rangle_R \\
& -2\langle \sigma_{33},\ u_{3,3}\rangle_R \\
& + 2\langle u_\alpha,\ f_\alpha\rangle_R + 2\langle u_3, f_3\rangle_R \\
& + \langle \sigma_{33},\ C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}\rangle_R \\
& + \langle 2\sigma_{\alpha 3},\ C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\
& + 2\langle \sigma_{\alpha\beta},\ (u_\alpha - \hat{u}_\alpha)\eta_\beta\rangle_{S_1} + 2\langle \sigma_{33},\ (u_3 - \hat{u}_3)\eta_3\rangle_{S_1} \\
& + 2\langle \sigma_{\alpha 3},\ (u_\alpha - \hat{u}_\alpha)\eta_3\rangle_{S_1} + 2\langle \sigma_{\alpha 3},\ (u_3 - \hat{u}_3)\eta_\alpha\rangle_{S_1} \\
& + 2\langle u_\alpha,\ \hat{t}_\alpha\rangle_{S_2} + 2\langle u_3,\ \hat{t}_3\rangle_{S_2} \qquad (36)
\end{aligned}
$$

Note that by this specialization Gateaux differential of the functional Ω_2 does not yield the kinematic relations of x_1-x_2 plane. This means that this functional is valid only for an elastic body which undergoes deformation in a way of satisfying constrained conditions in specialization procedure. This point is noteworthy in connection with application of this functional. If we assume further that the displacement boundary conditions on S_1 are identically satisfied, Ω_2 reduces to

$$
\begin{aligned}
\Omega_3 = & -\langle \sigma_{\alpha\beta},\ u_{\alpha,\beta}\rangle_R - 2\langle \sigma_{\alpha 3},\ (u_{\alpha,3}+u_{3,\alpha})\rangle_R \\
& -2\langle \sigma_{33},\ u_{3,3}\rangle_R \\
& + 2\langle u_\alpha,\ f_\alpha\rangle_R + 2\langle u_3,\ f_3\rangle_R \\
& + \langle \sigma_{33},\ C_{3333}\sigma_{33} + 2C_{33\alpha 3}\sigma_{\alpha 3} + C_{33\alpha\beta}\sigma_{\alpha\beta}\rangle_R \\
& + \langle 2\sigma_{\alpha 3},\ C_{\alpha 333}\sigma_{33} + 2C_{\alpha 3\gamma 3}\sigma_{\gamma 3} + C_{\alpha 3\gamma\delta}\sigma_{\gamma\delta}\rangle_R \\
& + 2\langle u_\alpha,\ \hat{t}_\alpha\rangle_{S_2} + 2\langle u_3,\ \hat{t}_3\rangle_{S_2} \qquad (37)
\end{aligned}
$$

Reissner[12, 13] recently presented two mixed variational principles similar to Ω_3, but without boundary terms, which were derived by using Lagrange multiplier technique and partial Legendre transform in the minimum potential energy principle and Hellinger-Reissner variational principle, respectively. The resulted governing functionals contain partial complementary energy density functions which are given in an implicit form. For some special types of elastic materials with certain symmetry of material properties, the procedure for obtaining the explicit form of the governing functional was discussed. However, for the general anisotropic material, derivation of the explicit form of the functional is difficult because of coupling in constitutive equation. For the general applicability, hence, it is obvious that the variational principle derived herein is more advantageous than Reissner's.

VARIATIONAL EQUATIONS FOR THE SPECIAL MATERIALS

For convenience in application of the mixed variational principle derived above, its variation is explicitly presented for some special materials with material symmetries. The mixed variational principle (37) may be rewritten as

$$\Omega_3 = \int_R \{\frac{1}{2}\sigma_{\alpha\beta}u_{\alpha,\beta} + \sigma_{\alpha3}(u_{\alpha,3} + u_{3,\alpha}) + \sigma_{33}u_{3,3}$$

$$-\sigma_{\alpha3}\bar{e}_{\alpha3} - \frac{1}{2}\sigma_{33}\bar{e}_{33} - u_i f_i\}dR - \int_{S_t} u_i \hat{t}_i \, ds \quad (38)$$

in which

$$\bar{e}_{\alpha3} = C_{\alpha3\gamma\delta}\sigma_{\gamma\delta} + 2C_{\alpha3\gamma3}\sigma_{\gamma3} + C_{\alpha333}\sigma_{33} \quad (39)$$

$$\bar{e}_{33} = C_{33\gamma\delta}\sigma_{\gamma\delta} + 2C_{33\gamma3}\sigma_{\gamma3} + C_{3333}\sigma_{33} \quad (40)$$

Recalling that in the derivation of the above functional the kinematic relation in $x_1 - x_2$ plane was assumed to be identically satisfied, with some algebra, vanishing of the Gateaux differential of Ω_3 may be stated as

$$O = \delta\Omega_3 = \int_R \{\sigma_{ij}\delta u_{i,j} + \delta\sigma_{\alpha3}(u_{\alpha,3} + u_{3,\alpha} - 2\bar{e}_{\alpha3})$$

$$+\delta\sigma_{33}(u_{3,3} - \bar{e}_{33})$$

$$-\delta u_i f_i + \delta(\sigma_{\alpha3}C_{\alpha333}\sigma_{33})\}dR$$

$$-\int_{S_t}\delta u_i \hat{t}_i \, ds \quad (41)$$

where δ means variation of the associated function. This variational equation is valid for general anisotropic materials. For obtaining variational equation of a special elastic body with material property symmetry with respect to certain plane, only thing to do is to set proper components of compliance tensor C_{ijkl} to zero in $\bar{e}_{\alpha3}$, $\bar{e}_{33}$ and the last term of the first integral in (41). For a monoclinic material having material symmetry about $x_3 = 0$, for instance, the stresses $\sigma_{\alpha3}$, σ_{33} are independent of $e_{\alpha3}$ and so $C_{ii\alpha3} = C_{12\alpha3} = 0$. .Also, for orthotropic, transversely isotropic and isotropic materials, normal stresses are not coupled with shear strains, so compliance components to be zero are $C_{ii\alpha\beta} = C_{ii\alpha3} = C_{12\alpha3} = C_{1223} = 0$.

DISCUSSION

A mixed variational principle for linear elasticity has been developed using a systematic procedure developed by Sandhu et al. The derived principle is applicable to general anisotropic case if deformation is small and the kinematic relations in $x_1 - x_2$ plane are satifsied. Even though it can be used as a basis for the development of approximate solution procedure such as finite element method, its usefulness may be more highlighted in the derivation of approximate theory for anisotropic and inhomogeneous plates and shells, e.g. laminated composites since the theories for such solids are often based upon the assumed displacement field, which violates transverse kinematic relations, while the in-plane kinematics are satisfied. It should be mentioned here that the derived principle is analogous to Reissner's mixed variational principle. But it has advantages over the later one because the present one is directly applicable to any types of anisotropic materials.

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