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Discrete Observation of a Continuous-Time Absorbing Markov Chain

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Abstract

For a continuous-time absorbing Markov chain, we apply and simplify the general method of finding the transition rates given the transition probabilities obtained from discrete observation of the continuous system. An example is given.

1. Introduction

If, for practical reasons, a continuous-time Marvkov chain is observed at equal intervals of time, then it is legitimate to treat it as if it were a discrete-time Markov chain [1, pp.94].

For convenience, let the length of the time interval be one time unit. In addition, let Λ denote the transition rate matrix of the underlying continuous-time Markov chain and $\mathbf{P}$ denote the transition probability matrix for the observed discrete-time Markov chain. Then the two matrices are related by

$$\mathbf{P}=\exp \Lambda. \tag{1}$$

Given $\mathbf{P}$, (1) has a unique solution for Λ, $\Lambda=\ln \mathbf{P}$, if the eigenvalues of $\mathbf{P}$ are real and positive [1].

For the case of absorbing Markov chain, we suggest in this paper an efficient and meaningful way to obtain Λ given $\mathbf{P}$.

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2. Finding Λ given P for an absorbing Markov chain

We consider **P** having real and positive eigenvalues denoted by μ_1, μ_2, μ_3, $\cdots$. Then a workable solution of (1) for Λ is

$$\Lambda = \mathbf{V}\ (\ln \mathbf{M})\ \mathbf{V}^{-1} \tag{2}$$

where **M** is the diagonal matrix whose i[th] entry is μ_i and

V is the matrix whose j[th] column is the eigenvector corresponding to μ_j.

Since **M** is diagonal so is (ln **M**). Moreover, the i[th] entry of (ln **M**) is simply ($\ln \mu_i$) [4].

Now for an absorbing Markov chain having **m** transient states and **n** absorbing states, we partition **P** as

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{O} & \mathbf{I}_n \end{array}\right]. \tag{3}$$

Where **Q** is the matrix of transition probabitities from transient states to transient states,

R is the matrix of transition probabilities from transient states to absorbing states,

$\mathbf{I}_n$ is the identity matrix of dimension n, and

O is the matrix with all entries zero.

Theorem 1. Let $\mathbf{M}_Q$ be the diagonal matrix with the eigenvalues of **Q** on the diagonal. Then

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_Q & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{I}_n \end{array}\right]. \tag{4}$$

Proof of Theorem 1. To find the eigenvalues of **P**, we set the determinant of ($\mu \mathbf{I}_{m+n} - \mathbf{P}$) equal to zero and solve for μ. Now expanding this determinant by minors, we get $(\mu-1)^n$ times the determinant of ($\mu \mathbf{I}_m - \mathbf{Q}$). Thus the eigenvalues of **P** consist of those of **Q** and 1 of multiplicity n.

Theorem 2. Let $\mathbf{V}_Q$ be the matrix whose columns are the eigenvectors of **Q**. Then

$$\mathbf{V} = \left[\begin{array}{c|c} \mathbf{V}_Q & \mathbf{A} \\ \hline \mathbf{O} & \mathbf{I}_n \end{array}\right]. \tag{5}$$

where $\mathbf{A} = (\mathbf{I}_m - \mathbf{Q})^{-1}\ \mathbf{R}$. (6)

Proof of Theorem 2. To find the eigenvector **X** corresponding to given eigenvalue μ, we set ($\mu \mathbf{I}_{m+n} - \mathbf{P}$)**X** equal to zero vector and solve for **X**. Partitioning this equation using (3) as

$$\left[\begin{array}{c|c} \mu \mathbf{I}_m - \mathbf{Q} & -\mathbf{R} \\ \hline \mathbf{O} & (\mu-1)\ \mathbf{I}_n \end{array}\right] \left[\begin{array}{c} \mathbf{X}_m \\ \mathbf{X}_n \end{array}\right] = \left[\begin{array}{c} \mathbf{O}_m \\ \mathbf{O}_n \end{array}\right],$$

where the subscripts attached to **X** and **O** are the dimensions of the column vectors, we get the following two sets of linear equations:

$$(\mu \mathbf{I}_m - \mathbf{Q})\mathbf{X}_m - \mathbf{R}\,\mathbf{X}_n = \mathbf{O}_m \tag{7}$$

$$(\mu - 1)\mathbf{X}_n = \mathbf{O}_n \tag{8}$$

We first consider the case $\mu=1$. For $\mu=1$, $\mathbf{X}_n$ in (8) is undetermined. This rather gives us freedom. For the first $\mu=1$ among n of such, we set $\mathbf{X}_n=(1, 0, \cdots, 0)^T$, where the superscript T means 'the transpose of'. Substituting this $\mathbf{X}_n$ together with $\mu=1$ into (7) and using (6), we get

$\mathbf{X}_m=(\mathrm{I}-\mathbf{Q})^{-1}\ \mathbf{R}\ (1, 0, \cdots, 0)^T=$(the 1st column of **A**).

Likewise, we set $\mathbf{X}_n=(0, 1, 0, \cdots, 0)^T$ for the second one and get $\mathbf{X}_m=$(**A**'s 2nd column). We continue setting $\mathbf{X}_n$ this way until we get $\mathbf{X}_m=$(the n^th^ column of **A**) for the last or the nth one. Then putting these eigenvectors together we see that the right half of partitioned **V** in (5) is the matrix whose columns are the eigenvectors associated with $\mu=1$ of multiplicity n.

Before proving the rest, we need to make a comment on $\mu=1$. Suppose the chain is not absorbing and there exist limiting steady-state probabilities, then the normalized eigenvector corresponding to $\mu=1$ gives the limiting steady-state probabilites[2]. An absorbing Markov chain does not have the limiting steady-state probabilities. Instead, it has the limiting absorption probabilities: The i,j entry of **A** is known to be the probability that the chain will eventually be absorbed in absorbing state j when the cain is currently in transient state i [3]. If, however, the current state is an absorbing state, then we have a trivial case of having **I** as the limiting absorption probability matrix. Thus it is intuitively appealing for **V** in (5) to have **A** as from-transient-to-absorbing component and to have **I** as from-absorbing-to-absorbing component.

We now consider the case that the eigenvalues of **P** come from the eigenvalues of **Q.** In this case none of the eigenvalues is 1 [2]. Thus we get $\mathbf{X}_n=\mathbf{O}_n$ from (8), and then from (7) we get $(\mu \mathbf{I}_m-\mathbf{Q})\mathbf{X}_m=\mathbf{O}_m$, which is the equation for the **Q**'s eigenvector corresponding to the the **Q**'s eigenvalue μ. Then putting these eigenvectors together, we have $\mathbf{V}_Q$ and **O** for the left half of **V** in (5).

Theorem 3.

$$\Lambda=\left[\begin{array}{c|c} \mathbf{V}_Q\ (\ln \mathbf{M}_Q)\ \mathbf{V}_Q^{-1} & -\mathbf{V}_Q(\ln \mathbf{M}_Q)\ \mathbf{V}_Q^{-1}\ \mathbf{A} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right]. \tag{9}$$

Proof of Theorem 3. We get (9) by substituting (4) and (5) into (2).

3. Example

Suppose discrete observation of a continuous-time absorbing Markov chain yields

$$P=\left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right]=\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 0.3 & 0.2 & 0.4 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

The eigenvalues of Q are $\mu_1=1/2$ and $\mu_2=1/5$; and we choose $(1, 1)^T$ and $(-2, 1)^T$, respectively, as corresponding eigenvectors. Thus we have

$$V_Q=\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix},\ V_Q^{-1}=\begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}, \text{ and}$$

$$\ln M_Q=\ln\begin{bmatrix} 1/2 & 0 \\ 0 & 1/5 \end{bmatrix}=\begin{bmatrix} \ln 1/2 & 1 \\ 0 & \ln 1/5 \end{bmatrix}.$$

Then we need the limiting absorption probability matrix, which is

$$A=(I-Q)^{-1}R=\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{cc} 3 & 4 \\ 3/4 & 1/4 \\ 5/8 & 3/8 \end{array}$$

For instance, if the chain is currently in state 1, then it will eventually enter either state 3 with probability 3/4 or state 4 with probability 1/4.

The transition rate matrix for the underlying continuous-time absorbing Markov chain is then

$$\Lambda=\left[\begin{array}{c|c} V_Q(\ln M_Q)\ V_Q^{-1} & -V_Q(\ln M_Q)\ V_Q^{-1}\ A \\ \hline 0 & 0 \end{array}\right]$$

$$\approx\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1.304 & 0.611 & 0.586 & 0.097 \\ 0.305 & -0.999 & 0.396 & 0.298 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

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