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Shortest Path Problems: A Parametric Study

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Abstract

Two important sensitivity issues over shortest path problems have been discussed. One is the problem of updating shortest paths when nodes are added and when the lengths of some arcs are increased or decreased. The other is the problem of calculating *arc tolerances*, that is, the maximum increase or decrease in the length of *a single arc without* changing a given optimal tree. In this paper, assuming that there exists a parameter of interest whose perturbation causes the *simultaneous* changes in arc lengths, we find the *invariance* condition on these simulthaneous changes such that the shortest path between two specified nodes remains unchanged.

The shortest path problem is one of the fundamental problems in the area of network programming and many efficient algorithms have been proposed in the literature (e.g. [3, 4]). Two important sensitivity issues have been discussed. One is the problem of updating shortest paths when nodes are added or deleted and when the lengths of some arcs are increased or decreased [6, 7, 11, 13]. The other is the problem of calculating arc tolerances, that is, the maximum increase or decrease in the length of a single arc without changing a given optimal tree [8, 12, 14].

However, in network applications, a node often serves as a decision point and the arcs

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emanating from this node represent possible alternatives, the costs of which naturally depend on some attribute or parameter of the decision point. In this situation, any perturbation in the parameter may affect the alternative costs in a dependent and simultaneous fashion. A proper sensitivity analysis should be capable of handling these simultaneous changes in arc costs.

In this paper, we assume that there exists a parameter of interest whose perturbation causes the *simultaneous* changes in arc lengths. We wish to find the *invariance* condition on these simultaneous changes such that the shortest path between two specified nodes remains unchanged. The invariance condition is then applied to derive the sensitivity range of the parameter. Computational aspects are discussed and numerical examples are presented.

1. Shortest Path Problems

Consider a directed acyclic network G=(J, A) with node-set J={1,⋯, n} and arc-set $A \subseteq J \times J$. In addition, there is a length(or travel time, or cost, etc). $c_{ij} \in R$ associated with each (i, j). Throughout the paper, it is assumed that the nodes in J are topologically ordered: that is, $(i, j) \in A$ implies $i<j$. This ordering is always possible for a directed graph. The problem of finding the shortest path from node 1 to n can be efficiently solved by either of the following recursions:

$$\begin{cases} f_1 = 0 \\ f_j = \min\limits_{(i,j)\in A} \{f_i + c_{ij}\},\ j=2,\cdots,\ n \end{cases} \tag{1}$$

or

$$\begin{cases} g_n = 0 \\ g_i = \min\limits_{(i,j)\in A} \{c_{ij} + g_j\},\ i=n-1,\cdots,\ 1 \end{cases} \tag{2}$$

where f_j represents the length of the shortest path from node 1 to j ;g_i, the length of the shortest path from node i to n.

Once both recursions (1) and (2) are solved, it is a simple matter to evaluate the relative penalty of an indivdual node or arc with respect to the shortest path $1 \rightarrow n$. Since any path $1 \rightarrow n$

passing through node j has at least the total length of f_j+g_j, the relative penalty of nodej can be expressed in terms of

$$f_j+g_j-f_n. \tag{3}$$

The value in (3) is zero if and only if node j is in the shortest path $1\rightarrow n$.
Any node with a positive value in (3) can be eliminated from the network without affecting the shortest path $1\rightarrow n$. Similarly, the relative penalty of arc (i, j) can be expressed in terms of

$$f_i+c_{ij}+g_j-f_n. \tag{4}$$

The value in (4) is zero if and only if arc (i, j) is in the shortest path $1\rightarrow n$.
As long as the nonnegativity of (4) is maintained, the arc length c_{ij} can be changed without affecting the shortest path $1\rightarrow n$.

Expressions (3) and (4) also can be obtained from an LP dual formulation of the shortest path problem. The optimal dual slack associated with arc (i, j) is given by

$$\bar{c}_{ij} = f_i + c_{ij} - f_j \tag{5}$$

which measures the minimum regret for taking the shortest path $1\rightarrow i$ and then arc(i, j) to reach node j. Intuitively, thare is no regret for taking arc (i, j) if it is in the shortest path from node 1 to k with $j \leq k \leq n$, or

$$\bar{c}_{ij} = 0 \text{ for } (i, j) \in P^*_k,\ j \leq k \leq \text{n} \tag{6}$$

where P^*_k is the set of arcs in the shortest path $1\rightarrow k$. Notice that $\bar{c}_{ij} \geq 0$ for $(i, j) \notin P^*_k$.

Let $\overline{\text{G}}$ denote the network G=(J, A) with slack $\bar{c}_{ij}$, instead of cost c_{ij}, for each (i, j) in A. We call G the cost network and $\overline{\text{G}}$ the slack network. Given $\overline{\text{G}}$, we define, $\bar{f}_j$ as the length of the shortest path $1\rightarrow j$ and $\bar{g}_i$ as the length of the shortest path $i\rightarrow n$. Then $\bar{f}_j = 0$, $j=1, \cdots\cdots, n$ by (6) and the values of $\bar{g}_i$ are obtained via a recursion similar to (2). Expressions (3) and (4) now can be rewritten as follows:

$$f_i + g_i - f_n = \bar{g}_i \tag{7}$$

and

$$f_i + c_{ij} + g_j - f_n = \bar{c}_{ij} + \bar{g}_j. \tag{8}$$

To verify (7), consider a path $[j, k_1, \cdots, k_l, n]$ in G. Its length is

$$\begin{aligned} & c_{jk_1} + c_{k_1k_2} + \cdots + c_{k_ln} \\ = & -f_j + (f_j + c_{jk_1} - f_{k_1}) + (f_{k_1} + c_{k_1k_2} - f_{k_2}) + \cdots + (f_{k_l} + c_{k_ln} - f_n) + f_n \\ = & -f_j + \overline{c_{jk_1}} - \overline{c_{k_1k_2}} + \cdots + \overline{c_{k_ln}} + f_n \end{aligned}$$

If$[j, k_1, \cdots, k_l, n]$ is a shortest path $j \rightarrow n$, it follows that $g_j = -f_j + \bar{g}_j + f_n$ or (7), which establishes the equivalence between G and $\bar{G}$ in terms of shortest path problems. Algebraically, this equivalence is a consequence of the fact that the *standard form* LP's objective gradient c can be replaced by $c - \pi A'$ for any conformable constraint matrix A' and row vector π without changing the primal optimal solution. Expression (8) follows immediately upon adding $f_i + c_{ij} - f_j = \bar{c}_{ij}$ to (7).

Note that $\bar{g}_j$ measures the minimum possible regret for passing through node j *en route* node n. This observation is useful in finding all ε-optimal paths when we solve a shortest path problem on G=(J, A). Specifically, if we let $J\varepsilon = \{j \in J \mid \bar{g}_j \leq \varepsilon\}$ and $A\varepsilon = \{(i, j) \in A \mid i, j \in J\varepsilon\}$, then network $(J\varepsilon, A\varepsilon)$ contains all ε-optimal paths $1 \rightarrow n$ (cf. [1]).

2. Invariance Conditions

We now suppose that some perturbation in the parameter of interest has increased the length of arc (i, j) by Δ_{ij} such that

$$c_{ij} \rightarrow c_{ij} + \Delta_{ij}$$

for all (i, j) in A. We wish to find the invariance condition on the unknowns Δ_{ij} such that the shortest path $1 \rightarrow n$ remains unchanged.

Let G_p denote the perturbed cost network $c_{ij} + \Delta_{ij}$, $\forall (i, j) \in A$

Similarly, let $\overline{G}_p$ denote the perturbed slack network with $\bar{c}_{ij}+\Delta_{ij}$, $\forall (i, j) \in A$ Then the equivalence between G and $\overline{G}$ remains valid for their perturbed versions, G_p and $\overline{G}_p$, since the quantities Δ_{ij}'s were simply superimposed for the construction of G_p and $\overline{G}_p$. This observation may be summarized as follows:

Result 1 *The shortest path $1 \to n$ on G is optimal on G_p if and only if the total of perturbed slacks on the shortest path $1 \to n$ on $\overline{G}_p$* is $\sum_{(i,j) \in P_n^*} \Delta_{ij}$.

In the following subsections, structural assumptions on Δ_{ij} are considered, on which Result 1 can be applied to derive invariance conditions for the shortest path.

2.1 Single arc length

A simplest assumption would be that the length of a single arc can be increased or decreased independently, or

$$\Delta_{ij} \neq 0 \text{ only if } (i, j) = (s, t) \tag{9}$$

for fixed (s, t). It suffices to consider only the paths $1 \to n$ *that* pass through arc(s, t). There are two cases in which $(s, t) \notin P_n^*$ and $(s, t) \in P_n^*$.

If $(s, t) \notin P_n^*$, then any path $1 \to n$ passing through arc (s, t) has at least the total of perturbed slacks.

$$\bar{f}_s + \bar{c}_{st} + \Delta_{st} + \bar{g}_t$$

which must remain nonnegative since $\sum_{(i,j) P_n^*} \Delta_{ij} = 0$.

Result 2 *Under the assumption (9), if $(s, t) \notin P_n^*$ the shortest path $1 \to n$* on G is optimal on G_p if and only if

$$\Delta_{st} \geq -\bar{c}_{st} - \bar{g}_t.$$

On the other hand, if $(s, t) \in P_n^*$, then the invariance condition on Δ_{st} can be simplified to

$$\Delta_{st} \leq \bar{f}_n(s, t)$$

where

$$\bar{f}_n(s, t) = \min_{\substack{(i,j) \in A \setminus \{(s, t)\} \\ i \leq s < j}} \{\bar{c}_{ij} + \bar{g}_j\}.$$

Note that $\bar{f}(s, t)$ represtents the length of the shortest path $1 \rightarrow n$ in the network $\bar{G}$ with arc(s, t) deleted. An efficient algorithm for calculating the values of $\bar{f}_n(s, t)$ is provided in the appendix, which runs in $O(1/6n^3)$ time.

Result 3 *Under the assumption (9), if $(s, t) \in P^*_n$, the shortest path $1 \rightarrow n$ on G is optimal on G_p if and only if*

$$\Delta_{st} \leq \bar{f}_n(s, t).$$

2.2 Node parameter

Another useful assumption would be that the lengths of all arcs emanating from a given node depend on its own parameter, or

$$\Delta_{ij} \neq 0 \text{ only if is } i = s \tag{10}$$

for fixed s. Since this is a simple extension of the assumption (9), the following results are easily understood and verified by the same reasoning.

In Result 5, $\bar{f}''_n(s)$ represents the length of the shortest path $1 \rightarrow n$ in the network $\bar{G}$ with node s deleted, or

$$\bar{f}_n(s) = \min_{\substack{(i,j) \in A \\ i < s < j}} \{\bar{c}_{ij} + \bar{g}_j\}.$$

The values of $\bar{f}_n(s)$ can be collected during the process of calculating the values of $\bar{f}_n(s)$. See the appendix for details. Also, notice that $\bar{f}''_n(s) = O$ if node s is *not* in the shortest path $1 \rightarrow n$.

Result 4 *Under the assumption (10), if $(s, j) \notin P^*_n$ for any j, the shortest path $1 \rightarrow n$ on G is optimal on G_p if and only if*

$$\Delta_{sj} \geq -\bar{c}_{sj} - \bar{g}_j \textit{ for each } (s, j) \in A.$$

Result 5 *With the assumption (10), if $(s, t) \in P^*_n$ for some t, the shortest path $1 \rightarrow n$ on G is optimal on G_p if and only if*

$$\Delta_{st} \leq \bar{f}''_n(s)$$

and

$$\Delta_{st} - \Delta_{sj} \leq \bar{c}_{sj} + \bar{g}_j \text{ for each } (s, j) \in A \text{ but } j \neq t.$$

2.3 Time parameter

There are important network applications in which nodes represent time points, and each arc represents an activity whose cost (length) depends on the parameters of those time points that are covered by the activity.

For example, see project sequencing problems [5], equipment replacement problems [p. 226, 15], and production planning problems [16]. A practical assumption then would be that

$$\Delta_{ij} \neq 0 \text{ only if } i \leq s < j \tag{11}$$

for fixed s. The parameter of node s having this property is called a *time parameter*.

If $s=n$, there is nothing to do for sensitivity analysis. If $1 \leq s < n$, every path $1 \rightarrow n$ contains some arc whose length is subject to change.

Hence

Result 6 *Under the assumption (11), if $(l,m) \in P^*_n$ with $l \leq s < m$, the shortest path $1 \rightarrow n$ on G is optimal on G_p if and only if*

$$\Delta_{lm} - \Delta_{ij} \leq \bar{c}_{ij} + \bar{g}_j \text{ for each } (i, j) \in A,\ i \leq s < j.$$

2.4 Stage parameter

Consider a network $G=(J, A)$ in which $J=\{1,\cdots, n\}$ can be partitioned into m disjoint sets $J_1=\{1\}$, $J_2, \cdots J_m=\{n\}$, and a one-step transition occurs from J_1 to J_2, then to J_3, and so on, finally to J_m so that $A=\{(i,j) \mid i \in J_k,\ j \in J_{k+1},\ k=1,\cdots m-1\}$. Subscript k of the subset J_k is called a stage variable [p.71.2]. Then we may assume that, for a fixed k,

$$\Delta_{ij} \neq 0 \text{ only if } i \in J_k \text{ and } j \in J_{k+1} \tag{12}$$

in order to investigate the sensitivity of the shortest path $1 \rightarrow n$ with respect to a parameter of stage k. Since every path $1 \rightarrow n$ contains some arc (i, j) with $i \in J_k$ and $j \in J_{k+1}$, the invariance condition on the Δ_{ij}'s is given as follows:

Result 7 *Under the assumption (12), if $(s, t) \in P_n^*$ with $s \in J_k$ and $t \in J_{k+1}$, the shortest path $1 \rightarrow n$ on G is optimal on G_p if and only if*

$$\Delta_{tm} - \Delta_{ij} \le \bar{c}_{ij} + \bar{g}_j \text{ for each } (i, j) \in A, \text{such that } i \in J_k,\ j \in J_{k+1}.$$

3. Computational Aspects

Given a shortest path $1 \rightarrow n$ with the values of f_i, the first step in finding the invariance conditions is to calculate the values of $\bar{c}_{ij}$ and g_i, whick is straightforward. In the second step, for each case of Results 2−7, the number of inequalities to be evaluated is proportional to the number of perturbed arcs. In particular, for Results 3 and 5, we need the values of $\bar{f}_n'(s)$ and $\bar{f}_n''(s)$ to complete the evaluation. The appendix lists a FORTRAN program which calculates the values of $\bar{f}_n(s, t)$ for all$(s, t) \in A$ in $O(1/6n^3)$ steps of comparisons. The values of $\bar{f}_n(s)$ can be obtained as a by-product.

In a practical viewpoint, each Δ_{ij} corresponds to an unknown in the system of inequalities to be solved. Consequently, without a proper functional form of Δ_{ij}, the usefulness of the invariance conditions in yieding the sensitivity range of a parameter would be diminished for large network applications. For notable exceptions, see the parametric study of production planning problems [9] and project sequencing problems [10] in which the simultaneous changes in arc lengths are captured into a closed−form function of the parameter. The following section illustrates a simple application.

4 Numerical Examples

Consider a network $G = (J, A)$ in which the arc cost c_{ij} is given as

$$c_{ij} = K_i + \sum_{k=i}^{j-1}(k - i + 1)t_k \tag{13}$$

where parameters K_i and t_k are any real number. We wish to find the sensitivity ranges of these parameters such that the shortest path $1 \to n$ remains unchanged for any value in the range.

Using the problem data of Table 1, we illustrate the use of the invariance conditions given in Section 2 in order to find the sensitivity range of each parameter. The corresponding cost network G is given in Figure 1. Note that node 5 is a dummy. By inspection, we have

j	K_i	t_i
1	3	1
2	5	2
3	2	4
4	4	3

Table 1 : The Problem Data

$$f_1=0,\ f_2=4,\ f_3=8,\ f_4=14,\ f_5=20$$

and the shortest path $1 \to 5$ is [1, 3, 5]. The slack network G is constructed as shown in Figure 2. Also, by inspection, we have

$$\bar{g}_1=0,\ \bar{g}_2=3,\ \bar{g}_3=0,\ \bar{g}_4=5,\ \bar{g}_5=0$$

and

$$\bar{f}_5(1)=\propto,\ \bar{f}_5(2)=0,\ \bar{f}_5(3)=6,\ \bar{f}_5(4)=0,\ \bar{f}_5(5)=\propto.$$

Examining the functional form of c_{ij} in (13), we see that any perturbation of parameter K_s causes the simultaneous changes in all the costs c_{ij} with $i=s$ and $j>s$, which is an instance of the assumption (10). Hence, the invariance conditions in Results 4 and 5 can be applied to find the sensitivity range of parameter K_s. Let $\varDelta K$ denote the change in K_s.

Then

$$\varDelta_{ij}=\varDelta K \text{ only if } i=s \text{ and } j>s. \tag{14}$$

As an example, we calculate the sensitivity range of parameter K_3. Since node 3 is in the shortest path 1→5, it follows from Result 3 that $\Delta_{35}=\Delta K \leq \bar{f}_5(3)=6$ or $\Delta K \leq 6$. Therefore, the current shortest path 1→5 remains unchanged for any value of K_3 not greater than 8.

Similarly, we see that any perturbation of parameter t_s causes the simultaneous changes in all the costs c_{ij} with $i \leq s < j$ so that Result 6 can produce the sensitivity range of t_s. Let Δt denote the change in t_s. Then

$$\Delta_{ij}=(s-i+1)\Delta t \text{ only if } i \leq s < j. \tag{15}$$

For example, for node 4, we have

$$\Delta_{15}=4\Delta t,\ \Delta_{25}=3\Delta t,\ \Delta_{35}=2\Delta t,\ \Delta_{45}=\Delta t$$

and

$$\begin{aligned} \Delta_{35}-\Delta_{15} &= -2\Delta t \leq 12+0 \\ \Delta_{35}-\Delta_{25} &= -\Delta t \leq 8+0 \\ \Delta_{35}-\Delta_{45} &= \Delta t \leq 1+0 \end{aligned}$$

so that

$$-6 \leq \Delta t \leq 1.$$

Therefore, the current shortest path 1→5 remains unchanged for any value of t_4 between −3 and 4. The complete list of the sensitivity ranges of parameters K_s and t_s is provided in Table 2. Verification of these sensitivity ranges is left as an exercise for the reader.

S	K_s		t_s	
	Lower limit	Upper limit	Lower limit	Upper limit
1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
2	2	$+\infty$	$-\infty$	4
3	$-\infty$	8	0.5	$+\infty$
4	3	$+\infty$	−3	4

Table 2: The Sensitivity Ranges of K_s and t_s

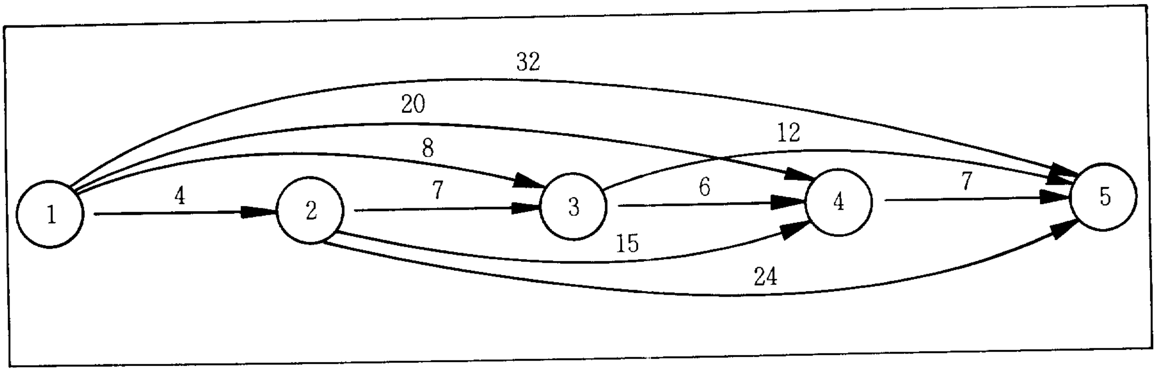

Figure 1. The Cost Network G

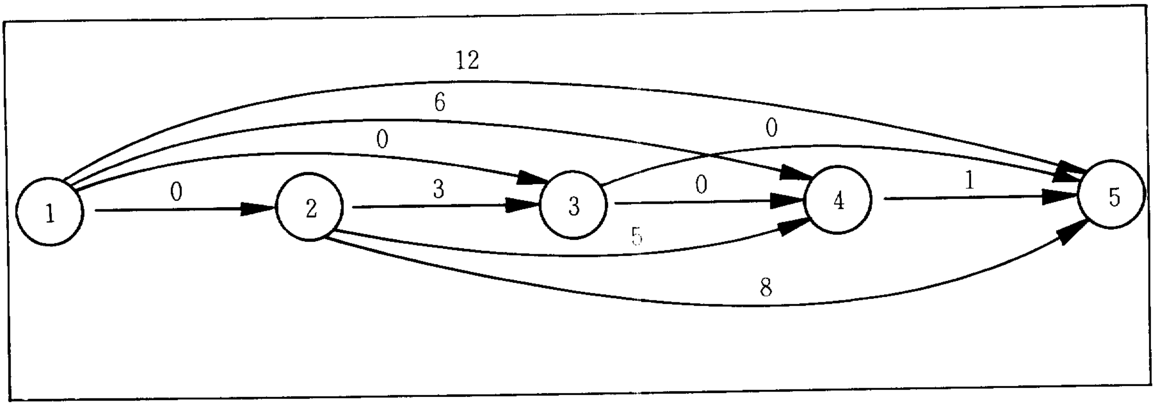

Figure 2. The Slack Network $\overline{G}$

5. Conclusion

In this paper, we conducted a parametric study of the shortest path between two specified nodes with respect to the simultanoues changes in arc lengths, while the traditional study has mainly focused on the problem of updating shortest paths or calculating arc tolerances. So-called invariance conditions were derived for each case of single arc length, node parameter, time parameter, and stage parameter. Although the whole procedure for finding the sensitivity range of each parameter is relatively simple as demonstrated in Section 4, we must admit that the simultaneous changes in arc lengths should be captured into a proper functional form for efficient implementation. Functional forms of Δ_{ij} as such can be found in important network applications, for instance, production planning problems [9] and project sequencing problems [10]. We hope many more applications would be found in near future.

Appendix

The following lists a FORTRAN program for calculating the values of $\bar{f}_n'(s,t)$ and $\bar{f}_n''(s)$, $1 \le s < t \le n$, given the values of $\bar{c}_{ij}$ and $\bar{g}_j$. In the program, variables F1(S, T) and F2(S) represent $\bar{f}_n'(s,t)$ and $\bar{f}_n''(s)$, respectively. Function XMIN(S) returns the value of $\min_{i<s<j}\{\bar{c}_{ij}+\bar{g}_j\}$, while suroutine YMIN(S, Y1, Y2, JY) returns the minimum(Y1) and second minimum(Y2) over $\bar{c}_{sj}+\bar{g}_j$, $j=s+1,\cdots,n$, and index JY with the value of Y1. The whole program runs in $O(1/6n^3)$ time.

```
      PROGRAM INVARIANCE
      INTEGER S, T
      COMMON N, C(10, 10), G(10)
      DIMENSION F2(10), F1(10, 10)
      OPEN(7, FILE='SHORT.DAT')
      OPEN(8, FILE='SHORT.OUT')
      READ(7,700) N
      DO 10 i=1, N-1
      READ(7,710)(C(I,J),J=I+1,N)
10    CONTINUE
      READ(7,710) (G(J), J=1,N)
      DO 20 S=1, N
      F2(S)=XMIN(S)
      WRITE(8,800) S,F2(S)
      CALL YMIN(S,Y1,Y2,JY)
      DO 20 T=S+1,N
      IF(T.EQ.JY) THEN
        Y=Y2
      ELSE
        Y=Y1
      ENDIF
      F1(S,T)=MIN(F2(S), Y)
      WRITE(8,810) S,T,F1(S,T)
20    CONTINUE
      STOP
700   FORMAT(I10)
710   FORMAT(5F10.0)
800   FORMAT(I5,5X,E20.6)
810   FORMAT(2I5,E20.6)
      END
C
      FUNCTION XMIN(S)
      INTEGER S
      COMMON N, C(10,10), G(10)
      XMIN=1.0E+37
      DO 10 I=1, S-1
      DO 10 I=S+1, N
      X=C(I,J)+G(J)
      IF(XMIN.GT.X) XMTN=X
      IF(XMIN.EG.0.0) RETURN
10    CONTINUE
      RETURN
      END
C
      SUBROUTINE YMIN(S,Y1,Y2,JY)
      INTRGER S
      COMMON N,C(10,10), G(10)
      Y1=1.OE+37
      Y2=1.OE+37
      JY=S
      DO 10 J=S+1,N
      X=C(S,J)+G(J)
      IF(Y1.GT.X) THEN
         Y2=Y1
         Y1=X
         JY=J
      ELSEIF(Y2.GT.X) THEN
         Y2=X
10    ENDIF
      CONTINUE
      RETURN
      END
```

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