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STRUCTURAL CHANGES IN DYNAMIC LINEAR MODEL

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BIOGRAPHY

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Abstract

The repeated occurrences of the structural changes in forecasting are formulated in a form of dynamic linear model by inclusion of a dummy variable representing the changes. The general statistics to detect such changes and estimate the size of the random change are derived. Assuming known that a structural change has occurred during a given a period, the posterior distribution of the change-point is also derived. The results are simplified for detecting and estimating the abrupt level changes frequently met in application of exponential smoothing. The derived statistic becomes a simple funtion of the one-step-ahead forecast errors obtained under assumption of no change, discounted by the same factor used in the exponential smoothing.

Gardner derived an approximate test statistic to detect mean changes in the constant mean model. When the present results are applied to this model, they give the exact statistic.

KEYWORDS

Structural change, Change detection, Change-point, Dynamic linear model, Kalman filter, Exponential Smoothing

INTRODUCTION

Any fixed forms of the forecasting models can hardly stand for variable situations since interventions of structural changes frequently occur by reasons. To meet such problems, Chernoff and Zacks(1964) and Gardner(1969) studied estimation and detection methods for the mean changes in the constant mean model, while Quandt(1960), Ferreira(1975), Broemeling(1985) and others have studied the two-phase regression model for the parameter changes in the regression model. Harrison and Stevens(1976) suggested the multi-process model to consider the structural changes in the dynamic linear model(DLM). Jun and Oliver (1985) and Jun(1989) introduced a dummy variable to the simple DLM for estimation and detecting a major level or a slope change. As a result, detection and estimation methods for a level or a sloppe change in application of exponential smoothing were also presented. Recently West and Harrison(1989), Young and Ng(1989) presented the methods to meet the structural changes in the DLM by relfecting forecasters' opinions on occurrences of the changes at every time point.

In this paper, random structural changes frequently occurred in forecasting are formulated in a form of DLM including such changes as a dummy variable. The statistics for estimating the size and the change-point of the structural change and detecting the changes are derived. The results are simplified for the level changes in the exponential smoothing and the mean changes in the constant mean model.

DYNAMIC LINEAR MODEL WITH RANDOM STRUCTURAL CHANGES

The effect of structural changes is studied, where the measurement equation and the underying processes are the dynamic linear model of Harrison and Stevens(1976), altered by inclusion of a dummy variable representing the structural change. Thus the euqations of motion for the random observation vector, Z_t, the state vector, S_t, and random noise terms, E_t, F_t, with known A, B, and C matrices are given by

$$Z_{t+1}=C\ S_{t+1}+E_{t+1}, \qquad t=0, 1, \cdots, n-1 \qquad (1a)$$

$$S_{t+1}=A\ S_t+\delta_t\ B\Delta+F_{t+1}, \qquad t=0, 1, \cdots, n-1 \qquad (1b)$$

Here we assume the random structural change, Δ, interrupts the system repeatedly, so that the effects are reflected in the system trough the linear relationship by the matrix of B. Let $\delta_t=1$ or 0 according to whether there is or is not a structural change between Z_t and Z_{t+1}. $\{E_t\}$ and $\{F_t\}$ are assumed to be serially uncorrelated gaussian noise inputs with mean 0 and known vriance–convariance matrices, $\sum_E$, $\sum_F$, respectively. To conduct the analysis in a Bayesian framework, it will be also assumed that the random variables S_t and Δ have the following prior distributions at time 0:

$$S_0 \sim N(\hat{S}_{0/0}, \hat{\sum}_{0/0});\Delta \sim N(\hat{\Delta}_05 , \hat{Q}_0), \qquad (2)$$

where S_0, Δ, $\{E_t\}$ and $\{F_t\}$ are also assumed mutually uncorrelated.

No Structural Change

When $\delta_t=0$ for all t in the above model, Kalman(1960) show the recursions for the posterior and prior mean and vraiance–covariance of the state:

$$S_t/Z_t, \cdots, Z_1 \sim N(S_{t/t}, \sum_{t/t});\ S_{t+1}/Z_t, \cdots, Z_1 \sim N(S_{t+1/t}, \sum_{t+1/t}) \qquad (3)$$

Thus the one–step–ahead forecast f_t made at time t(i.e. the conditional expectation of Z_{t+1} observing $\{Z_t, \cdots, Z_1\}$), the forecast error and the corresponding conditional variance–convariance ν_t are given by

$$f_t=C\hat{S}_{t+1/t};e_t=Z_{t+1}-f_t;\nu_t=C\hat{\sum}_{t/1/t}C'+\sum_E \qquad (4)$$

Known Change–Point(i.e. δ_t known)

Sometimes we might realize or be convinced that any structural changes have occurred at known times. In such a situation, the size of the structural changes can be estimated as follows:

$$\Delta / Z_n, \cdots, Z_1; \delta_{n-1}, \cdots, \delta_1 \sim N(\Lambda_n, Q_n) \tag{5a}$$

$$\hat{\Lambda}_n = \hat{Q}_n [\Sigma_{t=0}^{n-1} \{ \Sigma_{m=1}^{n-1} (\delta_m r'_{t/m}) C \nu_t^{-1} e_t \} + \hat{Q}_0^{-1} \hat{\Lambda}_0] \tag{5b}$$

$$\hat{Q}_n = [\Sigma_{t=0}^{n-1} \{ (\Sigma_{m=1}^{n-1} \delta_m r'_{t/m} C') \nu_t^{-1} (\Sigma_{m=1}^{n-1} \delta_m C r_{t/m}) \} + \hat{Q}_0^{-1}]^{-1} \tag{5c}$$

$$k_i = (A \hat{\Sigma}_{i/i-1} C')(C \Sigma_{i/i-1} C' + \hat{\Sigma}_E)^{-1} \tag{5d}$$

$$r_{t/m} = B \Pi_{i=m+1}^{t} (A - k_i C), \qquad t = m+1, \cdots, n-1 \tag{5e}$$

$$= B, \qquad t = m$$

$$= 0, \qquad t = 0, \cdots, m-1$$

where $\{e_t\}$ and k_i are one-step-ahead forecast errors and the Kalman gain obtained under the assumption of no structural change. The results are obtained using the Kalman filter and the multivariate normal analysis, where the one-step-ahead forecast of Z_{t+1} at time t assuming known structural changes becomes

$$Z_{t+1} / Z_t, \cdots, Z_1; \delta_t, \cdots, \delta_1; \Delta \sim N(f_t + \Sigma_{m=1}^{t} \delta_m C r_{t/m} \Delta; \nu_t). \tag{6}$$

Detection of Structural Changes

Sometimes we can be convinced in occurrences of structural changes, on the other hand, in most cases it is hard to assert whether such changes have occured. In the latter case, we might want to test if such changes have occurred or not during a given hypothetical time period. Thus the hypotheses come into $H_0: \Delta = 0$ at all times v.s. $H_1: \Delta \neq 0$ for a specified sequence $\delta = (\delta_1, \cdots, \delta_{n-1})$ of change times. A test of the hypothesis of structural change at no point against a set of alternatives $\{\delta\}$ having assigned nonzero prior probabilities $Pr[\delta]$ rejects the hypothesis for large values of

$$\Sigma_{\{\delta\}} Pr[\delta] \mid Q_n \mid^{0.5} \exp(0.5 \hat{\Lambda}_n' \ Q_n^{-1} \ \hat{\Lambda}_n) \tag{7a}$$

where $\hat{\Lambda}_n$ is the only data dependent term and $\hat{\Lambda}_n$ and $\hat{Q}_n$ given from the previous equations.

As in Gardner, if we would rather use the weighted sum of the log-likelihood ratios of the two hypotheses $\Delta \neq 0$ to $\Delta = 0$ at each specified sequence of change times by a prior distribution of δ than the exact posterior probability ratio of the hypotheses for the purpose of getting rid of the nuisance parameters, the change detection statistic becomes proportional to

$$\Sigma_{\{\delta\}}\Pr[\delta]\hat{\Lambda}_n{}'\hat{Q}_n{}^{-1}\hat{\Lambda}_n \tag{7b}$$

Detection of Change-Point When A Structural Change Occurs

Sometimes we might be convinced of occurrence of a structural change but we might not know when such a change have occurred. In such a case, we might consider the posterior probability distribution of the change-point, which depends only upon

$$\Sigma_{i=1}^{n-1}\Pr(M=i)\mid\hat{Q}_n\mid^{0.5}\exp(0.5\hat{\Lambda}_n{}'\ \hat{Q}_n{}^{-1}\ \hat{\Lambda}_n) \tag{8}$$

where $\Pr(M)$ is a given prior distribution of the change-point and Λ_n and Q_n are simpified from the previous equations as follows:

$$\hat{\Lambda}_n=Q_n\{\Sigma_{t=0}^{n-1}(r_{t/M}{}'C'\nu^{-1}{}_te_t)+\hat{Q}_0{}^{-1}\hat{\Lambda}_0\} \tag{9a}$$

$$\hat{Q}_n=[\sum_{t=0}^{n-1}(r_{t/M}{}'C'\nu_t{}^{-1}Cr_{t/M})+\hat{Q}_0{}^{-1}]^{-1}. \tag{9b}$$

LEVEL CHANGES IN EXPONHENTIAL SMOOTHING

If we apply the long-run equilibrium variance of the state to the updating equations for the posterior estimates of the state in the simple dynamic linear model i.e. Z_t is a scalar and $A=B=C=1$ and $\Delta=0$ in equation (1), the following updating equation gives optimal forecasts to simple dynamaic linear model, in the sense of minimum mean square errors:

$$f_t=f_{t-1}+(1-\alpha)e_{t-1}, \qquad t=1,\cdots,n \tag{10}$$

where $\alpha=\frac{\Sigma_E}{\nu^*}$ and $\nu^*=\lim_{t\to\infty}\nu_t$.

Thus the one-step-ahead forecast,

$$f_t=(1-\alpha)\Sigma_{k=0}^{\infty}\alpha^kZ_{t-k} \tag{11}$$

geometrically discounts the random variable by its age.

Once major level changes occur while we apply exponential smoothing to short-term forecasting, then we can estimate the size of the change by substituting the posterior state variance by the equilibrium state variance in the previous results as follows:

$$\hat{Q}_n=[1/\hat{Q}_0+1/\nu^*\{\delta_1{}^2+(\delta_1\alpha+\delta_2)^2+\cdots+(\delta_1\alpha^{n-2}+\delta_2\alpha^{n-3}+\cdots+\delta_{n-1})^2\}]^{-1} \tag{12a}$$

$$\hat{\Lambda}_n=\hat{Q}_n[\hat{\Lambda}_0/\hat{Q}_0+1/\nu^*\{\delta_1e_1+(\delta_1\alpha+\delta_2)e_2+\cdots+(\delta_1\alpha^{n-2}+\delta_2\alpha^{n-3}+\cdots+\delta_{n-1})e_{n-1}\}] \tag{12b}$$

where $\{e_t\}$ are the one-step-ahead forecast errors obtained under the assumption of no change. Here note that the posterior level estimate after observing n time series is obtained as a function of the forecast errors discounted by the same factor used in the exponential smoothing. That is, $\hat{\Lambda}_n$ becomes the function of $e_1+\alpha e_2+, \cdots, \alpha^{n-2}e_{n-1}$, if the only the one major level change occurs at time 1, that is, Z_2 is the first observation including a major level change.

To detect the changes, we might use the statistic in either equation (7a) or (7b). If we have no prior information of abrupt level changes i.e. $\tilde{Q}_0 \rightarrow \infty$, the dominant term of the change detection statistics, $\hat{\Lambda}_n^2/\hat{Q}_n$, becomes proportional to

$$\frac{[\{\delta_1 e_1+(\delta_1\alpha+\delta_2)e_2+, \cdots, +(\delta_1\alpha^{n-2}+\delta_2\alpha^{n-3}+, \cdots, \delta_{n-1})e_{n-1}\}]^2}{\{\delta^2_1+(\delta_1\alpha+\delta_2)^2+, \cdots, +(\delta_1\alpha^{n-2}+\delta_2\alpha^{n-3}+, \cdots, \delta_{n-1})^2\}} \tag{13}$$

The specific results for detecting a major level or slope change in general exponential smoothing and estimating the size of the change and the change-point are presented with examples in Jun(1989).

MEAN CHANGES IN THE CONSTANT MEAN MODEL

Gradner derives the change-detection statistic approximately as $\Sigma_{0/0} \rightarrow \infty$ and $Q_0 \rightarrow \infty$ for detecting the mean chnages in the constant mean model, which can be obtained exactly from equation (7b). However, in practics, we often meet a situation with no prior knowledge of mean and mean change. In that case i.e. as $\hat{\Sigma}_{0/0} \rightarrow \infty$ and $\hat{Q}_0 \rightarrow \infty$, the term of $\hat{\Lambda}^2_n/\hat{Q}_n$ in the change detection statistic of equation (7b) becomes

$$\{\Sigma_{m=1}^{n-1}\delta_m\Sigma_{t=m}^{n-1}(Z_{t+1}-\bar{Z}_n)\}^2/[\Sigma_E\Sigma_{t=1}^{n-1}\{(\Sigma_{m=1}^{t}\delta_m m)^2/(t(t+1))\}] \tag{14}$$

where $\bar{Z}_n$ is the arithmetic mean of the first n observations. The result is equivalent to Gardner's except the denominator.

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