論 文
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# 비선형 시스템의 근사 선형화

# Approximate Linearization of Nonlinear Systems

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*Abstract* - The ability to linearize a nonlinear system by feedback and coordinate change reduces to finding an integrating factor for a one-form which is determined from the system dynamics. Utilizing Taylor series expansion of this one-form, we characterize approximate linearizability. A constructive method is derived for approximate linearization up to order 2.

## 1 Introduction

Linearization of nonlinear systems by coordinate transformation and state feedback has been one of the most active research topics in recent years. Su [12] and Jakubczyk and Respondek [6] characterized feedback linearizability, i.e., the ability to linearize a system by a nonlinear state feedback and coordinate change, in terms of the involutiveness of vector fields, while Hermann [4] and Gardner [2] studied the dual characterizations in terms of differential forms. The linearizability of nonlinear discrete-time systems has also been studied extensively [3,8,11]. Feedback linearization offers a method of building a controller for the nonlinear system by designing one for the equivalent linear system and utilizing the transformation (from linear to nonlinear) along with its inverse. These linearization techniques, being robust, tolerate some deviation from perfect linearization [1]. This approach has already been applied to the design of automatic flight-control systems for aircraft [10], motor controller design [5], etc.

On the other hand, since less restrictive conditions are required for approximate linearization, this technique offers the means of enlarging the class of nonlinear systems to which linearizing techniques are applicable. It was shown in [7] that approximate linearization could be obtained by weakening the hypotheses required for feedback linearization. H. Lee and S. I. Marcus [9] obtained conditions for the approximate linearization of discrete-time systems through the expansion of higher order derivative terms into planar matrices.

In this work, we first show that feedback linearizability is equivalent to the existence of an integrating factor for a one-form which is determined from the system dynamics. Utilizing Taylor series expansion of this one-form, a characterization of approximate linearizability is obtained. We conclude with a complete analysis of approximate lin-

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earization up to order 2.

## 2 Preliminary Remarks and Definitions

Consider a single-input, single-output system

$$\dot{x} = f(x) + ug(x), \quad x \in M, \tag{1}$$

where $M$ is a smooth $n$-dimensional manifold and $f$, $g$ are smooth vector fields on $M$. Since the questions addressed in this work are local in nature, we identify $M$ with an open neighborhood of the origin in $\mathbf{R}^n$. We also assume that $f$ has an equilibrium (or fixed) point at $x = 0$. It is well known that (1) is feedback linearizable if and only if the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ are linearly independent, and $sp\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is an involutive distribution. The involutiveness of $sp\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is equivalent to the existence of a nonzero scalar function $h : M \to \mathbf{R}$ satisfying $< dh, ad_f^i g >= 0$, $i = 0, 1, \ldots, n-2$. Hence, feedback linearizability of the system in (1) reduces to the existence of a function $h : M \to \mathbf{R}$ such that

$$\begin{cases} < dh, ad_f^i g >= 0, & i = 0, 1, \ldots, n-2, \\ < dh, ad_f^{n-1} g > \neq 0. \end{cases} \tag{2}$$

If such a function $h$ is available, one may directly obtain a linearizing feedback $u$ and coordinate transformation map $T$. Specifically, if we let

$$u(t) \equiv (v(t) - L_f^n h)/L_g L_f^{n-1} h, \tag{3}$$

$$T(x) \equiv \left[h \ L_f h \ \ldots \ L_f^{n-1} h\right]^T \tag{4}$$

the system is transformed into a linear system $\dot{\xi} = A\xi + bv$, where $\xi = T(x)$, $v$ is the new input coordinate, and $(A, b)$ is a Brunovsky controllable pair. Obtaining a function $h$ satisfying (2), is not an easy task; in general, in order to do so, one needs to solve a set of partial differential equations. The following classical theorem is basic to our discussion.

**Theorem 1.** Let $M$ be an $n$-dimensional manifold, $TM$ the tangent bundle on $M$, and $\omega_1, \ldots, \omega_n \in T^*M$ be a basis of the cotangent bundle $T^*M$. Let $E \subset TM$ be a subbundle with $k$-dimensional fiber and $\mathbf{I}(E)$ be the associated ideal. Then, the following are equivalent:

i) $E$ is integrable.

ii) $E$ is involutive.

iii) $\mathbf{I}(E)$ is a differential ideal locally generated by $n - k$ linearly independent one-forms $\omega_1, \ldots, \omega_{n-k} \in T^*M$.

iv) $d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k} = 0$, for $1 \leq i \leq n$.

We will need the following definitions. Let $x = \{x_1, \ldots, x_n\}$ be coordinate functions in $\mathbf{R}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, i.e., a $n-$tuple of non-negative integers. The monomial $x^\alpha$ and the differential operator $D^\alpha$ are defined by

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \qquad \text{where } D_j = \frac{\partial}{\partial x_j}.$$

We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \ldots \alpha_n!$.

With these definitions, for a smooth function $\phi : \mathbf{R}^n \to \mathbf{R}$, Taylor's expansion formula takes the form

$$\phi(x) = \phi(0) + \sum_{|\alpha|=1}^{m} \frac{1}{\alpha!} D^\alpha \phi(0)\, x^\alpha + R_{m+1}(x), \tag{5}$$

where $R_{m+1}(x)$ is the remainder term and the summation is over all multi-indices $\alpha$.

**Definition:** System (1) is feedback linearizable up to order $i$, if there exists a coordinate transformation map $\xi = T(x)$ and a feedback $u = \beta(x)v + \alpha(x)$ such that, in the new coordinates,

$$\dot{\xi} = (A\xi + \mathcal{O}^{i+1}(\xi)) + (b + \mathcal{O}^i(\xi))\, v,$$

where $\mathcal{O}^i(\xi)$ is the class of functions $f$ such that $\limsup_{\|\xi\| \to 0} \frac{\|f(\xi)\|}{\|\xi\|^i} < \infty$.

**Remark:** From the infinitesimal linear approximation of (1) around the equilibrium point, we obtain the system

$$\dot{x} = Fx + Gu,$$

where $F = \partial f / \partial x|_{x=0}$ and $G = g(0)$. Hence, this linear system approximates (1) up to order 1.

## 3 Approximate Linearization

We define the function matrix $U(x)$ by

$$U(x) = \left[ad_f^{n-1} g \mid \cdots \mid ad_f g \mid g\right] \tag{6}$$

and we assume that $U(x)$ is full-rank in an open neighborhood of 0. Further, we let

$$\omega(x) = [\omega_1(x) \ldots \omega_n(x)] = [1 \ 0 \ldots 0]\, U^{-1}(x). \tag{7}$$

Since $\omega(x)$ is defined to be the first row of the inverse of the matrix $U(x)$, it follows that

$$\begin{cases} < \omega, ad_f^i g >= 0, & i = 0, 1, \ldots, n-2, \\ < \omega, ad_f^{n-1} g >= 1. \end{cases} \tag{8}$$

With a slight abuse of notation, we define a one-form $\omega$ by

$$\omega = \sum_{i=1}^{n} \omega_i dx_i. \tag{9}$$

Then, from (2) and (8) we observe that $dh$ is parallel to $\omega$ if the system is feedback linearizable. Therefore, if the system (1) is feedback linearizable, there must exist a scalar function, namely an integrating factor, $r : M \to R$ such that

$$dh = r\omega. \tag{10}$$

We say that $\omega$ is ***exact with the help of an integrating factor*** $r$ if the form $r\omega$ is exact. It follows from Theorem 1 that the necessary and sufficient condition for the exactness of the one-form $r\omega$ is $d(r\omega) = 0$ or $d(r\omega) \wedge (r\omega) = 0$. Hence, feedback linearizability is equivalent to the existence of an integrating factor $r : M \to R$ such that

$$\frac{\partial r\omega_i}{\partial x_j} = \frac{\partial r\omega_j}{\partial x_i}, \qquad 1 \le i < j \le n. \tag{11}$$

This conclusion is summarized in the following proposition.

**Proposition 1.** The system (1) is feedback linearizable if and only if there exists a function $r : M \to R$ such that (11) holds.

**Proof:** Necessity has already been discussed. It remains to prove sufficiency. If there exist $r$ such that (11) holds, there exist a potential function, namely $h$, of $r\omega$. Then with the input in (3) and the coordinate change (4), the system (1) is linearized. ■

Given an integrating factor $r$, one can obtain $h$ from (10) and, thus, the linearizing feedback (3) and the coordinate transformation map (4) can be easily constructed. Hence, the problem of obtaining the desired coordinate change and feedback reduces to that of obtaining an integrating factor. But, solving (11) for $r$ is a difficult problem. However, in the case of approximate linearization, the situation is quite different. We start our discussion of this problem with the analysis of approximate linearization up to order 2:

**Lemma 1.** The system (1) is feedback linearizable up to order 2 if and only if there exists a map $r : M \to R$ such that

$$\left.\frac{\partial r\omega_i}{\partial x_j}\right|_{x=0} = \left.\frac{\partial r\omega_j}{\partial x_i}\right|_{x=0}, \qquad i,j = 1,2,\ldots,n. \tag{12}$$

**Proof:** *(Sufficiency)* Let $\tilde{h} : M \to R$ be defined by

$$\tilde{h}(x) = \tilde{\omega}(0)x + \frac{1}{2}x^T \left.\frac{\partial \tilde{\omega}}{\partial x}\right|_{x=0} x,$$

where $\tilde{\omega}(x) = r(x)\omega(x)$. Then from (12), we obtain

$$d\tilde{h}(x) = \tilde{\omega}(0) + \left.\frac{\partial \tilde{\omega}}{\partial x}\right|_{x=0} x = \tilde{\omega}(x) + \mathcal{O}^2(x).$$

Therefore,

$$L_g\tilde{h} = <d\tilde{h}, g> = <\tilde{\omega}, g> + \mathcal{O}^2(x) = \mathcal{O}^2(x).$$

Since $f(0) = 0$, $L_f <d\tilde{h}, g> = \mathcal{O}^2(x)$. Thus,

$$L_gL_f\tilde{h} = L_f <d\tilde{h}, g> - <d\tilde{h}, ad_f g> = \mathcal{O}^2(x).$$

Similarly, $L_gL_f^i\tilde{h} = \mathcal{O}^2(x)$, for $i = 0,\ldots,n-2$. With $\xi = T(x) = \left[\tilde{h} \ \ldots \ L_f^{n-1}\tilde{h}\right]^T$ and $u = (v - L_f^n\tilde{h})/L_gL_f^{n-1}\tilde{h}$, we obtain the system

$$\begin{aligned} \dot{\xi} &= L_{f+ug}\xi = \begin{bmatrix} L_f\tilde{h} \\ \vdots \\ L_f^n\tilde{h} \end{bmatrix} + u \begin{bmatrix} L_g\tilde{h} \\ \vdots \\ L_gL_f^{n-1}\tilde{h} \end{bmatrix} \\ &= A\xi + bv + \mathcal{O}^2(x)u \\ &= A\xi + bv + \mathcal{O}^3(\xi) + \mathcal{O}^2(\xi)v, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

*(Necessity)* Any approximate linearizable system up to order 2 can be represented as $\dot{x} = (Ax + \mathcal{O}^3(x)) + (b + \mathcal{O}^2(x))v$. Then, $\omega(x) = dx_1 + \mathcal{O}^2(x)$. Hence, for $r(x) = 1$, (12) follows. ■

Lemma 1 may be extended, in a similar fashion, to an arbitrarily high order case.

**Proposition 2.** The system (1) is feedback linearizable up to order $m \ge 2$ if and only if there exists a function $r : M \to R$ such that

$$\left.\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}\frac{\partial r\omega_i}{\partial x_j}\right|_{x=0} = \left.\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}\frac{\partial r\omega_j}{\partial x_i}\right|_{x=0}, \tag{13}$$

for all $i,j = 1,2,\ldots,n$ and $|\alpha| = m-2$.

**Proof:** *(Sufficiency)* Let $\tilde{h} : M \to R$ be defined by

$$\tilde{h}(x) = \tilde{\omega}(0)\,x + \sum_{i=1}^{n}\sum_{|\alpha|=1}^{m-1} \frac{1}{(1+|\alpha|)\,\alpha!} D^\alpha \tilde{\omega}_i(0)\,x^\alpha x_i, \tag{14}$$

where $\tilde{\omega}(x) = r(x)\omega(x)$. For $i = 1,\ldots,n$, let $\delta_i$ denote the multi-index of length $n$ defined by $(\delta_i)_i = 1$ and $(\delta_i)_j = 0$, $i \neq j$. Then

$$\frac{\partial \tilde{h}(x)}{\partial x_k} = \tilde{\omega}_k(0) + \sum_{|\alpha|=1}^{m-1} \frac{\alpha_k + 1}{(1+|\alpha|)\,\alpha!} D^\alpha \tilde{\omega}_k(0)\, x^\alpha + \sum_{i\neq k} \sum_{\substack{|\alpha|=1 \\ \alpha_k>0}}^{m-1} \frac{\alpha_k}{(1+|\alpha|)\,\alpha!} D^\alpha \tilde{\omega}_i(0)\, x^{(\alpha+\delta_i-\delta_k)}. \quad (15)$$

Observe that, if $i \neq k$, then, for every multi-index $\alpha$, with $\alpha_k > 0$, there corresponds a multi-index $\tilde{\alpha} \equiv \alpha + \delta_i - \delta_k$, with $\tilde{\alpha}_i > 0$, and vice-versa. Then, by the hypothesis of the Proposition, if $\alpha_k > 0$,

$$D^\alpha \tilde{\omega}_i(0) = D^{\alpha+\delta_i-\delta_k} \tilde{\omega}_k(0) = D^{\tilde{\alpha}} \tilde{\omega}_k(0), \quad (16)$$

and, furthermore,

$$\frac{\alpha_k}{(1+|\alpha|)\,\alpha!} = \frac{\tilde{\alpha}_k + 1}{(1+|\tilde{\alpha}-\delta_i+\delta_k|)\,(\tilde{\alpha}-\delta_i+\delta_k)!} = \frac{\tilde{\alpha}_i}{(1+|\tilde{\alpha}|)\,\tilde{\alpha}!}\,. \quad (17)$$

Therefore, by (16) and (17) the second sum on the right hand side of (15) reduces to

$$\sum_{i\neq k} \sum_{\substack{|\alpha|=1 \\ \alpha_k>0}}^{m-1} \frac{\alpha_k}{(1+|\alpha|)\,\alpha!} D^\alpha \tilde{\omega}_i(0)\, x^{(\alpha+\delta_i-\delta_k)} = \sum_{i\neq k} \sum_{\substack{|\tilde{\alpha}|=1 \\ \tilde{\alpha}_i>0}}^{m-1} \frac{\tilde{\alpha}_i}{(1+|\tilde{\alpha}|)\,\tilde{\alpha}!} D^{\tilde{\alpha}} \tilde{\omega}_k(0)\, x^{\tilde{\alpha}}. \quad (18)$$

The second sum on the right hand side of (18) is over all multi-indices of order 1 to $m-1$ whose $i^{th}$ coordinate is positive. Thus, using (18) in (15), changing the variable $\tilde{\alpha}$ to $\alpha$, and combining the two summations in (15), we obtain

$$\frac{\partial \tilde{h}(x)}{\partial x_k} = \tilde{\omega}_k(0) + \sum_{|\alpha|=1}^{m-1} \frac{1}{\alpha!} D^\alpha \tilde{\omega}_k(0)\, x^\alpha. \quad (19)$$

Hence, by (19) and (5),

$$d\tilde{h}(x) = \tilde{\omega}(x) + \mathcal{O}^m(x).$$

Evaluating the Lie derivatives, we obtain,

$$L_g \tilde{h} = <d\tilde{h}, g> = <\tilde{\omega}, g> + \mathcal{O}^m(x),$$

$$L_g L_f \tilde{h} = L_f <d\tilde{h}, g> - <d\tilde{h}, ad_f g> = \mathcal{O}^m(x),$$

and proceeding, in a similar fashion, $L_g L_f^i \tilde{h} = \mathcal{O}^m(x)$ for $i = 0, \ldots, n-2$. Therefore,

$$\dot{\xi} = A\xi + bv + \mathcal{O}^m(x)u = A\xi + bv + \mathcal{O}^{m+1}(\xi) + \mathcal{O}^m(\xi)v.$$

*(Necessity)* Any approximate linearizable system up to order $m$ can be represented as

$$\dot{x} = Ax + bv + \mathcal{O}^{m+1}(x) + \mathcal{O}^m(x)v.$$

Then, $\omega(x) = dx_1 + \mathcal{O}^m(x)$. Hence, with $r(x) = 1$, (13) follows. ■

## 4 Solving for the Integrating Factor

Proposition 2 does not offer a constructive method of obtaining the integrating factor $r$. In this Section, necessary and sufficient conditions for the existence of an integrating factor $r$ are obtained for systems which are approximately linearizable up to order 2. Solving for $r$ in the general case is fairly complicated.

Without loss of generality, let

$$r(x) = 1 + \overline{r}_1 x_1 + \cdots + \overline{r}_n x_n + \mathcal{O}^2(x)$$

and

$$\omega(x) = \omega(0) + D_1\omega(0)\, x_1 + \cdots + D_n\omega(0)\, x_n + \mathcal{O}^2(x).$$

Simplifying the notation, let $\overline{\omega}_k = \omega_k(0)$ and $D_i\overline{\omega}_k = D_i\omega_k(0)$, $i, k = 1, \ldots, n$. Then, equation (12) reduces to

$$\overline{r}_i \overline{\omega}_j - \overline{r}_j \overline{\omega}_i = D_j\overline{\omega}_i - D_i\overline{\omega}_j, \qquad 1 \leq i, j \leq n. \quad (20)$$

Representing (20) in matrix form, we obtain

$$\begin{bmatrix} \overline{\omega}_2 & -\overline{\omega}_1 & 0 & 0 & \cdots & 0 \\ \overline{\omega}_3 & 0 & -\overline{\omega}_1 & 0 & \cdots & 0 \\ 0 & \overline{\omega}_3 & -\overline{\omega}_2 & 0 & \cdots & 0 \\ \overline{\omega}_4 & 0 & 0 & -\overline{\omega}_1 & \cdots & 0 \\ 0 & \overline{\omega}_4 & 0 & -\overline{\omega}_2 & \cdots & 0 \\ 0 & 0 & \overline{\omega}_4 & -\overline{\omega}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\omega}_n & 0 & 0 & 0 & \cdots & -\overline{\omega}_1 \\ 0 & \overline{\omega}_n & 0 & 0 & \cdots & -\overline{\omega}_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\overline{\omega}_{n-1} \end{bmatrix} \begin{bmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \vdots \\ \overline{r}_{n-1} \\ \overline{r}_n \end{bmatrix}$$

$$= \begin{bmatrix} D_2\overline{\omega}_1 - D_1\overline{\omega}_2 \\ D_3\overline{\omega}_1 - D_1\overline{\omega}_3 \\ D_3\overline{\omega}_2 - D_2\overline{\omega}_3 \\ D_4\overline{\omega}_1 - D_1\overline{\omega}_4 \\ D_4\overline{\omega}_2 - D_2\overline{\omega}_4 \\ D_4\overline{\omega}_3 - D_3\overline{\omega}_4 \\ \vdots \\ D_n\overline{\omega}_1 - D_1\overline{\omega}_n \\ D_n\overline{\omega}_2 - D_2\overline{\omega}_n \\ \vdots \\ D_n\overline{\omega}_{n-1} - D_{n-1}\overline{\omega}_n \end{bmatrix} \quad (21)$$

Let $\Omega_n = [\overline{\omega}_1 \dots \overline{\omega}_n]^T$ and $I_n$ denote the identity matrix of dimension $n$. For each $n = 2, 3, \dots$, we define a matrix $H_n$ of dimension $\binom{n}{2} \times n$ as follows: For $n = 2$, let $H_2 = [\overline{\omega}_2 \;\; -\overline{\omega}_1]$ and inductively for $n \geq 3$ by

$$H_n = \left( \begin{array}{c|c} H_{n-1} & 0 \\ \hline \overline{\omega}_n I_{n-1} & -\Omega_{n-1} \end{array} \right).$$

Also let $R_n = [\overline{r}_1 \dots \overline{r}_n]^T$ and $\Omega'_n$ denote the vector on the right hand side of (21). Then, equation (21) takes the form $H_n R_n = \Omega'_n$. Observe that (21) is invariant under permutations of the subscripts $\{1, \dots, n\}$. Therefore, since $\Omega_n \neq 0$, by (7), we may assume, without loss of generality for the arguments that follow, that $\overline{\omega}_n \neq 0$. Then, since $\overline{\omega}_n I_{n-1}$ is a minor of $H_n$, the rank of $H_n$ is at least $n-1$. On the other hand, $H_n \Omega_n = 0$ and it follows that $rank\,(H_n) = n-1$. The matrix $K_n = \begin{pmatrix} \overline{\omega}_n I_m \\ -H^T_{n-1} \end{pmatrix}$, where $m = \binom{n-1}{2}$, has rank $\binom{n-1}{2}$ and satisfies $K^T_n H_n = 0$. Thus, the span of $K_n$ is precisely the kernel of $H^T_n$. For (21) to have a solution in $R_n$ it is necessary and sufficient that $\Omega'_n$ be orthogonal to the kernel of $H^T_n$ or, equivalently, $K^T_n \Omega'_n = 0$. When $n = 3$, this reduces to the requirement that

$$\overline{\omega}_3(D_2\overline{\omega}_1 - D_1\overline{\omega}_2) + \overline{\omega}_2(D_1\overline{\omega}_3 - D_3\overline{\omega}_1) + \overline{\omega}_1(D_3\overline{\omega}_2 - D_2\overline{\omega}_3) = 0.$$

For arbitrary $n$, we have the following Proposition.

**Proposition 3.** Suppose that the dimension of the system is $n$ and let $k$ be any integer, $1 \leq k \leq n$, such that $\overline{\omega}_k \neq 0$. The system is feedback linearizable up to order 2 if and only if, for all $i < j \in \{1, \dots, n\}$, $i, j \neq k$,

$$\overline{\omega}_k(D_j\overline{\omega}_i - D_i\overline{\omega}_j) + \overline{\omega}_j(D_i\overline{\omega}_k - D_k\overline{\omega}_i) + \overline{\omega}_i(D_k\overline{\omega}_j - D_j\overline{\omega}_k) = 0. \quad (22)$$

**Proof:** If $k = n$, then a straightforward computation shows that the $\binom{n-1}{2}$ distinct equations in (22) are identical with the ones obtained from $K^T_n \Omega'_n = 0$. Otherwise, the same conclusion follows by observing that the system of equations (21) is invariant under permutations of the subscripts $\{1, \dots, n\}$. The rest follows from the previous discussion. ■

**Example:** Consider a nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \sin x_2 + x_2 x_4 \\ -x_3 + x_2^2 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u. \quad (23)$$

Then,

$$U(x) = \left[ ad^3_f g \mid ad^2_f g \mid ad_f g \mid g \right] = \begin{bmatrix} {2x_2(x_3 - x_2^2) \atop +2x_4 + \cos x_2} & (x_3 - x_2^2) & -x_2 & 0 \\ 2x_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $< ad_f g, ad^2_f g >= [-2 \;\; 0 \;\; 0 \;\; 0]^T$ does not lie in $sp\,\{g, ad_f g, ad^2_f g\}$, this distribution is not involutive. Therefore, system (23) is not feedback linearizable. In the following we establish that the system is feedback linearizable up to order 2. We have,

$$\omega(x) = [1 \;\; 0 \dots 0]\, U^{-1}(x) = \frac{1}{\det U(x)}[-1 \;\; x_3 - x_2^2 \;\; x_2 \;\; 0],$$

where $\det U(x) = 4x_2(x_3 - x_2^2) + 2x_4 + \cos x_2$. Since $\overline{\omega}_1 = -1 \neq 0$, it follows that the system is feedback linearizable up to order 2 if and only if for $(i, j) = (2,3), (2,4), (3,4)$,

$$\overline{\omega}_1(D_j\overline{\omega}_i - D_i\overline{\omega}_j) + \overline{\omega}_j(D_i\overline{\omega}_1 - D_1\overline{\omega}_i) + \overline{\omega}_i(D_1\overline{\omega}_j - D_j\overline{\omega}_1) = 0. \quad (24)$$

However, since the only nonzero derivative terms are $D_3\overline{\omega}_2 = D_2\overline{\omega}_3 = 1$ and $D_4\overline{\omega}_1 = 2$, condition (24) is satisfied. Solving (21), we obtain

$$[r_1 \;\; r_2 \;\; r_3 \;\; r_4] = [0 \;\; 0 \;\; 0 \;\; -2],$$

or, equivalently, an integrating factor $r(x) = 1 - 2x_4$. Finally, with

$$\tilde{\omega}(x) = \frac{(1 - 2x_4)}{\det U(x)}[-1 \;\; x_3 - x_2^2 \;\; x_2 \;\; 0],$$

and $\tilde{h}(x)$ as defined in (14), we can linearize the system up to order 2.

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