An Incomplete Information Structure and An Intertemporal General Equilibrium Model of Asset Pricing With Taxes

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〈요 약〉

一般均衡下의 資本資産의 價格決定

관측가능 확률과정, 관찰가능변수를 통한 확률과정의 형성과 조세를 중심으로

이 논문은 연속시간의 틀 속에서 재화시장의 수요 및 소비와 생산부문과 자본시장의 수요와 공급을 국민경제에 도입한 一般均衡의 經濟分析方法에 의하여 資本資産의 價格을 決定하는 一般模型을 제시한다. 이 모형에서는 특히 자본자산의 가격결정에 租稅가 미치는 영향을 심도있게 분석한다. 이 논문에서는 생산과 소비 그리고 자본자산의 수요와 공급 등을 결정하는 변수들이 確率過程을 따르는데, 이 변수들을 직접 관찰할 수 있는 경우에 형성되는 資本資産의 價格決定模型을 정립한다. 그리고 확률과정의 변수를 직접 관찰할 수 없고 간접적으로 관찰할 수 있을 때에는 간접관찰이 가능한 변수와 확률과정의 변수와의 관계를 정립한 확률과정을 형성하여 資本資産의 價格決定模型을 정립한다. 이 모형에는 자산의 가격과 확률적 성질이 모형내에서 결정된다. 이 모형은 證券의 價格決定, 利子率決定, 利子率의 期間構造分析, 利子率의 危險構造分析, 先物價格의 決定 등 다양하게 이용될 수 있다.

Abstract

This paper develops an intertemporal general equilibrium model of asset pricing with taxes under the noisy and the incomplete information structure and examines theoretically the stochastic behavior of general equilibrium asset prices in a one-good, production, and exchange economy in continuous time markets. The important features of the model are its integration of real and financial markets and the analysis

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of the effects of differential tax rates between ordinary income and capital gains. The model developed here can provide answers to a wide variety of questions about stochastic structure of asset prices and the effect of tax on them.

KEYWORDS : Noisy information, incomplete information, innovation process, contingent claim, Itô process, capital gains tax, ordinary income tax.

I. Introduction

This paper is a theoretical examination of the stochastic behavior of general equilibrium asset prices in a one-good, production, and exchange economy in the presence of personal taxes in continuous-time markets. Specifically, this paper develops an intertemporal general equilibrium model of asset pricing with taxes under the noisy and the incomplete or partial information structure. The important features of the model are its integration of real and financial markets and the analysis of the effects of differential personal tax rates between ordinary income and capital gains income realized from changes in asset prices. Another important feature is that the formulation of the perception of uncertainty is regarded as the framework of two types of information structure : noisy and incomplete.

The noisy information structure is the perfect knowledge of the state variables disturbed by some noise, which can be represented as a Wiener process random variable. On the other hand, the incomplete or partial information structure implies that the underlying state variables are unobservable, but that there exists an observable instrument that reveals partial information about the true state variables. In this case, the conditional process of the state variables, conditional on this instrument that the individuals continuously observe, gives them the information about the state variables that are continuously updated. Individuals can make consumption, production, and investment decisions instantly and continuously on the basis of this information. Thus the innovation process represents new information and hence is a sufficient statistic.

In this way we will devise a theory that is capable of addressing itself to general equilibrium questions such as (1) What is the impact of an increase in personal taxes upon the relative prices of riskless and risky assets ? (2) What is the impact of an increase in progressivity of the personal ordinary income and capital gains tax rates upon the relative price structure of risky assets ? (3) What is the relationship among the equilibrium expected rates of return on assets under incomplete or partial information when ordinary income and capital gains are taxed differently ? (4) What is the impact of incomplete information about the state variables upon the price structures of assets in the capital market ? (5) What is the impact of incomplete information upon the physical production processes ?

Cox, Ingersoll, and Ross(1985) developed a general equilibrium capital asset pricing model in a continuous-time framework by assuming perfect knowledge of the process of the state variables in the economy. They derived a relationship between asset prices and underlying variables in an economy. In their model, asset prices and their stochastic properties are determined endogenously. They did not, however, introduce the differential tax rates and the incomplete informational structure. The major difference between their paper and ours is that we will incorporate those two important aspects to develop an intertemporal general equilibrium model of asset prices.

In an intertemporal setting, Merton(1973) developed a relationship among the equilibrium expected rates of return on assets. He showed that if the investment opportunity set was constant, then the intertemporal model converged with the single-period capital asset pricing model, and that if the opportunity set was stochastic, then the intertemporal model would contain the effects that could not be captured by the static single-period model. Rhee(1987) derived the relationship among the equilibrium expected rates of return on assets in a continuous-time framework by assuming the presence of differential income and capital gains tax rates. His intertemporal capital asset pricing model with taxes was different from the static, single-period tax model under a constant investment opportunity set. He derived an intertemporal tax model in a stochastic investment opportunity set which captured the effect that could not be explained by a static, single-period model. He also showed the existence of the effects of taxes and dividends on the capital asset pricing. Lucas(1978) worked with a single-period, pure exchange economy with identical consumers. The martingale property determined the asset prices and provided a model to be used to test the efficient market hypothesis.

The paper proceeds as follows : In Section II the economy is described. Section III develops the endogenously determined equilibrium interest rate and characterized the equilibrium rate of return on assets both in the presence of differential tax rates and in the noisy information structure. A fundamental valuation partial differential equation for asset prices will be derived and interpreted in a number of ways. Furthermore, the stochastic structure of asset prices will be explored in detail. Section IV analyzes our economy both in the presence of differential tax rates and in the incomplete or partial information structure. The equilibrium interest rate and asset prices will be endogenously determined in terms of the underlying variables. A fundamental valuation equation for the asset prices will be derived in the uncertainty-incomplete information structure and the stochastic structure of asset prices will also be explored. Section V concludes the paper.

II. Description of the Economy

We consider the problem of individuals who wish to maximize their expected utility across time in the face of confronting the stochastic behavior of equilibrium asset prices in a one-good, production, and exchange economy with identical consumers. The single good in this economy is produced in a number of different productive units. Productivity in each unit fluctuates stochastically through time, so that equilibrium asset prices will fluctuate as well. Individuals can invest directly in physical production by creating their own firms. There are markets for a variety of contingent claims to amounts of goods. There are securities that are issued and purchased by individuals and firms to increase their wealth position. Both individuals and firms are competitive and act as price takers in all markets. Suppose that price and production processes are Itô processes. Taking these processes as given, an individual maximizes his lifetime expected utility. If an endogenously determined environment process and the price and production processes taken together form a diffusion process and if the environment process together with individual's wealth process constitutes a sufficient statistic for his optimal dynamic choice of consumption over time, then the Markovian stochastic dynamic programming technique can be used to characterize the individual's consumption-investment decision.

1. The Probabilistic Structure

Let (Ω, B, P) be a complete probability space that is specified and fixed. The sample space Ω has a finite number of elements. Each $\omega \in \Omega$ represents a complete description of the exogenous uncertain environment or the possible state of the world. We assume $P(\omega) > 0$ for all $\omega \in \Omega$. Consummer-investors are endowed with a common probability measure P on measure space (Ω, B). Thus a community of consumer-investors agree on which states of nature are possible and further agree on their probability assessments.

A time horizon $T < \infty$ is also specified and fixed, which is a terminal date for all economic activity under consideration. The filtration $\eta = \{B_t : 0 \leq t \leq T$ is the collection of distinguishable events and is specified exogenously. The information structure is a family of increasing sub-Borel fields of B. Each B_t is an algebra of subsets of Ω with $B_s \subseteq B_t$ for $s \leq t$, B_0 containing only Ω and the null set, and $B_T = B$, which is the set of all subsets.

Securities are traded at time t, $0 \leq t \leq T$ and the filtration η describes how information is revealed to the investors. Taken as primitive in our model are the n-dimensional production stochastic process and the m-dimentional stochastic process for contingent claims prices. Each component process is strictly positive and adapted to η. The adaption of the price to η implies that investors know at time t the past and current price of the security. Specifically, we assume that in Section III individuals and firms are endowed with a common probability measure on (Ω, B), while in Section IV they are endowed with partial or incomplete beliefs.[1)]

Let $y(t)$ be an *n-dimensional stochastic process that satisfies the system of stochastic defferential equation*

$$dy(t) = \alpha(y, t)dt + \sigma(y, t)dz(t), \tag{1}$$

where $\alpha(y, t)$ is an $(n \times 1)$ vector valued function ; $\alpha(y, t)$ is an $(n \times m)$ matrix valued function ; and $z(t)$ is an n-dimensional Wiener process.[2)] That is

$$\alpha(y, t) : R^n \times T \rightarrow R^n \text{ and } \sigma(y, t) : R^n \times T \rightarrow R^{n \times n}$$

are given functions, continuous in *y* and *t*, and satisfy the following conditions :

(i) (The growth condition) There exists a constant k_1, such that

$$|\alpha(y, t)|^2 \leq k_1^2(1+|y|^2) \text{ and } |\sigma(y, t)|^2 \leq k_1^2(1+|y|^2).$$

(ii) (The Lipschitz condition) There exsits a constant k_2 such that

$$|\alpha(y, t)-\alpha(\bar{y}, t)| \langle k_2 |y-\bar{y}| \text{ and } |\sigma(y, t)-\sigma(\bar{y}, t)| \langle k_2 |y-\bar{y}|.$$

A solution of the system (1) is an *n*-dimensional process satisfying the Itô integral equation

$$y(t)=y(0)+\int_0^t \alpha(y(s), s)ds+\int_0^t \sigma(y(s), s)dz(s) \quad \text{for all } t \in T \text{ a.s.} \quad (2)$$

where $y(0)$ is an *n*-dimensional vector of the initial position at time 0. Arnold(1974, Theorem 9.3.1, pp. 152-154) ensures that *y* in (2) is a unique solution of the integral equation (1) and that the solution $y(t)$ is an *n*-dimensional diffusion process with drift vector $\alpha(y, t)$ and diffusion(covariance) matrix $\sigma(y, t)\ \sigma'(y, t)$. The drift vector $\alpha(y, t)$ is the vector of expected rate of change(percentage change) of *y* and the diffusion matrix is the covariance matrix of rates of change(percentage change) of *y*. Individuals and firms in this economy have the information structure *B*. The exogenous uncertainty in the economy can be described by the Brownian motion or Wiener process $z(t)$. Agents in the economy can observe a vector process *y* whose evolution over time depends on *z*. In Section III it is assumed that agents can observe the Wiener process *z* directly while in Section IV they cannot have the direct observation of the process *z*.

In a probabilistic sense, our economy is governed by the Wiener process $z(t)$. Since the Wiener process is a homogeneous *n*-dimensional Markov process with stationary transition probability, stationary in the sense that the distribution of $z(t)-z(s)$ depends only $t-s$, sudden discontinuous changes in the variables in the economy are precluded.[3)]

2. The Consumption Space

There is a single physical good available for consumption or investment. Formally, we take the consumption space to be

$$\tilde{C}=L^2(\Omega\times T),$$

the space of equivalence classes of process : $\tilde{C}:(\Omega\times T)\rightarrow R$ such that $E\int_0^t |\tilde{C}(s)|\,ds<\infty$. All values are measured in terms of units of this single commodity.

3. Agents

There is a finite number of individuals in the economy indexed by $i\in\{1, 2, \ldots, I\}$, identical in their endowments and preferences. Each individual is characterized by a consumapation set C_+, and endowment vector $\tilde{C}\in C_+$. A representative individual wishes to maximize a von Neumann-Morgenstern time additive utility function $U: C_+\rightarrow R$:

$$E\int_t^T U(C(s),\ Q(s),\ s)ds \qquad \text{for all } C\in C_+, \tag{3}$$

where E is the conditional expectation operator, conditional on the current state ; C_+ is the subspace of $\tilde{C}$; Q denotes the state variables that will be defined shortly ; $U(C(s), Q(s), s): R_+\times R^m\times T\rightarrow R$ is continuous, strictly concave, strictly increasing, twice differentiable ; and U satisfies the condition $|U(C(s), Q(s), s)|\leq k'_2$. for some constant k'_1 and k'_2. We assume that $\tilde{C}>0$ for all $i\in\{1, 2, \ldots, I\}$.

4. Production Processes

The production opportunities consist of a set of n linear activities or technologies. The vector of the instantaneous return on the investment in production technologies follows a system of stochastic differential equations of the form

$$d\delta(t)=\alpha(Q,\ t)dt+H(Q,\ t)dz(t), \quad (4)$$

where $\alpha(Q, t)$ is a bounded n-dimensional vector valued function of Q and t, which is the vector of the expected rates of return on the production activities ; Q is an m-dinentional vector of state variables which is changing randomly over time and will be described in Subsection II.5 ; and $H(Q, t)$ is a bounded $(n \times m)$ matrix valued function of Q and t. The covariance matrix of physical rates of return on the production processess, HH', is positive definite ; and $z(t)$ is an n-dimensional Wiener process. The development of the state vector Q thus determines the production opportunities that will be available to the economy in the future. Equation (4), which is a complete description of the available production processess, specifies the growth of an initial investment when the output of each process is continually reinvested in the same process. Physical investment takes place continually in time with no transaction costs, and with limited liability. The production market is always in equilibrium.

5. The State Process

The dynamics for the changes in the state variables follow the m-dimensional vector Itô processes of a system of stochastic differential equations of the form

$$dQ(t)=\theta(Q,\ t)dt+S(Q,\ t)dz(t) \quad (5)$$

where $\theta(Q, t)$ is an m-dimensional vector of the instantaneous expected rates of change in the state variables : and $S(Q, t)$ is an $(m \times n)$ dimensional matrix. The covariance matrix of instantaneous changes in the state variables, SS', is nonnegative definite.

6. The Capital Market

There exits an exchange market for borrowing and lending at the same rate of interest. The market equilibrium risk-free rate is determined endogenously as part of the competitive equilibrium of the economy. The capital market is always in equilibrium.

The instantaneous movement of the value of the jth asset is governed by a system of the stochastic differential equations of the form

$$dF^j = F^j(\beta_j - \delta_j)dt + F^j g_j dz(t), \tag{6}$$

where $\beta_j F^j$ is the total expected return on the jth asset, i.e., the expected price change(capital gains) plus the payout received ; $\delta_j F^j$ is the payout received that may be dividends, interest, or the value of stock repurchase. The variance of the rate of return on asset j is $g_j g_j'$.

7. Taxes

There exist differential tax rates between capital gains income and ordinary income. Dividends and interest are treated as subject to income tax. In the presence of taxes, the return on investment must be studied on an after-tax basis. In addition, the before-tax rate of return may reflect investors' attitudes toward taxation of the returns on assets. Moreover, there are nonlinearities in tax schedules that are essential to understanding the relation between investors' behavior in the capital market and taxation.

III. The Equilibrium Valuation Model under a Noisy Information Structure

Consider an individual's allocation problem in the economy described in Section II Investors can allocate their wealth into three parts : consumption, the physical production processess and the contingent claims. It is assumed that in this Section individuals have knowledge of stochastic information and have homogeneous expectations about the probability distributions of the physical production processess, the state variables and the contingent claims. The noisy information structure to be used in this Section is expressed as a Wiener process. Investors can directly observe this process, which gives information about the evolution of the state variables over time.

1. Budget Equation Dynamics and the Equation of Optimality

There are I consumer-investors with preference structures described in Subsection II.3 : namely, the ith investor acts so as to maximize his expected utility that is equation (3). An individual will allocate his wealth among the n physical production processes, the m contingent claims, and the riskless borrowing or lending. Let a_j be the fraction of the wealth invested in the jth production process ; b_k be the fraction invested in the kth contingent claim ; and W is the amount of his wealth. Let γ be ordinary income tax rate and λ capital gains tax rate. The individual will choose a_j and b_k so as to maximize his lifetime utility subject to his budget constraints.

We can write the budget equation[4] for the ith investor as

$$dW=\Big[\sum_{j=1}^{n} a_j\ W(\alpha_j-r)(1-\gamma)+\sum_{j=1}^{K} b_jW(\beta_j-\delta_j-r)(1-\lambda)$$
$$+\sum_{j=1}^{K} b_jW(\delta_j-\gamma)(1-\gamma)+rW(1-\gamma)-C\Big]dt$$
$$+\sum_{j=1}^{n} a_jW\ h_jdz_j(1-\gamma)+\sum_{j=1}^{K} b_j\ W\ g_jdz_j(1-\lambda) \qquad (7)$$

It is assumed here that there exists a derived utility of wealth function, $J(W, Q, t)$, which is twice continuously differentiable in its arguments, strictly concave and strictly increasing.

The necessary and sufficient optimality conditions for an investor who acts according to (3) subject to (7), his budget equation, in choosing his consumption-investment program, are that for the derived utility of wealth function J, at each point in time,

$$0=\underset{\{C,a,b\}}{Max}\Big[U(C,\ Q,\ t)+J_t+J_w\ \Big\{\sum_{j=1} a_j\ W(\alpha_j-r)(1-\gamma)+\sum_{j=1}^{K} b_jW(\beta_j-\delta_j-r)(1-\lambda)$$
$$+\sum_{j=1}^{K} b_jW\delta_j(1-\gamma)+rW(1-\gamma)-C\Big\}+\sum_{j=1}^{m} J_{Q_j}\theta_j+$$
$$\frac{1}{2}J_{ww}\Big\{\sum_{j=1}^{n}\sum_{k=1}^{n} a_ja_k\ W^2h_{jk}(1-\gamma)^2+\sum_{j=1}^{K}\sum_{k=1}^{K} b_jb_k\ W^2\ g_{jk}(1-\lambda)^2+2\sum_{j=1}^{n}\sum_{k=1}^{K} a_jb_k\ W^2\ h_jg_k$$

$$\rho_{jk}(1-\gamma)(1-\lambda)\} + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} J_{Q_jQ_k} s_{jk}$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{m} J_{WQ_k}\, \alpha_j W h_j g_k \bar{\rho}_{jk}(1-\gamma)$$

$$+\sum_{j=1}^{K}\sum_{k=1}^{m} J_{WQ_k} b_j W g_{jk} \rho'_{jk}(1-\lambda)^2], \tag{8}$$

where subscripts on the derived utility of wealth function J denote partial derivatives. The h_{jk} is the instantaneous covariance batween the return on the jth and kth physical production processes and the g_{jk} is the covariance between the instantaneous changes in the jth and kth contingent claim. The ρ_{jk} is the instantaneous coefficient of correlation between the return on the jth physical production process and the changes in the kth state variable. Equation (8) is Bellman equation.[5)]

To derive a meaningful and tactable general equilibrium model we now make an assumption of a purely technical nature that makes it possible for us to employ results from stochastic optimal control theory to this problem.

ASSUMPTION 1 : *To maximize the von Neumann-Morgenstern utility function investors make decisions within a class of admissible feedback controls, V. An admissible feedback control, v, is σ-algebra measurable, defined on* $[t, T]\times R^{n+m+1}$ *and satisfies the growth and Lipschitz conditions Furthermore, admissible equilibrium* β *and* r *are bounded and satisfy the growth and Lipschitz conditions. There exists a unique function J and control* $\hat{v}$ *satisfying the Bellman equation and the stated regularity condition.*[6)]

Equation (8) is the valuation model as well as the portfolio selection model for the investor. Assuming the existence of a unique maximizing control policy, the optimal portfolio policies and decision rules can be determined from (8) and the boundary conditions. However, this parabolic type of partial differential equation often has no closed-form solution. In this case the differential equation can be solved numerically.[7)] Equation (8) will be used to describe and evaluate investors' optimal control policies of portfolio decisions as well as to derive the optimal portfolio decision rules and normative behavior of investors in the capital markets.

Define all the expressions in the right-hand side of equation (8) except the first and the second terms as *L(t)J*, which is said to be differential generator of *J* or the Dynkin operator where *C(t)* is replaced by an admissible feedback control *v(t)*. Then equation (8) can be written compactly as

$$0=\underset{\{v(t)\}}{Max}[U(v(t),\ Q(t),\ t)+J_t+L(t)J]$$

Now define

$$B(v(t),\ W(t),\ Q(t),\ t)=\underset{\{v(s)\}}{Max}[E\int_t^T U(v(s),\ Q(s),\ s)\ ds]$$

where *v(t)* is an admissible feedback control. Then Bellman equation (8) becomes

$$0=\underset{\{v(t)\}}{Max}[U(v(t),\ Q(t),\ t)+B_t+L(t)B]$$

where *L(t)B* is the differential generator of *B*.

The following lemma provides the feedback optimal control

LEMMA 1 : *Let J(t, W, Q) be a solution of the dynamic programming equation (the Bellman equation)*

$$0=\underset{v\in V}{Max}\ [L(t)J+U(t,\ Q,\ v)]+J_t \quad \text{for } (t,\ W,\ Q)\in D=(t,\ T)\times(0,\ \infty)\times R^n,$$

with boundary conditions

$$J(0,\ Q,\ t)=\underset{Q,t}{E}\int_t^T U(0,\ Q(s),\ s)ds\ ;\ \text{and } J(T,\ W,\ Q)=0$$

such that J(t, W, Q) is continuously differentiable with respect to t ; twice continuously differentiable with respect to W, and Q continuous on the closure of D ; and satisfies a polynomial growth condition on D, that is,

$$|J(t,\ W,\ Q)|\le\tilde{K}_1(1+|W,\ Q|^{\tilde{K}_2}),$$

where $\tilde{K}_1$ and $\tilde{K}_2$ are some constants. Then

(i) J(T, W, Q)≥B(T, W, Q, v) for any admissible feedback control v and any initial

position $(W, Q) \in D$; *and*

(ii) if v^* *is an admissible feedback control such that*

$$L(t)J+U(v^*, Q, t)=\underset{v \in V}{Max}\,[L(t)J+U(v^*, Q, t)] \quad \textit{for all } (t, W, Q) \in D \textit{; then}$$
$$J(W, Q, t)=B(v^*, W, Q, t) \qquad \textit{for all } (W, Q, t) \in D.$$

Thus v^* *is optimal.*

PROOF : See Flemming and Rishel(1975, p.159) *Q.E.D.*

Since consumer-investors can invest their wealth in the physical production processes and cannot obtain any satisfaction or can produce only disutility from nonnegative consumption, $a_j \geq 0$ and $C \geq 0$. The necessary and sufficient optimality conditions for (8) under these constraints are in matrix notation,

$$\phi = U_c - J_w \leq 0, \tag{9a}$$
$$C\phi_c = 0, \tag{9b}$$
$$\phi_a = (1-\gamma)\ (\alpha - r\underline{1})WJ_w + [(1-\gamma)^2\ HH'a + (1-\gamma)(1-\lambda)HG'b]W^2J_{ww} + (1-\gamma)HS'WJ_{WQ} \leq 0. \tag{9c}$$
$$a'\phi_a = 0, \tag{9d}$$
$$\phi_b = (1-\lambda)(\beta - \delta - r\underline{1})WJ_w + \delta(1-\gamma)\ WJ_w + [(1-\gamma)(1-\lambda)GH'a + (1-\lambda)^2\ GG'b]\ W^2J_{ww} + (1-\lambda)\ GS'WJ_{WQ} = 0, \tag{9e}$$

where double subscripts on J denote second partial derivatives, J_{WQ} is an m-dimensional vector whose ith element is J_{WQi}, and $\underline{1}$ is a unit vector.

The system can be solved for the optimal quantities C, a, and b in terms of W, Q, and t as a function of the partial derivatives of J, the derived utility of wealth function. Substituting the solution into Bellman equation (8), we have a partial differential equation with unknown J. With the boundary condition, we obtain a solution for J. Plugging the solution of J back into the system of the simultaneous equations, we have the solution for C^*, a^* and b^* as functions of only t, W, and Q, which are the current wealth and the current state variables at time t. Thus the optimal investment in production processes, optimal consumption and portfolio are derived, together with the optimal portfolio decision rules.

2. A General Equilibrium Valuation Model

The analysis and conclusion of the preceding Subsection state that given α, β and r, γ and λ, with J explicitly determined, the optimality conditions for the problem with contingent claims and borrowing and lending can be combined with the market clearing conditions to give the equilibrium interest rate and expected rates of return on contingent claims. Equilibrium also determines the optimal physical production plan and the optimal consumption plan.

In equilibrium in this homogeneous economy, the interest rate and the expected rates of return on the contingent claims must adjust until all wealth is invested in the physical production processes. In aggregate, the net supply of contingent claims and riskless lending must be zero. Formally, we have the following definition.

DEFINITION : *An equilibrium is characterized by the process having the specification (6) and by a set of stochastic processes (r, β ; a, C) satisfying (9) and the market clearing conditions* $a\underline{1}=1$ *(capital market clearing) and* $b=0$ *(claims market clearing).*

The endogenous determination of the equilibrium interest rate and its properties are of great interest. To see the characteristics of the riskless interest rate, solve (9*d*) for the equilibrium interest rate using the market clearing conditions. Then we can write the general equilibrium rate of interest of the riskless asset as

$$r(W, Q, t)=a^{*\prime}\alpha+(1-\gamma)a^{*\prime}HH'a^{*\prime}W\frac{J_{WW}}{J_W}+a^{*\prime}HS'\frac{J_{WQ}}{J_W}$$

$$=a^{*\prime}\alpha-(1-\gamma)\left[\frac{-J_{WW}}{J_W}\right]\left[\frac{Var(W)}{W}\right]-\sum_{j=1}^{m}\left[\frac{-J_{WQ_i}}{J_W}\right]\left[\frac{cov(W, Q_i)}{W}\right] \quad (10)$$

where *var(W)* stands for the variance of changes in optimally invested wealth and *cov(W, Q_j)* represents the covariance of changes in optimally invested wealth with changes in the state variable Q_j. Here * denotes the optimal value.

Suppose that this world is one with no risk at all. Then under this condition of perfect certainty all variance and covariance terms vanish. Consequently $r=a^{*\prime}\alpha$, that is, the riskless rate of interest is identically equal to the market rate of return on the market(aggregate) physical production process or the rate of return on any physical production process due to the fact that the rate of return on a physical production process is equal to that on any other physical production process in the world of certainty. It is interesting to notice that the presence of taxes does not affect the equilibrium interest rate. The interest rate in the presence of taxes is identically equal to the rate that would obtain in the absence of taxes.

With perfect foresight, individuals have perfect knowledge of their wealth position at any time in their life span, and are willing to belong to a particular clientele with the same tax bracket in order to reduce the effective tax rate in such a way that they may ignore the presence of taxes and act as if the taxes have no impact on their investment activities. If investors can form a clientele favorable to them such that the effective tax rate is the same for each investor, then the presence of taxes clearly has no effect on the behavior of the investors toward their investment activities.

Equation (10) states that the equilibrium interest rate may be either less or greater than $a^{*\prime}\alpha$, the market rate of return on the physical production process. Since the derived utility of wealth function is strictly concave and strictly increasing in W and the variance of changes in optimally invested wealth is allways nonnegative, the second term in (10) is nonnegative. The third term reflects the investor's demand for the physical production processes to hedge against unfavorable shifts in investment opportunity. If the ex post opportunity set is less favorable than anticipated, the investor will expect to be compensated by a higher level of wealth through the positive correlation of the returns. Similarly, if ex post returns are lower, he will expect a more favorable investment enviornment. This fact reflects an attempt to minimize the unanticipated variability in consumption over time in order to achieve an interte-mporal smooth pattern of consumption. An individual investing only in locally riskless lending would be unprotected against unfavorable shifts in the stochastic investment opportunity set or the state variables. Then the equilibrium interest rate is less than the market rate of return on the physical production processes.

The above argument leads to the fact that the presence of taxes should not have any impact on the covariance of changes in optimally invested wealth with changes in the state variables Q_j because the tax rates reduce nonlinearly if the rate of return decreases. The unfavorable shifts in the stochastic investment opportunity set is perfectly positively correlated with the tax rates. Thus taxes should not affect the covariance term. The tax term, however, is in the second term in (10). It is natural that taxes affect the changes in the investor's wealth. Since the second term is non negative and $\gamma \in [0, 1]$, the amount of this term is less than that of the term which would have achieved with no taxes.

Thus the equilibrium interest rate in the presence of differential taxes is greater than the rate derived in the absence of taxes. Investors, investing in riskless lending, want to be compensated for paying taxes on their interest earnings. It is desirable to discuss in some detail some properties of the general equilibrium riskless rate of interest. By Itô's lemma[8], J_w will satisfy the stochastic differential equation to $J_w(W, Q, t)$. It is assumed that the second partial derivatives on J_w and J_t are continuous and differentiable with respect to W. Then we have the following theorem for the general equilibrium interest rate.

THEOREM 1 : *(i) In equilibrium the interest rate is equal to the inverse of (1-γ) times the negative of the expectation of the rate of change in the derived marginal utility of wealth with respect to wealth, i.e.,*

$$r(W, Q, t) = -\frac{(1-\gamma)^{-1}E[dJ_W]}{J_W} \tag{11}$$

where E denotes the expectation operator and d is the symbol for the derivative.

(ii) The equilibrium interest rate is equal to the expected market rate of return on the physical production process plus the inverse of (1-γ) multiplied by the covarince of the rate of return on wealth with the rate of change in the marginal derived utility of wealth function divided by wealth times the marginal utility of wealth, i.e.,

$$r(W, Q, t) = a^{*\prime}\alpha + (1-\gamma)^{-1}\frac{\mathrm{cov}(dW, dJ_W)}{WJ_W} \tag{12}$$

PROOF : (i) Use Itô's lemma on $J_w = J_w(W, Q, t)$ to get the stochastic differential equation of the form

$$(1-\gamma)^{-1}dJ_W = \tfrac{1}{2}(1-\gamma)J_{WWW}\mathrm{var}(W) + \tfrac{1}{2}(1-\gamma)^{-1}\sum_{j=1}^{m}\sum_{k=1}^{m}J_{WQ_jQ_k}\mathrm{cov}(Q_j, Q_k)$$

$$+\sum_{j=1}^{m} J_{wwQj}\mathrm{cov}(W,\ Q_j)+(1-\gamma)^{-1}\sum_{j=1}^{m} J_{wQj}\theta_j+J_{wt}$$

$$+J_{ww}(a^{*\prime}\alpha W-(1-\gamma)^{-1}C)\ dt+[\ \cdot\ ]\ dz, \tag{13}$$

where [·] is the collection of the terms containing dz. When we take the expectation on $(1-\gamma)^{-1}\ dJ_w$, the terms containing dz become zero. Dividing dJ_w by J_w yields the expected rate of change in the derived marginal utility of wealth. Next we differentiate (8) with respect to W and use *(9c)* and the market clearing conditions to obtain

$$r=a^{*\prime}a-(1-\gamma)\left[\frac{-J_{WW}}{J_W}\right]\left[\frac{\mathrm{var}(W)}{W}\right]-\sum_{j=1}^{m}\left[\frac{-J_{WQj}}{J_W}\right]\left[\frac{\mathrm{cov}(W,Q_j)}{W}\right]$$

$$=-[\tfrac{1}{2}(1-\gamma)J_{WWW}\mathrm{var}(W)+\tfrac{1}{2}(1-\gamma)^{-1}\sum_{j=1}^{m}\sum_{k=1}^{m} J_{WQjQk}\mathrm{cov}(Q_j,\ Q_k)+\sum_{j=1}^{m} J_{WWQj}\mathrm{cov}(W,\ Q_j)$$

$$+(1-\gamma)^{-1}\sum_{j=1}^{m} J_{WQj}\theta_j+J_{wt}+J_{ww}(a^{*\prime}\alpha W-(1-\gamma)^{-1}C)]\ /J_w$$

$$=-[(1-\gamma)^{-1}EdJ_w\ /\ J_w), \tag{14}$$

which is the desired result. This completes the proof of part (i).

(ii) By Itô's lemma, the stochastic part of dJ_w is

$$[J_{WW}a^{*\prime}HW(1-\gamma)+J'_{WQ}S]dz(t)$$

and the stochastic part of dW is $\alpha^{*\prime}HW(1-\gamma)$. Thus the covariance of the rate of return on wealth with the rate of change in the marginal utility of wealth is

$$\frac{\mathrm{cov}(dW,\ dJw)}{WJw}=\frac{1}{WJ_w}[J_{WW}a^{*\prime}HW(1-\gamma)+J'_{WQ}S]\ [a^{*\prime}HW(1-\gamma)]'$$

$$=(1-\gamma)^2\left[\frac{J_{WW}}{J_W}\right]\left[\frac{\mathrm{var}(W)}{W}\right]+(1-\gamma)\sum_{j=1}^{m}\left[\frac{J_{WQj}}{J_W}\right]\left[\frac{\mathrm{cov}(W,\ Q_j)}{W}\right].$$

Dividing both sides by$(1-\gamma)$ yields

$$\frac{(1-\gamma)^{-1}\mathrm{cov}(dW,\ dJ_w)}{WJw}=-[(1-\gamma)\left(\frac{-J_{WW}}{J_W}\right)\left(\frac{\mathrm{var}(W)}{W}\right)$$

$$+\sum_{j=1}^{m}\left(\frac{-J_{WQ_j}}{J_W}\right)\left(\frac{\mathrm{cov}(W,\ Q_j)}{W}\right)\Big]. \tag{15}$$

This equation is exactly the same as the second part of (10). Substituting this into (10), we have that

$$r=a^{*\prime}\alpha+(1-\gamma)^{-1}\ \frac{\mathrm{cov}(dW,\ dJ_W)}{WJ_W},$$

which is the desired result. This completes the proof of the second part of the Theorem. *Q.E.D.*

The Theorem states that the interest rate in the presence of taxes is determined in equilibrium by adjusting the expected derived marginal utility of wealth through the inverse of one minus the income tax rate, or by adjusting the covariance of dW with J_W through the inverse of one minus the income tax rate. When investors hedge against the risk of unfavorable shifts in production technologies, the covariance of dW with dJ_W will become negative. The presence of taxes will aggravate the situation, and hence will in turn hurt the smooth pattern of the standard of living. Thus, investors require a higher return for the riskless lending by dividing $(1-\gamma)$ into the covariance between dW and dJ_W. The riskless rate of interest is a function of the returns on the physical production processes and the covariance of the rate of return on production technologies and the rate of change in the marginal utility of wealth, which is adjusted by the inverse of one minus income tax rate. This Theorem clearly shows that the riskless rate is determined in the capital market by the interactions of the dynamics of assets prices and the utility function.

We now turn to the equilibrium expected return on contingent claims. The equilibrium return is characterized by the following theorem.

THEOREM 2 : *Equilibrium determines the equilibrium expected return on any contingent claim as follows :*

$$(\beta_i-r)F^i=\xi\delta_iF^i+[\phi_W,\ \phi_{Q_1},\cdots,\ \phi_{Q_m}]\ (F^i_W,\ F^i_{Q_1},\cdots,\ F^i_{Q_m}]', \tag{16}$$

where

$$\zeta = \frac{\gamma - \lambda}{1 - \lambda}$$

$$\phi_W = (1-\gamma)^2 \left[\frac{-J_{WW}}{J_W} \right] [\text{var}(W)] + (1-\gamma) \sum_{j=1}^{m} \left[\frac{-J_{WQj}}{J_W} \right] [\text{cov}(W, Q_j)],$$

$$\phi_{Qj} = (1-\gamma) \left[\frac{-J_{WW}}{J_W} \right] [\text{cov}(W, Q_j) + \sum_{k=1}^{m} \left[\frac{-J_{WQk}}{J_W} \right] [\text{cov}(W, Q_j, Q_k)].$$

PROOF : Using the market clearing conditions and substituting a* and r into (9e), we obtain

$$\beta_i = \delta_j \left(1 - \frac{1-\lambda}{1-\lambda}\right) + a^{*\prime} \alpha \underline{1} + \frac{W J_{WW}}{J_W} [\{(1-\gamma) a^{*\prime} H H' a^*\} \underline{1} - (1-\gamma) G H' a^*]$$

$$+ \frac{1}{J} [a^{*\prime} H S' J_{WQ}) \underline{1} - G S' J_{WQ}]. \qquad (17)$$

To link (17) with F and eliminate G, we use Itô's lemma for $F(W,Q,t)$, which yields the stochastic terms as follows :

$$dF^i = [F_W (1-\gamma) a^{*\prime} H W dz(t) + F_Q S dz(t)] + \{ \cdot \} dt,$$

where { · } denotes the collection of the terms containing dt. Recalling from (6) that Gdz is the stochastic term of the system of the stochastic differential equations which is the instantaneous evolution of the state variables, we can equate $F'Gdz$ to the bracketed term of the above equation and obtain

$$FG = F_W (1-\gamma) a^{*\prime} H W + F_Q S. \qquad (18)$$

Substituting (18) into (17) and rearranging, we can write the equilibrium expected return on the ith contingent claim as

$$\beta_i F^i = r F^i + \left(\frac{\gamma-\lambda}{1-\lambda}\right) \delta_i + F_W \left[\left(\frac{-J_{WW}}{J_W}\right) (1-\gamma)^2 \text{var}(W) + \sum_{j=1}^{m} \left(\frac{-J_{WQj}}{J_W}\right) (1-\gamma) \text{cov}(W, Q_j) \right]$$

$$+ \sum_{j=1}^{m} F^i_{Qj} \left[\left(\frac{-J_{WW}}{J_W}\right) (1-\gamma) \text{cov}(W, Q_j) + \sum_{k=1}^{m} \left(\frac{-J_{WQk}}{J_W}\right) \text{cov}(Q_j, Q_k) \right], \qquad (19)$$

which is the desired result. This completes the proof of the Theorem. *Q.E.D.*

The Theorem states that the equilibrium expected return for any contingent claim can be expressed as the riskless return plus the tax-adjusted payout plus a linear combination of the first partials of the tax-adjusted asset price with respect to W and F. Equation (16) or (19) shows that the equilibrium return on a contingent claim is linearly related to the variance-covariance between W and Q. Note that the absolute risk aversion of the derived utility of wealth function affects the determination of the equilibrium return on the claim. Notice also that all the factors affecting the return must be adjusted by the income tax rate. Furthermore the payout is adjusted by both the income and capital gains tax rates.

Let us examine equation (16) in some more detail. From (10) follows that $\phi=(1-\gamma)(a^{*\prime}\alpha-\gamma)W$, which is the tax-adjusted expected excess return on optimally invested wealth over the riskless return. The tax-adjusted expected excess returns are, in turn, adjusted by the marginal values of the contingent claim with respect to the wealth and each of the state variables. The total payout is taxed by the rate ξ. Therefore, the total return of the contingent claim is a function of the tax-adjusted total payout, tax-adjusted excess return, the marginal values with respect to wealth and state variables, and tax rates.

3. The Fundamental Valuation Equation

In the preceding Subsection, we have obtained the equilibrium expected return for the contingent claim, which provides great insights into the process of determination of the expected return. On the other hand, the solution for price function F is important, whether it is of a closed-form type or not. The following theorem gives the explicit price functions for the contingent claims.

THEOREM 3 : *The price of any contingent claim is given by the fundamental partial differential equation of the form* :

$$\frac{1}{2}(1-\gamma)^2\mathrm{var}(W)F_{WW}+(1-\gamma)\sum_{j=1}^{m}\mathrm{cov}(W,\ Q_j)F_{WQ_j}+\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\mathrm{cov}(Q_j,\ Q_k)\ F_{Q_jQ_k}$$

$$+[r(W,Q,t)W-C^*(W,Q,t)]F_W+\gamma[(1-\gamma)(\frac{-J_{WW}}{J_W})\ \mathrm{var}(W)$$

$$+\sum_{j=1}^{m}(\frac{-J_{WQj}}{J_W})\ \mathrm{cov}(W,\ Q_j)]F_W$$

$$+\sum_{j=1}^{m}[\theta_j-(1-\gamma)\ (\frac{-J_{WW}}{J_W})\mathrm{cov}(W,\ Q_j)$$

$$-\sum_{k=i}^{m}(\frac{-J_{WQk}}{J_W})\mathrm{cov}(Q_j,\ Q_k)]F_{Qj}+F_t$$

$$-r(W,Q,t)F(W,Q,t)$$

$$+\delta(W,Q,t)F(W,Q,t)-(\gamma-\lambda)(1-\lambda)^{-1}\delta(W,Q,t)=0. \qquad (20)$$

PROOF : By Ito's lemma, the drift term of *F(W,Q,T)* is given by

$$\beta F=\delta F+\tfrac{1}{2}(1-\gamma)^2\mathrm{var}(W)F_{WW}+(1-\gamma)\sum_{j=1}^{m}\mathrm{cov}(W,Q_j)F_{WQj}$$

$$+\tfrac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\mathrm{cov}(Q_j,\ Q_k)\ F_{QjQk}$$

$$+[(1-\gamma)a^{*\prime}\alpha W-C]F_W$$

$$+\sum_{j=1}^{m}\theta_jF_{Qj}+F_t. \qquad (21)$$

Combining (21) and (19) yields (20). *Q.E.D.*

Note in (20) that taxes are not involved in the terms regarding the state variables. Total payout is subject to the tax rate ξ.

Recall that γ is given by

$$r=a^{*\prime}\alpha-(1-\gamma)(\frac{-J_{WW}}{J_W})\ (\frac{\mathrm{var}(W)}{W})-\sum_{j=1}^{m}(\frac{-J_{WQj}}{J_W})\ [\frac{\mathrm{cov}(W,\ Q_j)}{W}].$$

Substituting r into (20) and using the definition of ϕ_W and ϕ_Q, we can rewrite (20) as

$$\tfrac{1}{2}(1-\gamma)^2\mathrm{var}(W)F_{WW}+(1-\gamma)\sum_{j=1}^{m}\mathrm{cov}(W,\ Q_j)F_{WQj}+\tfrac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\ \mathrm{cov}(Q_j,\ Q_k)\ F_{QjQk}$$

$$+[a^{*\prime}\alpha W-C^{*}]F_W+\sum_{j=1}^{m}\theta_j F_{Qj}+F_t-rF+\delta F-(\gamma-\lambda)(1-\lambda)^{-1}\delta$$

$$=[1-\gamma(1-\gamma)]\phi_W F_W+\phi_Q F_Q. \qquad (22)$$

Equation (20) is of great importance due to the fact that it gives the equilibrium prices for contingent claims whose closed form solution will be given shortly, but equation (22) provides an easy interpretation for Theorem 3. The left-hand side of (22) gives the tax-adjusted excess expected return on the security above the risk-free return, while the right-hand side gives the tax-adjusted risk premium that the security must have in equilibrium. Both the excess return and risk premium must be adjusted by tax rates. Note that in (22) ϕ_W, which is already tax-adjusted, receives again the tax adjustment.

To solve the fundamental partial differential equation for the price of any contingent claim, we need boundary conditions, which can be written as

$$F[W(T),\ Q(T),\ T]=\Upsilon[W(T),\ Q(T)] \qquad \text{for } W(T),\ Q(T)\in\Delta\ ;\ \text{and}$$

$$F[W(\tau),\ Q(\tau),\ \tau]=\Phi[W(\tau),\ Q(\tau),\ \tau] \qquad \text{for } W(\tau),\ Q(\tau)\in\partial\Delta. \qquad (23)$$

The boundary conditions above may be interpreted as follows : T is the maturity date of the contingent claim. Therefore, F is defined on $[t,\ T]\times\Delta$, where $\Delta\subset(0,\infty)\times R^m$ is an open set and $\partial\Delta$ is its boundary. Hence F belongs to a well-defined time dimension or domain until the maturity date of the contingent claim. At the time of the maturity, the asset has the terminal value. In the mean time, if F happens to reach the boundary of the domain at time before T, i.e. $\tau<T$, then F is to pass through the boundary of the domain. Hence, the contingent claim can no longer stay in the capital market. As a result, the contract is terminated. Thus, τ is the first passage time or the first hitting time that F leaves Δ if such a time exists, and $\tau=T$ otherwise. $\partial\Delta$ is the set of all accessible boundary points. If F hits $\partial\Delta$, then the contingent claim is liquidated and its value will be Φ.

Now define System I as follows :

$$dW(t)-[a^{*\prime}\alpha W(1-\gamma)-C^{*}]dt+a^{*\prime}HW(1-\gamma)dz(t)\ ;$$

$$dQ(t)=\theta(Q,\ T)+S(Q,\ t)dz(t). \tag{24}$$

Using System I and boundary conditions, we can solve (20) for the price of the contingent claim. The next theorem shows this.

THEOREM 4 : *The general equilibrium fundamental valuation partial differential equation for the price of the contingent claim, with the boundary conditions defined above, has a unique solution. The equilibrium price for the contingent claim that is the unique solution to (20) is given by*

$$\begin{aligned}F(W,\ Q,\ \lambda,\ \gamma,\ t,\ T)=E[&\Upsilon(W(T),Q(T))\cdot[\exp\{-\int_t^T \beta(W(u),\ Q(u),\ u)du\}]I(\tau\geq T)\\&+\Phi(W(\tau),\ Q(\tau),\ \tau)\cdot[\exp-\{-\int_t^\tau \beta(W(u),\ Q(u),\ u)\}du]\ I(\tau<T)\\&+\int_t^{\tau\wedge T}\delta(W(s),\ Q(s),\ s)\cdot[\exp-\{\int_t^s \beta(W(u),\ Q(u),\ u)du\}]ds],\end{aligned} \tag{25}$$

where $I(\cdot)$ is an indicator function, and τ is the first hitting (passage) time to $\partial\Delta$, *and the dot* $(\cdot)$ *represents multiplication. Expectation is taken with respect to System I.*

PROOF : By Itô's lemma, equation (20) can be expressed as $LF+F_t+\delta F=\beta F$. Use Theorem 5.2 of Friedman (1975, vol. 1, p.147). *Q.E.D.*

The Theorem provides significant implications for the investors' optimal behavior of the contingent claim. The price of any contingent claim is determined by the three important factors : First, if the underlying variables do not leave the defined dimensional space before the maturity date, a payment of Υ is received at the maturity date, and the contingent claim is fully compensated. Second, if the variables leave the space before the maturity date, Φ will be paid at that time[9]. And third, the holder of the contingent claim will be compensated for his accumulated payout received. Furthermore, all the future values are discounted at the discount rate $\beta(\cdot)$.

Note that equation (25) does not contain an explicit tax term in it. Taxes are incorporated in the expectations, as shown in System I, and in the definition of

β, as shown in (19). The presence of taxes has an impact on the price of the contingent claim through the expectation and the rate of return on the claim. Therefore, F is a function of W, Q, γ, λ, t and T.

The Theorem provides significant insights into the determination processes of the contingent claim prices and the wealth-maximizing behavior of investors. However, there are two unknown variables F and β in it. Unless the equilibrium expected rate of return of that particular contingent claim is known in advance, we do not have an explicit solution. To avoid this difficulty, consider System II

$$dW(t)=[a^{*\prime}\alpha W(1-\gamma)-\phi_w-C^*]dt+a^{*\prime}HW(1-\gamma)dz(t);$$

$$dQ(t)=[\theta(Q,\ t)-(\phi_{Q1},\ \phi_{Q2},\ \ldots,\ \phi_{Qm})']dt+S(Q,\ t)dz(t). \tag{26}$$

Then we have the following theorem.

THEOREM 5 : *The unique solution to (20) is the equilibrium price for any contingent claim that is also written as*

$$\begin{aligned} F(W,\ Q,\ \lambda,\ \gamma,\ t,\ T)=\hat{E}[\Upsilon W(T),\ Q(T))\cdot[\exp\{-\int_t^T r(W(u),\ Q(u),\ u)du\}]\ I(\tau\geq T) \\ +\Phi(W(\tau),\ Q(\tau),\ \tau)\cdot[\exp\{-\int_t^\tau r(W(u),\ Q(u),\ u)du\}]\ I(\tau<T) \\ +\int_t^{\tau\prime\ T}\delta(W(s),\ Q(s),\ s)\cdot[\exp\{-\int_t^s r(W(u),\ Q(u),\ u)du\}]ds], \end{aligned} \tag{27}$$

where $\hat{E}$ denotes expectation with respect to System II.

PROOF : By Theorem 5.2 of Friedman(1975, Vol.1, p.147), we obtain the above result. *Q,E,D.*

Though equation (27) has no explicit term about taxes, the expectation and the riskless rate of return determined endogenously by equation (10) reflect the taxes. Equation (27) states that the equilibrium price of a claim is given by its expected discounted value adjusted by taxes through the expectation that is taken with respect to a risk- and tax-adjusted process for the optimally invested wealth and state variables.

IV. A General Equilibrium Valuation Model Under Incomplete Information

In Section 3, we have developed a general equilibrium model of asset prices with taxes assuming perfect knowledge of the economy, or the direct observation of the process that provides the information about the dynamics of the state of nature. In this Section we will relax this assumption. We assume that the underlying state process is unobservable, but that there exists a process which is observable and provides partial information about the state of nature. Formally, we have the following assumption :

ASSUMPTION 2 : *Let* $(\varepsilon ; Q)$ *be a partially observable process of state variables where* ε *is unobservable and* Q *is observable. The joint process is Gaussian :*

$$d\varepsilon(t)=\pi(t)\varepsilon(t)dt+A(t)dz_1(t) ; \tag{28}$$

$$dQ(t)=\theta(t)\varepsilon(t)dt+S(t)dz(t) \tag{29}$$

where $z_1(t)$ *and* $z(t)$ *are the n-dimensional independent Wiener processes ;* $\pi(t)$ *is an m-dimensional vector valued function ; and* $A(t)$ *is an* $(m\times n)$ *dimensional matrix. The covariance matrix of instantaneous changes in the unobservable part of the state variables,* AA', *is nonnegative definite.*

Since investors have only partial information about the state variables, they need conditional distribution to infer knowledge of the evolution of the state variables from the observable component. Assume that the initial conditional distribution is $N(\mu_0, \sigma_0^2)$. Then the conditional distribution of the unobservable part of the state variables $(\varepsilon(t) \mid Q(t))$ is normally distributed with $N(\mu(t) ; \psi(t))$, for which $\mu(t)$ and

$\psi(t)$ are determined by solving

$$d\mu(t)=\pi(t)\mu(t)dt+\psi(t)\theta'(t)(SS')\ (dQ(t)-\theta(t)\mu(t)dt)\ ;$$

$$\mu(0)=\mu_0\ ;$$

$$\dot{\psi}(t)=\pi(t)\psi(t)+\psi(t)\pi'(t)+AA'-\psi(t)\theta'(t)(SS')^{-1}\theta(t)\psi'(t)\ ;$$

$$\psi(0)=\sigma_0^2, \tag{30}$$

where $\dot{\psi}(t)$ is the derivative of $\psi(t)$ with respect to time t. Define $dv(t)=dQ(t)-\theta(t)\mu(t)dt$. Then $v(t)$ is an innovation process with instantaneous variance $V=SS'$. Since all values are known together with the initial conditional distribution, $\psi(t)$ is deterministic. Therefore, the conditional density is

$$\zeta(\varepsilon(t) \mid Q(s) : 0\leq s\leq t)=\zeta(\varepsilon(t) \mid \mu(t)). \tag{31}$$

From (4) and (6), the physical production processes and the contingent claim processes can be written as

$$d\delta=\alpha(\varepsilon(t),\ t)dt+H(\varepsilon(t),\ t)dz_1(t)\ ; \tag{32}$$

$$dF^j=F^j(\beta_j-\delta_j)dt+F^jg_jdv(t)-F^j\eta_jdz_1(t). \tag{33}$$

Since incomplete information structure determines the physical production processes and the contingent claim processes, we can obtain an intertemporal general equilibrium model of asset pricing in the structure of partial information about the state variables.

1. A General Equilibrium Asset Pricing Model

Under the incomplete in formation structure, wealth dynamics as the time t is suppressed for notational simplicity, can be expressed as

$$dW=[\sum_{j=1}^{n}a_jW(\alpha_j-r)(1-\gamma)+\sum_{j=1}^{k}b_jW(\beta_j-\delta_j-r)(1-\lambda)+\sum_{j=1}^{k}b_jW(\delta_j-r)(1-\gamma)$$

$$+rW(1-\gamma)-C]dt+\sum_{j=1}^{n}a_jWh_jdz_1(t)(1-\gamma)+\sum_{j=1}^{k}b_jWg_jdv(t)(1-\lambda)$$

$$+\sum_{j=1}^{k}b_jW\eta_jdz_1(t)(1-\lambda). \tag{34}$$

The investor observes the returns, $d\delta(t)$, on the phy͡sical production processes and uses $d\delta(t)$ as an inference process through the process $dQ(t)$. Thus $\delta(t)$ is a subcomponent of $Q(t)$. And then he will allocate his wealth in order to maximize his lifetime utility of consumption

$$\text{Max } \underset{Q(t)}{E}\int_t^T U(C(s),\ s)ds \tag{35}$$

subject to his bugdet equation (34).

The Bellman equation under the incomplete information structure or the indirect observation of the state variables becomes

$$0=\underset{\{C,a,b\}}{\text{Max}}\ \{U(C,t)+J_t+J_W[a'(\hat{\alpha}-r\underline{1})(1-\gamma)W+b'(\beta-\delta-r\underline{1})(1-\lambda)W+b'\delta(1-\gamma)W$$

$$+r(1-\gamma)W-C]+J_\mu\pi\mu+J_\psi\Box[\pi\psi+\psi\pi'+AA'-\psi\theta'(SS')^{-1}\theta\psi']$$

$$+\tfrac{1}{2}J_{WW}\{\underset{\mu(t)}{E}[a'H(1-\gamma)+b'\eta(1-\lambda)][(H'a(1-\gamma)+\eta'b(1-\lambda)]+b'gVg'b\}W^2$$

$$+\tfrac{1}{2}J_{\mu\mu}\Box\psi\theta'(SS')^{-1}V(SS')^{-1}\theta\psi'+J_{W\mu}[\psi\theta'(SS')^{-1}Vg'b(1-\lambda)]W\}, \tag{36}$$

where J_μ is a $(1\times n)$ dimensional vector ; J_ψ is an $(n\times n)$ dimensional matrix ; $J_{\mu\mu}$ is an $(n\times n)$ *dimensional matrix* ; $J_{W\mu}$ is a $(1\times n)$ dimensional vector ; $\Box$ is direct multiplication symbol for matrices followed by addition of all the components ; and $V=SS'$ is the covariance matrix of dv. Hatted variables are conditional expectations. The conditional expectations can be obtained by the following lemma.

LEMMA 2 : *Suppose that the signal process*[10] $\{X(t)\}$ *is the unique strong solution of the stochastic differential equation*

$$dX(t)=N(X(t))+\sigma(X(t))dz(t), \qquad 0\leq t\leq T$$

with initial condition X_0. Suppose that M is a twice continuously differentiable function on R^m so that

$$M(X(t))=M(X_0)+\int_0^t LM(X(u))du+\int_0^t \nabla M\cdot\sigma(X(u))dz(u)$$

The observation process y(t) will be with $K(t)=\tilde{h}$ (X(t)), a bounded function, and suppose

$$\langle Z^i, W^j\rangle(t)=\int_t^T \rho^{ij}(u)du, \qquad 1\leq i\leq m,\ 1\leq j\leq n.$$

For $f\in L^2_{loc(R^m)}$, *Write* $\Pi(t)(f)=E[f(X(t))\mid y_t]$.

Then

$$\Pi(t)(M)=\Pi_0(M)+\int_0^t \Pi(u)(LM)du$$

$$+\int_0^t\{\Pi(u)(\nabla M,\sigma,\rho)+\alpha^{-1}(y(u))(\Pi(u))(M\tilde{h})-\Pi(u)(M)\Pi(u)(\tilde{h}))\}'dv(u).$$

PROOF : See Theorem 18.12 of Elliot (1982, p.283). *Q.E.D.*

The necessary and sufficient optimally conditions for the Bellman equation (33) are given by

$$\phi_c=U_c-J_w\leq 0 \tag{37a}$$

$$C\phi_c=0 \tag{37b}$$

$$\phi_a=J_W W(\hat{\alpha}-r\underline{1})(1-\gamma)+J_{WW}W^2[EHH'a(1-\gamma)^2+EH\eta'b(1-\gamma)(1-\lambda)]\leq 0 \tag{37c}$$

$$a'\phi_a=0 \tag{37d}$$

$$\phi_b = J_W W[(\hat{\beta} - \delta - r\underline{1})(1-\lambda) + \delta(1-\gamma)] + J_{WW} W^2 [gVg'b + E\{\eta\eta' b(1-\lambda)^2$$

$$+ \eta H' a(1-\lambda)(1-\gamma)\}] + J_{W\mu} W[gV(SS')^{-1}\theta\psi](1-\lambda) = 0, \tag{37e}$$

where E represents expectation, which is taken with respect to $\mu(t)$.

This system can be solved for C, a, and b in terms of W, μ c, and t, and partials of J. Substituting back into the Bellman equation, we obtain a partial differential equation for J. Plugging J back into the system, we can obtain the desired values.

Solving (37*d*) for the equilibrium interest rate, we have

$$r = a^{*\prime}\hat{\alpha} + \frac{WJ_{WW}}{J_W} a^{*\prime} + HH' a^*(1-\gamma). \tag{38}$$

Since the solution to the Bellman equation (36) is a function of μ and ψ, the equilibrium interest rate is a function of μ and ψ. We can rewrite the equilibriun rate as

$$r = a^{*\prime}\hat{\alpha} - (1-\gamma)\left(\frac{-J_{WW}}{J_W}\right) E_{\mu(t)}\left[\frac{\text{var}[W/\varepsilon(t)\,;\,\mu(t)]}{W}\right] \tag{39}$$

Equation (39) is different from (10) in two respects. First, it does not contain the covariance of the change in optimally invested wealth with changes in the state variables. The reason is that the information structure consists of independence of $z(t)$ and $z_1(t)$. Put another way, investors have only partial information of the evolution of the state variables. Second, from the same reason (39) has the variance term with expectation, conditional on the information $\varepsilon(t)$, inferred from $\mu(t)$. Therefore, we present the following theorem without proof (which is similar to that for Theorem 1).

THEOREM 6 : *In the rational expectations equilibrium the equilibrium interest rate is equal to the one minus income tax rate times the negative of the conditional expectation of the rate of change of marginal utility with respect to wealth, i.e.,*

$$r = -\frac{(1-\gamma)^{-1} E_{\mu(t)}[dJ_W]}{J_W} \tag{40}$$

where $E_{\mu(t)}$ denotes the conditional expectation, conditional on $\mu(t)$, *i.e.,*

$E[\cdot\ \underline{1}\varepsilon(t)\ ;\ \mu(t)]$.

We now turn to the equilibrium expected return on contingent claims. The following theorem gives the equilibrium expected return in terms of the underlying variables.

THEOREM 7 : *The expected return for any contingent claim in the rational expectations equilibrium is given by*

$$(\beta_j - r)F^j = \left(\frac{\gamma-\lambda}{1-\lambda}\right)\delta_j F^j + (1-\gamma)^2 F^j_w \left[\left(\frac{-J_{WW}}{J_W}\right) \underset{\mu(t)}{E}\left(\mathrm{var}(W \mid \varepsilon(t)\ ;\ \mu(t)\right)\right]$$

$$+ (1-\lambda)\sum_{j=1}^{m} F^j_{\mu j}\left[\sum_{k=1}^{m}\left(\frac{-J_{W\mu k}}{J_W}\right)\left(\mathrm{cov}(\mu_j,\ \mu_k \mid \varepsilon(t)\ ;\ \mu(t)\ ;\ \psi(t)\right)\right]. \qquad (41)$$

PROOF : By employing Itô's lemma to $F(W,\ \mu,\ \psi,\ t)$ we can express g *and* η explicitly. Plugging the results into (34*e*) yields (41). *Q.E.D.*

The main difference of (41) from (16) is that equation (41) does not contain the covariance terms of changes in optimally invested wealth with changes in the state variables due to the information structure. The inference of knowledge about the state variables from the partial information and the market clearing condition $b=0$ for all j dictate that wealth diffuses independently of the innovation process $d\nu(t)$, because $d\nu(t)$ is independent of $dz_1(t)$ through the independence between $dz_1(t)$ and $dz(t)$.

Equation (16) has both the variance of optimally invested wealth and the covariance of wealth with changes in the state variables without expectation, while the variance and the covariance in (41) are conditional expected ones. This discussion leads to the following theorem about the value of information under situations of incomplete information in the continuous market.

THEOREM 8 : *The ex post value of information under the situation of incomplete information in the continuous market with taxes is given by*

$$\Psi = F_W(1-\gamma)\sum_{j=1}^{m}\left(\frac{-J_{WQj}}{J_W}\right)\left[\mathrm{cov}(W,\ Q_j)\right] + (1-\gamma)\sum_{j=1}^{m} F_{Qj}\left[\frac{-J_{WW}}{J_W}\right]\left[\mathrm{cov}(W,\ Q_j)\right].$$

PROOF : Obvious by comparing (16) to (41). *Q.E.D.*

Equation (42) says that the value or cost of information to translate the incomplete to the noisy information is a function of five factors : the absolute risk aversion ($-J_{WW}/J_W$) with respect to wealth ; the hedging ($-J_{WQ_j}/J_W$) against unfavorable shifts in the stochastic investment opportunity set ; the tax-adjusted covariance of optimally invested wealth with the state variables ; the marginal value of the claim with respect to wealth ; and the marginal value of the claim with respect to the state variables. Thus the marginal values of the claim and the covariance between wealth and state variables are important factors which determine the value of information.

2. A Valuation Equation for a Contingent Claim

The development of the preceding Subsection allows us to obtain a partial differential equation which will serve as a general equilibrium fundamental valuation equation for contingent claims. The equilibrium prices for contingent claims under incomplete information is specified in the following theorem.

THEOREM 9 : *The price of any contingent claim satisfies the partial differential equation of the form*

$$\tfrac{1}{2}(1-\gamma)^2 \underset{\mu(t)}{E} \left[\mathrm{var}(W \mid \varepsilon(t) ; \mu(t))\right] F_{WW} + (1-\gamma) \sum_{j=1}^{m} \left[\mathrm{cov}(W, \mu_j \mid \varepsilon(t) ; \mu(t))\right] F_{W\mu_j}$$

$$+ \tfrac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} \left[\mathrm{cov}(\mu_j, \mu_k) \mid \varepsilon(t) ; \mu(t))\right] F_{\mu_j\mu_k} + [rW - C] F_W + \sum_{j=1}^{m} F_{\mu_j} \Big[\sum_{k=1}^{m} \pi_{jk}\mu_k$$

$$-(1-\lambda)^{-1} \sum_{k=1}^{m} \left(\frac{-J_{W\mu_k}}{J_W}\right) \mathrm{cov}(\mu_j, \mu_k \mid \varepsilon(t) ; \mu(t))\Big] + F_t - rF + \left(\frac{1-\gamma}{1-\lambda}\right) \underset{\mu(t)}{E} \delta F + \sum_{j=1}^{m} F_{\psi_j} \dot{\Psi}_j, \quad (43)$$

where

$$r = a^{*\prime}\hat{\alpha} - (1-\gamma)\left(\frac{-J_{WW}}{J_W}\right) \underset{\mu(t)}{E} \frac{\mathrm{var}(W/\varepsilon(t) ; \mu(t))}{W} ;$$

$$\dot{\psi} = \pi\psi + \psi\pi' + AA' - \psi\theta'(SS')^{-1}\theta\psi'.$$

PROOF : Itô's lemma employed for $F(W, \mu, \psi, t)$ gives the drift term of dF. Equate the resulting term to the drift term of dF given by (33). Taking the conditional expectation for the resulting equation, conditional on $\mu(t)$ and substituting into (41) we obtain the desired result. *Q.E.D.*

The important difference between (20) and (43) is that the fundamental valuation equation (20) contains the term $\gamma[(1-\gamma)(-J_{WW}/J_W)var(W)+\sum_{j=1}^{m}(-J_{WQ_j}/J_W)\,cov(W,Q_j)F_W$ which cannot be found in (43), while the fundamental valuation partial differential equation (43) contains the term $F_\psi\square[\pi\psi+\psi\pi'+AA'-\psi\theta'(SS')^{-1}\theta\psi']$, which does not belong to (20). In addition, equation (43) is formed by taking the conditional expectations, conditional on $\mu(t)$. The term in question in (20) is adjusted by income tax rates due to the fact that the variance is the variance of optimally invested wealth and the covariances are those of wealth with the state variables. The term in (40), however, does not have the tax factors in it. Both terms represent risk premium. The premium in (20) comes from the uncertainty about both the fluctuations of optimally invested wealth and the comovement of wealth with changes in the state variables. On the other hand, the risk premium under the situations of incomplete information is derived from the drift terms of both the observable and unobservable processes of the state variables, the variance of the both processes, and the given value $\psi(t)$. This quantity is needed as the adjustment factor for the uncertainty caused by the conditional expectations taken in (43).

As in Section 3, we impose boundary conditions as follows :

$$F(W(t),\ \mu(T),\ \psi(T),\ T)=\Gamma(W(T),\ \mu(T),\ \psi(T)) \qquad \text{for } (W(T),\ \mu(T),\ \psi(T))\in\Delta\ ;$$

$$F(W(\tau),\ \mu(\tau),\ \psi(\tau),\ \tau)=\Xi(W(\tau),\ \mu(\tau),\ \psi(\tau),\ \tau) \qquad \text{for } (W(\tau),\ \mu(\tau),\ \psi(\tau))\in\partial\Delta. \quad (44)$$

To obtain the solution to (43) , we define the following two systems of stochastic differential equations. The System Ⅲ is

$$dW(t)=[a^{*\prime}\hat{\alpha}W(1-\gamma)-C]dt+(1-\gamma)a^{*\prime}HWdz_1(t)\ ;$$

$$d\mu(t)=\pi\mu dt+\psi\theta'(SS')^{-1/2}dz(t)\ ;$$

$$d\psi(t)=[\pi\psi+\psi\pi'+AA'-\psi\theta'(SS')^{-1}\theta\psi']dt, \tag{45}$$

The System Ⅳ is

$$dW(t)=[(1-\gamma)rW-C]at\times[1-\gamma]a^{*\prime}HWdz_1(t)\ ;$$

$$du(t)=[\pi\mu+V_{\mu\mu}\frac{J_{w\mu}}{J_w}]dt+\psi\theta'(SS')^{-1/2}\,dz(t)\ ;$$

$$d\psi(t)=[\pi\psi+\psi\pi'+AA'-\psi\theta'(SS')^{-1}\theta\psi']dt, \tag{46}$$

where

$$r=a^{*\prime}\hat{\alpha}+(1-\gamma)\ \frac{Jww}{Jw}\ \ \frac{\underset{E(t)}{E}[var(W\mid e(t)\ ;\ m(t))]}{W}$$

Using the boundary conditions and System Ⅲ, we can obtain the solution to (43), stand by the following theorem.

THEOREM 10 : *The price of a contingent claim, which is the unique solution to the fundamental valuation partial differential equation (43) with boundary conditions (44), is given by*

$$F(W,\ \mu,\ \psi,\ \lambda,\ \gamma,\ t,\ T)=$$

$$E[\Gamma(W(T),\ \mu(T),\ \psi(T))\cdot\exp\{-\int_t^T\beta(W(u),\ \mu(u),\ \psi(u),\ u)du\}\ I(\tau-\geq T)$$

$$+\Xi(W(\tau),\ \mu(t),\ \psi(\tau),\ \tau)\cdot\exp\{-\int_t^\tau\beta(W(u),\ \mu(u),\ \psi(u),\ u)du\}\ I(\tau<T)$$

$$+\int_t^{\tau\wedge T}\delta(W(s),\ \mu(s),\ \psi(s),\ s)\cdot\exp\{\int_t^s\beta(W(u),\ \mu(u),\ \psi(u),\ u)du\}ds] \tag{47}$$

where E denotes expectation with respect to System Ⅲ.

PROOF : Use Theorem 5.2 of Friedman(1975, vol.1, p.147) to equation (43)

Q.E.D.

In order to express the price of a contingent claim as a function of the known interest rate instead of the unknown β, we can use System Ⅳ and obtain the following theorem.

THEOREM 11 : *The unique solution to (40) with boundary condition (41) when the risk-free rate is used is also given by*

$$F(W, \mu, \psi, \lambda, \gamma, t, T)=$$

$$E[\Gamma(W(T), \mu(T), \psi(T)) \cdot \exp\{-\int_t^T r(W(u), \mu(u), \psi(u), u)du\} \ I(\tau \geq T)$$

$$+\Gamma(W(\tau), \mu(\tau), \psi(\tau), \tau) \cdot \exp\{-\int_t^\tau \beta(W(u), \mu(u), \psi(u), u)du\} \ I(\tau < T)$$

$$+\int_t^{\tau \wedge T} \delta(W(s), \mu(s), \psi(s), s) \cdot \exp\{\int_t^s r(W(u), \mu(u), \psi(u), u)du\}ds] \qquad (48)$$

where we take the expectations with respect to System Ⅳ.

PROOF : Use Theorem 5.2 of Friedman(1975, vol.1, p.147) to equation (44).

Q.E.D.

Equations (47) and (48) reflect μ and ψ instead of Q in equations (23) and (24). Otherwise, they are the same. These equations ensure that both the fundamental valuation equations have unique solutions if we can adequately and appropriately specify the utility function, the boundary conditions, and the state variables. No explicit terms about taxes can be found in both (47) and (48). Taxes are implicitly reflected in the expectation and the definition of β and r.

The fundamental valuation partial differential equation (43) can give more useful information about the process of determining the prices for the contingent claims. The solution to (43) can be interpreted in terms of marginal-utility-weighted expected values.

Let

$$M' = [(\frac{-J_{WW}}{J_W}), (\frac{-J_{WQ1}}{J_W}), \cdots, (\frac{J_{WQm}}{J_W})]$$

and

$$\Sigma=\begin{bmatrix} a^{*\prime}HW(1-\gamma) \\ S \end{bmatrix}$$

Assume that there exist $\tilde{y}<0$ and $Y>0$ such that $E_{W,Q,t}\ \exp(Y \mid M'(S)\Sigma(S) \mid^2 \leq \tilde{y}$ for all s, $t \leq s \leq T$. Then we have the following theorem.

THEOREM 12 : *The price of any contingent claim that is a unique solution to the fundamental valuation equation (21) , is given by*

$$F(W,\ Q,\ \lambda,\ \gamma,\ t,\ T,)=\underset{W,Q,t}{E}\ [\Upsilon(W(T),\ Q(T))\cdot(\frac{J_W(W(T),\ Q(T),\ T)}{J_W(W(t),\ Q(t),\ t)})\ I(\tau\geqq T)$$

$$+\Phi(W(\tau),\ Q(\tau),\ \tau)\ (\frac{J_W(W(\tau),\ Q(\tau),\ \tau)}{J_W(W(t),\ Q(t),\ t)})\ I(\tau<T)$$

$$+\int_t^{\tau\wedge T}\delta(W(s),\ Q(s),\ s)\ (\frac{J_W(W(s),\ Q(s),\ s)}{J_W(W(t),\ Q(t),\ t)})\ ds], \qquad (49)$$

where E represents expectation with respect to System I.

PROOF : Recall from (13) that

$$dJ_W=[\tfrac{1}{2}(1-\gamma)^2]J_{WWW}var(W)+\tfrac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}J_{WQ_jQ_k}cov(Q_j,\ Q_k)+(1-\gamma)\sum_{j=1}^{m}J_{WWQ_j}cov(W,\ Q_j)$$

$$+\sum J_{WQ_j}\theta_j+J_{Wt}+J_{WW}(a^{*\prime}\alpha W(1-\gamma)-C)]dt+(1-\gamma)J_{WW}a^{*\prime}HWdz(t)+J'_{WQ}Sdz(t)$$

$$=(LJ_W+J_{Wt})dt+[(1-\gamma)J_{WW}a^{*\prime}HW+J'_{WQ}S]dz(t).$$

By Itô's lemma,

$$\frac{J_W(W(s),\ Q(s),\ s)}{J_W(W(t),\ Q(t),\ t)}=\exp(\log J_W[W(s),\ Q(s),\ s)-\log J_W(W(t),\ Q(t),\ t)]$$

$$=\exp[\int_t^S(L\ \log J_W(u)+\frac{\partial \log J_W(u)}{\partial t})\ du+\int_t^S(-M'\Sigma)dz(u)]$$

Now

$$L \log J_W = (\frac{1}{J_W}) LJ_W - ½(1-\gamma)^2 var(W) (\frac{-J_{WW}}{J_W})^2 - (1-\gamma) \sum cov(W, Q_j) (\frac{-J_{WQj}}{J_W}) (\frac{-J_{WW}}{J_W})$$

$$- ½ \sum\sum cov(Q_j, Q_k) (\frac{-J_{WQj}}{J_W}) (\frac{-J_{WQk}}{J_W})$$

$$= (\frac{1}{J_W}) LJ_W = ½ M' \sum\sum' M.$$

From Theorem 1, we know that

$$(\frac{1}{J_W}) (LJ_W + {}_{Wt}) = -r$$

Thus we have that

$$\frac{J_W(W(s), Q(s), s)}{J_W(W(t), Q(t), t)}$$

$$= [\exp(-\int_t^S r(W(u), Q(u), u) du] \cdot [\exp[\int_t^S (-M' \sum) dz(u) - \frac{1}{2} \int_t^S | M \sum |^2 du]. \quad (50)$$

From the Girsanov Theorem proved by Gihman and Skorlhod (1979, vol. III, pp. 250-251) it follows that equation (24) is the same as (50). *Q.E.D.*

Equation (50) provides another insight into the pricing of contingent claims. It states that the price of any contingent claim is equal to the expectation of the product of its tax-adjusted random amount, a time-discount factor, and a risk adjustment. The time-discount factor reflects the accumulated effects of locally anticipated percentage changes in the marginal utility of wealth. The risk-adjustment factor represents the accumulated effects of locally unanticipated percentage changes in the marginal utility of wealth. Both time and risk factors are incorporated in the model for the prices of contingent claims to be correctly determined in the continuous market.

If we employ the same procedures to the situations of incomplete information, we have the following theorem determining the price of any contingent claims. To do this, assume that there exist $\tilde{y}>0$ and $\tilde{Y}'>0$ such that $E_{W,\mu,\psi}[\exp(\tilde{y} \mid \tilde{Y} \mid^2) \leq \tilde{Y}'$ where

$$| \tilde{Y} |^2 = (\frac{J_{WW}}{J_W})^2 a^{*\prime} HH' a^{*\prime} W_2 + \frac{J_{W\mu}}{J_W} \psi \theta' (SS')^{-1} \theta \psi' \frac{J'_{W\mu}}{J_W}$$

Then we have the following theorem.

THEOREM 13 : *The price of any contingent claim, which is the solution to the partial differential equation (43) derived under the situation of partial information(incomplete information) is given by*

$$F(W, \mu, \psi, \lambda, \gamma, t, T)$$

$$=E[\Gamma(W(T), \mu(T), \psi(T)) \left(\frac{J_W(W(T)), \mu(T), \psi(T), T)}{J_W(W(t), \mu(t), \psi(t), t)}\right) I(\tau \geqq T)$$

$$+\Xi(W(\tau)), \mu(\tau), \psi(\tau)) \left(\frac{J_W(W(\tau)), \mu(\tau), \psi(\tau), \tau)}{J_W(W(t), \mu(t), \psi(s), s)}\right) I(\tau < T)$$

$$+\int_t^{\tau\wedge T} \delta(W(s), \mu(s), \psi(s), s) \left(\frac{J_W(W(s), \mu(s), \psi(s), s)}{J_W(W(t), \mu(t), \psi(t), t)}\right) ds] \qquad (51)$$

where the expectation is taken with respect to system III.

PROOF : Follow the same procedure applied to (40) as the proof of Theorem 12.

Q.E.D.

As noted previously, the tax effects are implicitly reflected in (50) and (51). The main difference between (50) and (51) is that equation (50) is a direct function of the state variables Q while equation (51) is a function of $\mu(t)$ and $\psi(t)$ which reveal the information of the state variables. This arises due to the fact that the underlying state variables are unobservable under the incomplete information structure but that there exists an obsrevable instrument that reveals partial information about the state variables, thus enabling investors to infer about the states of nature from the partial information revealed. The conditional process of the state variables, conditional on this instrument, provides the information about the state variables that are updated continuously.

V. Conclusion

In this paper we have developed a general equilibrium model of asset pricing with taxes under the situation of incomplete or partial knowledge of state variables as well as in the presence of knowledge of state variables. We have derived the equation for the equilibrium riskless return which is endogenously determined in the economy. This gives more insights into the process of determining interest rates than the Fisher model.

The main results of the paper are the fundamental valuation partial differential equations for asset prices in the presence of differential tax rates. The solutions to these partial differential equations determine the general equilibrium asset prices under the nonlinear tax system as a function of the underlying wealth and state varlables in the economy. The value(price) of information has also obtained.

Two types of uncertainty-information structure have been analyzed . One is a noisy information structure and the other is an incomplete or partial information structure. These two structures determine the two fundamental valuation partial differential in the presence of differential tax rates, one equation for one information structure. The combination of the solutions to the fundamental valuation equations with these information structures can provide answers to a wide variety of questions about the stochastic structure of asset prices and tax effects on them. The taxes have impact on the asset prices and the riskless interest rate. The riskless return is hedged against the unfavorable shifts in the investment opportunity set through income tax rates.

The intertemporal general equilibrium model of asset pricing derived in the presence of differential tax rates can be extended to the case of a heterogeneous information structure. Under the rational expectations equilibrium concept, this extension would give interesting results about the aggregation across all investors of heterogeneous private information held by individuals and the relevation of private information through the prices of capital assets. Then this model would determine an equation for the cost of information acquisition, which is of great interest to all investors. Furthermore, it would clarify in some detail the efficient market hypothesis.

Footnotes

1) Each specific information structure will be described in detail in Sections 3 and 4, respectively.
2) A Wiener process or a Brownian motion process $\{z(t)\}$ is a stochastic process on a probability space (Ω, B, P) with the following properties.
   (i) The process starts at 0, i.e., $z(0)(\omega)=0$ a.s.
   (ii) The increments of the process $z(t)-z(s)$ are independent random variable, i.e.,

$$P[z(t_j)-z(T_{(j-1)} \in \Lambda_j \text{ for } j\leq n]=\prod_{j\leq n} P[z(t_j)-z(t_{j-1}) \in \Lambda_j].$$

   (iii) The increment $z(t)-z(s)$, $t\geq s$ is normally distribued with mean 0 and variance $(t\text{-}s)$.
   (iv) For each $\omega \in \Omega$, $z(t)(\omega)$ is continuous in t for $t\geq 0$.
   A stochastic process is said to be continuous if its possible realizations or sample paths are continuous with probability one.
3) Sudden changes in the economic variables can be introduced into the model by using the Poisson process. Merton(1976) and Cox and Ross(1976) used both a Wiener process and a Poisson process to develop the option pricing model.
4) For the detailed explanation of the form of the budget equation, see Merton(1971, 1973).
5) Suppose we have a stochastic dynamic programming problem

$$J(X, t)=\text{Max } E\int_t^T H(X(t), v(t), t)\, dt$$

s.t.

$$dX=\alpha(X(t), v(t), t)+\sigma(X(t), v(t), t)dz(t_j)$$

$$X(0)=X_0\,;$$

$$X(T)=X_T\,;$$

$$\{v(t)\} \in V,$$

where state variables $X(t)$ is continuous and control variable $v(t)$ is piecewise continuous. V is an admissible control. Then the Bellman equation is

$$0=\mathrm{Max}\left[H(\cdot)+\frac{\partial J}{\partial t}dt+\sum_i \frac{\partial J}{\partial x_j}dx_i(\cdot)+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_{ij}(\cdot)\right].$$

For the details of the stochastic dynamic programming and existence of the optimal control, see Kushner(1967, Chapter Ⅳ).

6) As described in Section 2, if those conditions are satisfied, we can have the unique optimal solution to our decision problem.

7) For a survey of methods, see Brennan and Schwartz(1978) and Geske and Shastri (1985).

8) Itô's lemma is a stochastic differential rule for a function of a diffusion process. The stochastic differential equations and the stochastic integrals do not obey the rule of ordinary differential calculus. The main reason for this is that the increment of a Wiener process dz(t) is of magnitude dt in the mean square metric.

Let *W(t,x)* be of class[1,2] where *x* is an *n*-dimensional vector. Let the *n*-dimensional vector *x* satisfy the stochastic differential equationwhere

$$dx=\alpha(x,\ t)+\sigma(x,\ t)dz$$

where stochastic process $\{z(t),\ t\in T\}$ is a Wiener process with incremental covariance *vdt*. Then Itô's lemma states that *W(t,x)* satisfies the following differential equation :

$$dW=\frac{\partial W}{\partial t}dt+\sum_1^n\frac{\partial W}{\partial x_i}dx_i+\frac{1}{2}\sum_{i,j,k=1}^n\frac{\partial^2 W}{\partial x_i\partial x_j}\sigma_{ik}\sigma_{jk}dt$$

$$=\left[\frac{\partial W}{\partial t}+\sum_1^n\frac{\partial W}{\partial x_i}\alpha_i+\frac{1}{2}\sum_{i,j,k=1}^n\frac{\partial^2 W}{\partial x_j\partial x_j}\sigma_{ik}\sigma_{jk}\right]dt+\sum_1^n\frac{\partial W}{\partial x_i}(\sigma dz)_i$$

For proof, see Lipster and Shiryayev(1977, vol.1, pp.118-122).

9) Consider the situations where a firm issues callable bonds. The Theorem states that the maturity data *T* and the calable time must be specified in the contract. Investors have knowledge about the price of the bond that will be called at time τ and the price of the bond that will not.

As another example, consider a time deposit in a bank. Suppose that the deposit is insured. Then the second term of the Theorem guarantees that the depositors will receive Φ with no uncertainty when the bank goes into bankruptcy. If the marginal consumption is zero, then Φ discounted at the discount rate is a sure thing for the depositors to receive under any situation. As a consequence the Theorem may imply that perhaps the insurance policy is the best policy if and only if the regulatory agencies believe that the failure of a single bank would have tremendous impact on the economy and that any measures to prevent such a thing must be taken at any costs. The Theorem clearly shows that the contract between the bank and the depositors is necessary and sufficient conditions for both parties to maximize their own welfare. Depositors will get maximum possible payment if they retain their deposits in the bank until the maturity date. On the other hand, if they make premature withdrawals, then they must compensate the bank for their early actions due to the premature termination of the contract.

10) A singal process describes the state of a system, but cannot be observed directly. Instead, we can only observe some noisy function $\{y(t)\}$ of $\{x(t)\}$

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