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Optimal Control Policy for Replacements Involving Two Machines and One Repairman

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ABSTRACT

There has been a great deal of research dealing with the optimal replacement of stochastically deteriorating equipment and research in queueing systems with a finite calling population. However, there has been a lack of research which combines these two areas to deal with optimal replacement for a fixed number of machines and a limited number of repairmen.

In this research, an optimal control policy for replacement involving two machines and one repairman is developed by investigating a class of age replacement policies in the context of controlling a $G/M/1$ queueing system with a finite calling population. The control policy to be imposed on this problem is an age-dependent control policy, described by the control limit t^*. The control limit is operational only when the repairman is idle; that is, if both machines are working, as soon as a machine reaches the age t^* it is taken out of service for replacememt.

We obtain the ε - optimal control age which will minimize the long-run average system cost. An algorithm is developed that is applicable to general failure time distributions and cost functions. The algorithm does not require the condition of unimodality for implementation.

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1. Introduction

Much research has been done dealing with the optimal replacement (maintenance) of systems that are subject to stochastic failures. The initial replacement problems dealt with the question of determining the optimal replacement age of equipment when the two major cost considerations were the replacement cost incurred during a planned replacement and the larger cost incurred when the replacement was made due to a machine failure.

McCall [1965], Jardine [1973], Pierskalla and Voelker [1976], and Sherif and Smith [1981] have extensively surveyed the works published up to 1981. Recently Valdez-Flores and Feldman [1989] have presented a comprehensive survey of replacement models published subsequent to Pireskalla and Voelker's survey.

Much of the early literature on replacement problems assumed that sufficient repair crews were available so that replacment could begin immediately after a failure. However, there are often situations in which some of the failed machines must wait to be fixed or replaced. In such circumstances each machine can be found in one of the following three cases: running, idle and waiting for repair, or idle during repair. When this situation is formulated as a queueing problem in which a repair crew is the server and the machines are customers, it is referred to as the machine interference problem.

Recently, Stecke and Aronson [1985] extensively surveyed work dealing with the machine interference problem published up to 1983. Most of the machine interference models presented in their survey can be classified into two groups according to the types of machine failure and repair: deterministic models and probabilistic models. Bunday and Scraton [1980] derived the steady-state probabilities for the $G/M/r$ machine inteference model, and they are identical to the steady-state probabilities for the $M/M/r$ machine interference model case as long as their means are equal. Hence much of the work completed for the Markov machine interference problem would seem to apply to the general failure distribution case. (Keller [1980] uses a reversibility argument to indicate that the steady-state probabilities for the finite population $G/M/r$ queue are insensitive to the distribution. However, we focus on the work of Bunday and Scraton, because we will take advantage of their methodology.)

Even though there is a large body of literature for optimal replacement models and machine interference models, there appears to be no research results when these two areas are combined. Our purpose is to investigate optimal replacement for the two-machine interference problem. By investigating the two-machine case, we are restrictive in the generality of the immediate application of this research, but we are able to develop an approach for optimal replacement in a setting that has not received much attention.

Section 2 contains the problem statement. The derivation of the steady-state probability and expected number of visits to each state are discussed in Sections 3 to 5. Finally, in Section 6, we derive an algorithm for finding the optimal replacement policy. We illustrate the algorithm by applying it to a problem where the failure times are Phase-type distribution.

2. Problem statement

Consider a system consisting of two identical, independent machines under the care of one repairman. The time to failure of each machine is a random variable governed by a general distribution G. When a machine fails, the machine is replaced and the replacement (or repair) time is exponentially distributed with mean time $1/\mu$. The control policy to be imposed on this problem is an age-dependent control policy described by the control limit t^*. The control limit is operational only when the repairman is idle, that is, if both machines are working, as soon as a machine reaches the age t^* it is taken out of service for replacement. When a machine is taken out of service due to its age reaching the control limit t^*, it is replaced with a new, probabilistically identical machine. This replacement time is also exponentially distributed with mean $1/\mu$. If the repairman is busy, the control limit is ignored and the machine is replaced only when it fails.

Whenever a machine is taken out of service due to its age reaching the control limit t^* (i.e., a planned replacement), a cost c_p is incurred. A cost c_f (with $c_f > c_p$) is incurred when a machine is replaced because of failure. Furthermore, there is an addtitional cost of c_d incurred per unit time while a machine is down.

Our problem is to find the control limit that minimizes the long-run average cost. This system cost, for a fixed control limit t^*, is given by

$$\bar{C}(t^*) = \lim_{t\to\infty}\frac{1}{t}\ \{c_pE[T_s(t)] + c_fE[N_f(t)] + c_pE[N_p(t)]\}, \tag{2.1}$$

where $T_s(t)$ is the machine down time during the interval [0, t], $N_f(t)$ is the number of machine failures during the interval [0, t], and $N_p(t)$ is the number of machine planned replacements during the interval [0, t]. (Note that the right-hand-side of Eq. (2.1) is a function of t^* although for notational simplicity we do not explicitly show it.)

The typical state space used to describe a two-machine replacement problem is {0, 1, 2} where a state represents the number of working machines. Howerver, for this problem, it is necessary to know the reason for a machine being taken out of service. Before descibing our state space, consider the ways that the system may evolve to the state of having only one machine working. There are four possibilities : (1) both machines were working and one machine fails, (2) both machines were working and the older machine reaches the control limit, (3) no machines were woking and a machine replacement was finished, and (4) one machine whose age was greater than the control limit was working at the time that replacement was finished. If all that is known about the system is the number of working machines, then being in state 1 does not contain enough information to determine costs. Thus the state space will have to be expanded.

Let E denote the state space, where E is defined by

$$E = \{(0),\ (2),\ (1,f),\ (1,p),\ (1,p,0)\}.$$

State (0) represents no machines working; state (2) represents two mahines working; state (1, f) represents one machine working such that the transition into (1, f) was caused by a machine failure and not by the control limit; state (1, p) was planned replacement (i.e., caued by the control limit) from state (2); and, state (1, p, 0) was a planned replacement from state (1, f), (1, p), or (1, p, 0). The machine working at the arrival epoch of state (1, p, 0) is a brand new machine (see Figure 2.1).

The first part of the system cost is the cost incurred due to machine unavailability (machine down time cost). Let $q_i(t^*)$ represent the steady-state probability of state $i \in E$ under control limit t^*. Then the expected machine down cost per unit time is

$$\lim_{t\to\infty}\frac{1}{t}\{c_dE[T_s(t)]\} = c_d\{2q_0(t^*) + q_{1,f}(t^*) + q_{1,p}(t^*) + q_{1,p,0}(t^*)\}. \tag{2.2}$$

The second part of the system cost (machine failure cost) is the product of c_f and the expected number of machine failures per unit time. All visits to state (0) result from machine failures. By the definition of state (1, f), visits (1, f) are due to failure from state (2) or are from state (0). Since the number of visits from stake (0) must equal the number of visits to state (0), we have that all visits to state (1, f) are (eventually) from failures. Thus, in the long run, the number of visits to state (1, f) per unit time equals the number of failures (see Figure 2.1). Let $\eta_i(t^*)$ be the number of visits to state $i \in E$ per unit time under the control limit t^*. Then the expected machine failure cost per unit time is

$$\lim_{t \to \infty} \frac{1}{t}\{c_f E[N_f(t)]\} = c_f\{\eta_1(t^*)\}. \tag{2.3}$$

The third part of the system cost (machine planned replacement cost) is the product of c_p and the expected number of machine planned replacements per unit time. The expected number of machine planned replacements is the sum of expected number of visits to state (1, p) and (1, p, 0). then the expected machine planned replacement cost per unit time is

$$\lim_{t \to \infty} \frac{1}{t}\{c_p E[N_p(t)]\} = c_p\{\eta_{1,p}(t^*) + \eta_{1,p,0}(t^*)\}. \tag{2.4}$$

From Eqs. (1.1), (2.2), (2.3), and (2.4), the cost function is

$$\begin{aligned} \tilde{C}(t^*) &= c_d\{2q_0(t^*) + q_{1,f}(t^*) + q_{1,p}(t^*) + q_{1,p,0}(t^*)\} + c_f\{\eta_{1,f}(t^*)\} \\ &+ c_p\{\eta_{1,p}(t^*) + \eta_{1,p,0}(t^*)\}. \end{aligned} \tag{2.5}$$

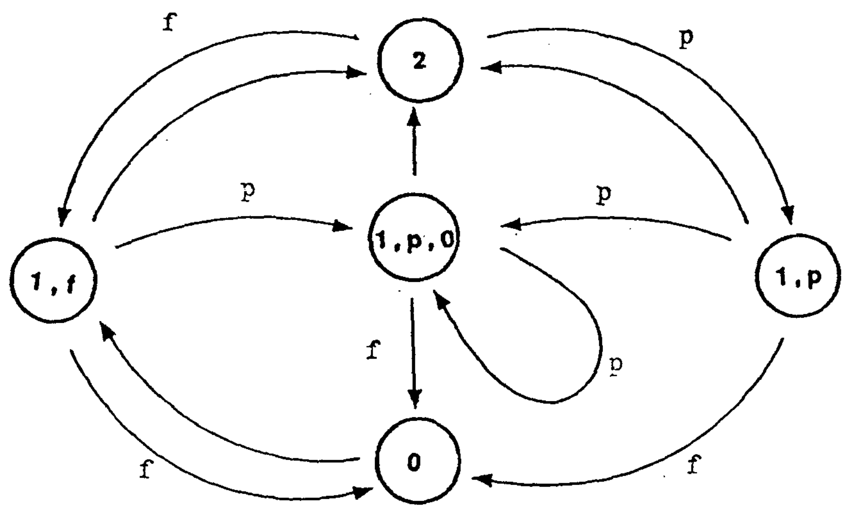

Figure 2.1. Feasible state transitions diagram. The arrows with a f represent transition due to machine failure.

For $t \in R_+ = (0, \infty)$, let Y_t denote the state of system for fixed t, then $Y = \{Y_t; t \in R_+\}$ is a stochastic process with a continuous time parameter. We shall assume that process $\{Y_t; t \in R_+\}$ is ergodic; therefore, the limiting probability that the process will be in state $i \in E$ at a arbitrary point of time exists. For fixed state $i \in E$, let the length of successive sojourn times of state i be denoted by $\tau_{i,1}$, $\tau_{i,2}$, $\cdots$, Let $N_i(t)$ be the random variable representing the number of vsits to state i during the time interval $(0, t]$. By the ergodic property (Parzen[62], pp.72-75), the steady-state probability of state i under control limit t^* has the following property :

$$\lim_{t\to\infty} \frac{\sum_{j=1}^{N_i(t)} \tau_{i,j}}{t} = q_i(t^*). \qquad (2.6)$$

Define the mean sojourn time of state i (namely, $E[\tau_{i,j}]$) under control limit t^* as $m_i(t^*)$. Then,

$$E[\sum_{j=1}^{N_i(t)} \tau_{i,j}] = E[N_i(t)] m_i(t^*). \qquad (2.7)$$

By definition, we also have that

$$\lim_{t\to\infty}\frac{1}{t}E[N_i(t)] = \eta_i(t^*). \tag{2.8}$$

Therefore combining Eqs. (2.6) to (2.8), the expected number of visits to state i per unit time under control limit t^* is given by

$$\eta_i(t^*) = \frac{q_i(t^*)}{m_i(t^*)}. \tag{2.9}$$

The problem of calculating the cost Eq. (2.5) is now reduced to the determination of the steady-state probabilities, q_i, and the mean sojourn times, m_i for $i \in E$.

3. Derivation of steady-sate equations

Let G the distribution function of the failure time for a machine, with $\bar{G} = 1 - G$ and g its density. We derive the steady-state equation by adding a control limit to the machine interference problem of Bunday and Scraton [1980] restricted to two machines. Their approach uses a "delta-t" argument and includes derivatives of the functions Q_1, Q_2, and G. Since imposing a control on G results in a discontinuous function, we approximate the control function with a differentiable function that is arbitrarily close to control. That is for a small $\epsilon > 0$, define the following function :

$$\bar{G}_\epsilon(t) = \begin{cases} \bar{G}(t) & \text{for } t < t^*, \\ \varphi(t) & \text{for } t^* \le t < t^* + \epsilon, \\ 0 & \text{for } t \ge t^* + \epsilon. \end{cases}$$

where φ is a (steeply decreasing) function such that $\bar{G}_\varepsilon$ is continuous and once defferentiable. Thus, $\bar{G}_\varepsilon$ limits to the complement of the failure time distribution under the control limit t^* as ε approaches zero. Also, let g_ε denote the negative of the derivative of $\bar{G}_\varepsilon$.

Let $Q_2(t_1, t_2; t)\, dt_1 dt_2$ denote the probability that at time t both machines are operating and the age of one machine is in the interval $(t_1, t_1 + dt_1)$ and the age of of the second machine is in the interval $(t_1, t_2 + dt_2)$. Let $Q_{1,f}(t_1; t)\, dt_1$ be the probability that any one particular machine is operating under state (1, f) and the age of the operating machine is the interval $(t_1, t_1 + dt_1)$. Let $Q_{1,p}(t_1; t)\, dt_1$ be the probability that any one particular machine in running under state (1, p, 0) and its age is in the interval $(t_1, t_1 + dt_1)$. Let $Q_{1,p,0}(t_1; t) dt_1$ be the probability that any one particuar machine is running under state (1, p, 0) and its age is in the interval $(t_1, t_1 + dt_1)$. Finally, the probability that no machines are operating at time t is given by $Q_0(t)$. (The functions Q_2, $Q_{1,f}$, $Q_{1,p}$, $Q_{1p,0}$ and Q_0 are dependent on ε but we have not in cluded ε in the subscript for ease of notation.)

Since repair times are exponentially distributed, we do not need to consider the current state of each repair in progress at time t. Hence, the state of the system is adequately described by the ages of the machines and the time t to predict its future behavior. By relating the state of the system at time $t + \triangle t$ to the state at time t, we obtain a set of equations. In the following, μ is the repair rate, $g(t)/\bar{G}(t)$ is the failure rate for a machine of age t when only one machine is operating, and $g_w(t)/\bar{G}_w(t)$ is the failure rate for a machine of age t when both machines are operating. Thus, we have

$$Q_2(t_1 + \Delta t, t_2 + \Delta t; t + \Delta t) = Q_2(t_1, t_2; t)\left[1 - \Delta t\frac{g_\epsilon(t_1)}{\bar{G}_\epsilon(t_1)} - \Delta t\frac{g_\epsilon(t_2)}{\bar{G}_\epsilon(t_2)}\right] \qquad (3.1)$$

for $0 < t_1,\ t_2 < t^* + w$,

$$\begin{aligned} Q_{1,f}(t_1 + \Delta t; t + \Delta t) &= Q_{1,f}(t_1; t)\left[1 - \Delta t\frac{g(t_1)}{\bar{G}(t_1)} - \mu\Delta t\right] \\ &+ \Delta t\int_0^{t^*} Q_2(t_1, s; t)\frac{g_\epsilon(s)}{\bar{G}_\epsilon(s)}ds \quad \text{for} \quad t_1 > 0, \end{aligned} \qquad (3.2)$$

$$\begin{aligned} Q_{1,p}(t_1 + \Delta t; t + \Delta t) &= Q_{1,p}(t_1; t)\left[1 - \Delta t\frac{g(t_1)}{\bar{G}(t_1)} - \mu\Delta t\right] \\ &+ \Delta t\int_{t^*}^{t^*+\epsilon} Q_2(t_1, s; t)\frac{g_\epsilon(s)}{\bar{G}_\epsilon(s)}ds \quad \text{for} \quad t_1 > 0, \end{aligned} \qquad (3.3)$$

$$Q_{1,p,0}(t_1+\Delta t;t+\Delta t)=Q_{1,p,0}(t_1;t)\left[1-\Delta t\frac{g(t_1)}{\bar{G}(t_1)}-\mu\Delta t\right] \quad \text{for } t_1>0, \quad (3.4)$$

$$Q_0(t+\Delta t)=$$

$$Q_0(t)(1-\mu\Delta t)+2\Delta t\int_0^\infty (Q_{1,f}(s;t)+Q_{1,p}(s;t)+Q_{1,p,0}(s;t))\frac{g(s)}{\bar{G}(s)}ds. \quad (3.5)$$

When a repair is completed in $(t, t+\Delta t)$, we get another set of equations as following;

$$\Delta t Q_2(t_1+\Delta t,0;t+\Delta t)=$$

$$\begin{cases} \mu\Delta t(Q_{1,f}(t_1;t)+Q_{1,p}(t_1;t)+Q_{1,p,0}(t_1;t)) & \text{for } t_1<t^*+\epsilon, \\ 0 & \text{for } t_1\ge t^*+\epsilon, \end{cases} \quad (3.6)$$

$$2\Delta t Q_{1,f}(0;t+\Delta t)=\mu\Delta t Q_0(t), \quad (3.7)$$

$$\Delta t Q_{1,p,0}(0;t+\Delta t)=\mu\Delta t\int_{t^*}^\infty (Q_{1,f}(s)+Q_{1,p}(s)+Q_{1,p,0}(s))ds. \quad (3.8)$$

In all of Eqs. (3.1) to (3.6), the values of t_i are greater than zero. As $t\to\infty$, $Q_2(t_1,t_2;t)$, $Q_{1f}(t_1;t)\to Q_{1f}(t_1)$; $\to Q_1,f(t_1)$; $Q_{1,p}(t_1;t)\to Q_{1,p}(t_1)$; $Q_{1,p,0}(t_1;t)\to Q_{1,p,0}(t_1)$; $Q_{0(t)}\to Q_0$. Divide Eqs. (3.1) to (3.4) by Δt, and let $\Delta t\to 0$ and take the limit as $t\to\infty$. Then, the steady-state equations desribing the system are obtained as following :

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)Q_2(t_1,t_2)=-Q_2(t_1,t_2)\left[\frac{g_\epsilon(t_1)}{\bar{G}_\epsilon(t_1)}+\frac{g_\epsilon(t_2)}{\bar{G}_\epsilon(t_2)}\right] \quad (3.9)$$

for $0<t_1,\ t_2<t^*+\epsilon$,

$$\frac{\partial}{\partial t_1}Q_{1,f}(t_1)=-Q_{1,f}(t_1)\left(\frac{g_1(t_1)}{\bar{G}(t_1)}+\mu\right)+\int_0^{t^*}Q_2(t_1,s)\frac{g_\epsilon(s)}{\bar{G}_\epsilon(s)}ds \quad \text{for } t_1>0, \quad (3.10)$$

$$\frac{\partial}{\partial t_1}Q_{1,p}(t_1)=-Q_{1,p}(t_1)\left(\frac{g_1(t_1)}{\bar{G}(t_1)}+\mu\right)+\int_{t^*}^{t^*+\epsilon}Q_2(t_1,s)\frac{g_\epsilon(s)}{\bar{G}_\epsilon(s)}ds \quad \text{for } t_1>0, \quad (3.11)$$

$$\frac{\partial}{\partial t1}Q_{1,p,0}(t_1)=-Q_{1,p,0}(t_1)\left(\frac{g(t_1)}{\bar{G}(t_1)}+\mu\right) \quad \text{for } t_1>0. \quad (3.12)$$

And from Eqs. (3.5) to (3.8) we get another set of equations :

$$0 = -\mu Q_0 + 2\int_0^{\infty}(Q_{1,f}(s) + Q_{1,p}(s) + Q_{1,p,0}(s))\frac{g(s)}{\bar{G}(s)}ds. \tag{3.13}$$

$$Q_2(t_1,0) = \begin{cases} \mu(Q_{1,f}(t_1) + Q_{1,p}(t_1) + Q_{1,p,0}(t_1)) & \text{for } t_1 < t^* + \epsilon, \\ 0. & \text{for } t_1 \geq t^* + \epsilon, \end{cases} \tag{3.14}$$

$$2Q_{1,f}(0) = \mu Q_0. \tag{3.15}$$

$$Q_{1,p,0}(0) = \mu\int_{t^*}^{\infty}(Q_{1,f}(s) + Q_{1,p}(s) + Q_{1,p}(s))ds. \tag{3.16}$$

To solve this system of equations, let $Q_2(t_1,t_2) = R_2(t_1,t_2)\bar{G}_\epsilon(t_1)\bar{G}_\epsilon(t_2)$; $Q_{1,f}(t_1) = R_{1,f}(t_1)\bar{G}(t_1)$; $Q_{1,p}(t_1) = R_{1,p}(t_1)\bar{G}(t_1)$; $Q_{1,p,0}(t_1) = R_{1,p,0}(t_1)\bar{G}(t_1)$; $Q_0 = R_0$. The derivative of $Q_2, Q_{1,f}, Q_{1,p}, Q_{1,p,0}$, and Q_0 can be expressed by

$$\frac{\partial}{\partial t_i}Q_2(t_1,t_2) = \frac{\partial}{\partial t_i}R_2(t_1,t_2)\bar{G}_\epsilon(t_1)\bar{G}_\epsilon(t_2) + R_2(t_1,t_2)\frac{\partial}{\partial t_i}(\bar{G}_\epsilon(t_1)\bar{G}_\epsilon(t_2)) \tag{3.17}$$

for $i = 1,2$ and

$$\frac{\partial}{\partial t_1}Q_{1,f}(t_1) = \frac{\partial}{\partial t_1}R_{1,f}(t_1)\bar{G}(t_1) - R_{1,f}(t_1)g(t_1), \tag{3.18}$$

$$\frac{\partial}{\partial t_1}Q_{1,p}(t_1) = \frac{\partial}{\partial t_1}R_{1,p}(t_1)\bar{G}(t_1) - R_{1,p}(t_1)g(t_1), \tag{3.19}$$

$$\frac{\partial}{\partial t_1}Q_{1,p,0}(t_1) = \frac{\partial}{\partial t_1}R_{1,p,0}(t_1)\bar{G}(t_1) - R_{1,p,0}(t_1)g(t_1). \tag{3.20}$$

By substituting Eqs. (3.17) to (3.20) into Eqs. (3.9) to (3.12), the following equations are obtained in terms of R_2, $R_{1,f}$, $R_{1,p}$, $R_{1p,0}$ and R_0.

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)R_2(t_1,t_2) = 0 \quad \text{for } t_1,\ t_2 < t^* + \epsilon, \tag{3.21}$$

$$\frac{\partial}{\partial t_1}R_{1,f}(t_1) = -\mu R_{1,f}(t_1) + \int_0^{t_*} R_2(t_1,s)g(s)ds \quad \text{for } t_1 > 0, \tag{3.22}$$

$$\frac{\partial}{\partial t_1}R_{1,p}(t_1) = -\mu R_{1,p}(t_1) + \int_{t_*}^{t^*+\epsilon} R_2(t_1,s)\frac{G_\epsilon(t_1)}{\bar{G}(t_1)}g_\epsilon(s)ds \quad \text{for } t_1 > 0, \tag{3.23}$$

$$\frac{\partial}{\partial t_1}R_{1,p,0}(t_1) = -\mu R_{1,p,0}(t_1) \quad \text{for } t_1 > 0. \tag{3.24}$$

And from Eqs. (3.13) to (3.16),

$$0 = \mu R_0 - 2\int_0^\infty (R_{1,f}(s) + R_{1,p}(s) + R_{1,p,0}(s))g(s)ds, \tag{3.25}$$

$$R_2(t_1,0) = \begin{cases} \mu[R_{1,f}(t_1) + R_{1,p}(t_1) + R_{1,p,0}(t_1)] & \text{for } t_1 < t^* + \epsilon, \\ 0 & \text{for } t_1 \geq t^* + \epsilon, \end{cases} \tag{3.26}$$

$$R_{1,f}(0) = \frac{1}{2}\mu R_0, \tag{3.27}$$

$$R_{1,p,0}(0) = \mu\int_{t^*}^\infty (R_{1,f}(s) + R_{1,p}(s) + R_{1,p,0}(s))\bar{G}(s)ds. \tag{3.28}$$

4. The solution of steady - state equations

To solve the set of differential Eqs. (3.21) to (3.28), first, solve Eq. (3.21). The equation has a general solution, $R_2(t_1,t_2) = u(t_1 - t_2)$ where u is an arbitrary function. Thus, a constant is also the solution to Eq. (3.21). As $\epsilon \to 0$, the solution of the differential equation is given by

$$R_2(t_1,t_2) = \begin{cases} \kappa(t^*) & \text{for } t_1,\ t_2 < t^*, \\ 0 & \text{otherwise}, \end{cases} \tag{4.1}$$

where $k(t^*)$ is a constant to be determined by boundary conditions.

For $t < t^*$, $\bar{G}_\epsilon(t) = \bar{G}(t)$, $\int_{s=0}^{t^*} g_\epsilon(s)ds = G(t^*)$,

and $\int_{s=t^*}^{t^*+\epsilon} g_\epsilon(s)ds = \bar{G}(t^*)$.

As $\epsilon \to 0$, the solutions of the differential Eqs. (3.22) to (3.24) are given as following :

$$R_{1,f}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu} G(t^*) + a_0 e^{-\mu t_1} & \text{for } t_1 < t^*, \\ a_1 e^{-\mu t_1} & \text{for } t_1 \geq t^* \end{cases} \tag{4.2}$$

$$R_{1,p}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu} \bar{G}(t^*) + b_0 e^{-\mu t_1} & \text{for } t_1 < t^*, \\ b_1 e^{-\mu t_1} & \text{for } t_1 \geq t^*, \end{cases} \tag{4.3}$$

$$R_{1,p,0}(t_1) = c_0 e^{-\mu t_1} \qquad \text{for } t_1 > 0. \tag{4.4}$$

We assume that functions $Q_{1,f}(t_1)$ and $Q_{1,p}(t_1)$ are bounded and functions $g(t_1)$, $g_\varepsilon(t_1)$, $\bar{G}(t_1)$ and $\bar{G}_\varepsilon(t_1)$ are bounded and continuous functions over an admissible control zone. Set $S(t^*) = \int^{\infty} \mu e^{-\mu s} G(s) ds$. Then as seen in Appendix A, R_0 and the coefficients a_0, a_1, b_0, b_1 and c_0 of Eqs. (4.2) to (4.4) are obtained as $\varepsilon \rightarrow 0$.

$$R_0 = \frac{2\kappa(t^*)}{\mu^2}(1 - e^{\mu t^*} S(t^*)), \tag{4.5}$$

$$a_0 = \frac{\kappa(t^*)}{\mu}(\bar{G}(t^*) - e^{\mu t^*} S(t^*)), \tag{4.6}$$

$$a_1 = \frac{\kappa(t^*)}{\mu}(\bar{G}(t^*) + e^{\mu t^*}(G(t^*) - S(t^*))), \tag{4.7}$$

$$b_0 = -\frac{\kappa(t^*)}{\mu}\bar{G}(t^*), \tag{4.8}$$

$$b_1 = \frac{\kappa(t^*)}{\mu}(e^{\mu t^*}\bar{G}(t^*) - \bar{G}(t^*)), \tag{4.9}$$

$$c_0 = \frac{\kappa(t^*)}{\mu} e^{\mu t^*} S(t^*). \tag{4.10}$$

By substituting the coefficients a_0, a_1, b_0, b_1 and c_0 into Eqs. (4.2) to (4.4), we get the solution of the steady-state equations as following :

$$R_2(t_1,t_2) = \begin{cases} \kappa(t^*) & \text{for } t_1 < t^*, \\ 0 & \text{otherwise,} \end{cases} \tag{4.11}$$

$$R_{1,f}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu}(G(t^*) + \bar{G}(t^*)e^{-\mu t_1} - S(t^*)e^{\mu(t^*-t_1)}) & \text{for } t_1 < t^*, \\ \frac{\kappa(t^*)}{\mu}(\bar{G}(t^*) + G(t^*)e^{\mu t^*} - S(t^*)e^{\mu t^*})e^{-\mu t_1} & \text{for } t_1 \geq t^*, \end{cases} \tag{4.12}$$

$$R_{1,p}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu}\bar{G}(t_*)(1 - e^{-\mu t_1}) & \text{for } t_1 < t^*, \\ \frac{\kappa(t^*)}{\mu}\bar{G}(t^*)(e^{\mu t^*} - 1)e^{-\mu t_1} & \text{for } t_1 \geq t^*, \end{cases} \tag{4.13}$$

$$R_{1,p,0}(t_1) = \frac{\kappa(t^*)}{\mu}e^{\mu(t^*)}S(t^*)e^{-\mu(t_1)} \quad \text{for } t_1 \geq 0, \tag{4.14}$$

$$R_0 = \frac{2\kappa(t^*)}{\mu^2}(1 - e^{\mu t^*}S(t^*)). \tag{4.15}$$

This solution set satisfies all of Eqs. (3.21) to (3.28). Since these equations completly describe the system, this is the required solution. Thus, the basic probability density function for a particular state are obtained as following :

$$Q_2(t_1,t_2) = \begin{cases} \kappa(t^*)\bar{G}_\epsilon(t_1)\bar{G}_\epsilon(t_2) & \text{for } t_1 < t^*, \\ 0 & \text{otherwise,} \end{cases} \tag{4.16}$$

$$Q_{1,f}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu}(G(t^*) + \bar{G}(t^*)e^{-\mu t_1} - S(t^*)e^{\mu(t^*-t_1)})\bar{G}(t_1) & \text{for } t_1 < t^*, \\ \frac{\kappa(t^*)}{\mu}(\bar{G}(t^*) + G(t^*)e^{\mu t^*} - S(t^*)e^{\mu t^*})e^{-\mu t_1}\bar{G}(t_1) & \text{for } t_1 \geq t^*, \end{cases} \tag{4.17}$$

$$Q_{1,p}(t_1) = \begin{cases} \frac{\kappa(t^*)}{\mu}\bar{G}(t_*)(1 - e^{-\mu t_1})\bar{G}(t_1) & \text{for } t_1 < t^*, \\ \frac{\kappa(t^*)}{\mu}\bar{G}(t^*)(e^{\mu t^*} - 1)e^{-\mu t_1}\bar{G}(t_1) & \text{for } t_1 \geq t^*, \end{cases} \tag{4.18}$$

$$Q_{1,p,0}(t_1) = \frac{\kappa(t^*)}{\mu}e^{\mu(t^*)}S(t^*)e^{-\mu t_1}\bar{G}(t_1) \quad \text{for } t_1 \geq 0, \tag{4.19}$$

$$Q_0 = \frac{2\kappa(t^*)}{\mu^2}(1 - e^{\mu t^*} S(t^*)). \tag{4.20}$$

The steady-state probability for a particular state is obtained by integrating the density function for a patricular state. Note that $Q_{1,f}(t_1)$, $Q_{1,p}(t_1)$, and $Q_{1,p,0}(t_1)$ are the probability density function associated with one particular machine. Since we have assumed that both machines are identical, the steady-state probability of states $(1,f)$, $(1, p)$ and $(q, p, 0)$ are $2\int_0^\infty Q_{1,f}(s)ds$, $2\int_0^\infty Q_{1,p}(s)ds$, and $2\int_0^\infty Q_{1,p,0}(s)ds$, respectively. Set $S(t) = \int_0^t \mu(1 - e^{\mu s})\bar{G}(s)ds$ and $\int_0^t \bar{G}(s)ds$ and $\bar{m}(t) = \int_0^t \bar{G}(s)ds$ Then the steady-state probability for each state are as following :

$$q_2(t^*) = \int_0^{t^*}\int_0^{t^*} Q_2(x,y)dxdy = \kappa(t^*)\{\bar{m}(t^*)\}^2, \tag{4.21}$$

$$\begin{aligned} q_{1,f}(t^*) &= \frac{2\kappa(t^*)}{\mu}\{\int_0^{t^*} (G(t^*) + \bar{G}(t^*)e^{-\mu s} - S(t^*)e^{\mu(t^*-s)})\bar{G}(s)ds \\ &+ (\bar{G}(t^*) + G(t^*)e^{\mu t^*} - S(t^*)e^{\mu t^*})\frac{1}{\mu}S(t^*)\}, \end{aligned} \tag{4.22}$$

$$q_{1,p}(t^*) = \frac{2\kappa(t^*)}{\mu^2}\bar{G}(t^*)\{\hat{S}(t^*) + (e^{\mu t^*} - 1)S(t^*)\}, \tag{4.23}$$

$$q_{1,p,0}(t^*) = \frac{2\kappa(t^*)}{\mu^2}e^{\mu t^*}S(t^*)S(0), \tag{4.24}$$

$$q_0(t^*) = \frac{2\kappa(t^*)}{\mu^2}\{1 - e^{\mu t^*}S(t^*)\}. \tag{4.25}$$

The coefficient $k(t^*)$ is determined by the norming equation

$$q_2(t^*) + q_{1,f}(t^*) + q_{1,p}(t^*) + q_{1,p,0}(t^*) + q_0(t^*) = 1, \tag{4.26}$$

thus,

$$\kappa(t^*) = \{\frac{2}{\mu^2} + \frac{2}{\mu}\bar{m}(t^*) + \bar{m}^2(t^*)\}^{-1}. \tag{4.27}$$

5. Expected number of visits

The cost function defined in Eq. (2.5) is the function of steady-state probability of each state $i \in E$ and expected number of visits to each state per unit time is the function of mean sojourn time of each state as seen in Eq. (2.9). Thus the expected number of visits to each state per unit time of state $i \in E$ under control limit $t^*(m_i(t^*))$, we need the information about age distribution at the arrival epoch of each state $i \in E$.

Now, let us consider the mean sojourn time for each state. Mean sojourn time at state (0) is simple, since the service time distribution is assumed to be an exponential distribution with mean μ, therefore,

$$m_0(t^*) = \frac{1}{\mu}. \tag{5.1}$$

For the other states, the mean sojourn time is slightly more complicated because the age distribution at an arrival epoch of each state has to be considered. For the mean sojourn time at state (2), first consider the age distibution at the arrival epoch of state (2). State (2) can be entered through the state (1, f), (1, p), and (1, p, 0) as soon as the repair service of a down machine has been completed and the age of running machine is less than t^*. Therefore, at the arrival epoch of state (2), one of the machines is a brand new machine and the other machine is age of between 0 and t^*. Thus the age density at the arrival epoch of state (2) is

$$Q_2^*(.) = Q_2(t_1, 0) \quad \text{for} \quad t_1 < t^*. \tag{5.2}$$

Let $H_{t,0}(s)$ denote the conditional probability distribution of failure time given that two machines are working, one machine being of age t and the other machine beimg of age 0. and let $\bar{H}_{t,0}(s) = 1 - H_{t,0}(s)$, then

$$\bar{H}_{t_1,0}(s) = \frac{\bar{G}(t_1 + s)\bar{G}(s)}{\bar{G}(t_1)\bar{G}(0)}. \tag{5.3}$$

The conditional mean sojourn time that two machines of age t and 0 are working is defined by

$$m_{2/t_1,0}(t^*) = \int_{s=0}^{t_*-t_1} \bar{H}_{t_1,0}(s)ds. \tag{5.4}$$

Since both machines are assumed to be identical, the mean sojourn time of state (2) is given by substituting Eq. (4.16)

$$\begin{aligned} m_2(t^*) &= \frac{\int_{t_1=0}^{t^*}\int_{s=0}^{t^*-t_1} \bar{H}_{t_1,0}(s)Q_2(t_1,0)dsdt_1}{\int_{t_1=0}^{t^*} Q_2(t_1,0)dt_1} \\ &= \frac{1}{\bar{m}(t^*)}\int_{s=0}^{t^*} \bar{G}(s)\int_{t_1=s}^{t^*} \bar{G}(t_1)dt_1ds. \end{aligned} \tag{5.5}$$

Next, consider the age distibution at an arrival epoch of state $(1,p)$. State $(1,p)$ represents the situation where one machine is running and can be entered from state (2) due to the control limit t^*. Thus, just before the machine replacement operation, one of the machines is at age of t^* and the other machine is at age less than t^*. Thus, the age distribution at arrival epoch of state $(1,p)$ is given by

$$Q^*_{1,p}(\cdot) = Q_2(t_1,t^*) \text{ for } t_1 < t^*. \tag{5.6}$$

Let $\bar{F}_{t_1}(s)$ denote the conditional probability distrbution of survival time given that one machine is working, and the age of the machine is t_1 that is $\bar{F}t_{t_1}(s) = \frac{\bar{G}(t_1+s)}{\bar{G}(t_1)}$, then the conditional mean sojourn time at age t_1 at an arrival epoch of state $(1,p)$ is given by

$$m_{1,p/t_1}(t^*) = \int_{s=0}^{\infty} \bar{F}_{t_1}(s)e^{-\mu s}ds. \tag{5.7}$$

Thus, the mean sojourn time of state $(1,p)$ is given by

$$\begin{aligned} m_{1,p}(t^*) &= \frac{\int_{t_1=0}^{t^*}\int_{s=0}^{\infty} \bar{F}_{t_1}(s)e^{-\mu s}Q_2(t_1,t^*)dsdt_1}{\int_{t_1=0}^{t^*} Q_2(t_1,t^*)dt_1} \\ &= \frac{1}{\bar{m}(t^*)}\int_{t_1=0}^{t^*}\int_{s=0}^{\infty} e^{-\mu s}\bar{G}(t_1+s)dsdt_1. \end{aligned} \tag{5.8}$$

By the definition of state(1, p, 0), the age of the working machine at an arrival epoch of state(1, p, 0) is always 0, ie, a new machine. Therefore, the mean sojourn time of state(1, p, 0) is given by

$$m_{1,p,0}(t^*) = \int_0^\infty e^{-\mu s}\bar{G}(s)ds. \tag{5.9}$$

Thus the expected number of visits to each state per unit time, except for state (1, *f*), can be computed by using Eq. (2.9).

The age distribution at the arrival epoch of state (1, *f*) is more complicated than those of other states. Our interest is to find the expected number of visits to state(1, *f*). Therefore, a different approach will be used. Let us examine the graph in Figue (2.1) which describes the flow of the system. By the definition of state (1, *f*) in section 2, visits to state (1, *f*) are from state (2) due to machine failure, or are from state (0). Since the number of visits to state (1, *f*) from state (0) must equal the number of visits to state (0), we have that the expected number of visits to state (1, *f*) per unit time ($\eta_{1,f}(t^*)$) is

$$\eta_{1,f}(t^*) = \eta_0(t^*) + \eta_2(t^*) - \eta_{1,p}(t^*). \tag{5.10}$$

VI. Algorithm for obtaining the optimal control limit

In this section, we develop an algorithm that will yield an ε - optimum control limit. That is, for a fixed $\varepsilon > 0$, the algorithm will determine a control limit whose objective function value is within ε of the optimum value. The algorithm uses a bounding technique developed by Ritchken and Wilson [1988].

The idea of the procedure is to identify an admissible control zone in which the optimum must exist. The admissible control zone is then partitioned and, through the use of a bound, some of the intervals of the partition are discarded; thus reducing the admissible control zone. With the reduced admissible control zone, the partitioning and further reduction is repeated.

Assume we start with an admissible control zone given by the interval $D_0 = [a, \infty)$. Let

$\Delta_0 > 0$ be given and define a partition $a = t_0 < t_1 < \cdots < t_n < \infty$ where $t_i - t_{i-1} = \Delta_0$ for $i = 1 \cdots, n$. Associated with this partition are bounding function B_1 and B_2, defined in Appendix B. (The computational procedure for the bounding functions, although straight forward, is tedious so we have given them in the appendix.) the key properties of the bounding functions are

$$\bar{C}(t) \geq \bar{C}(t_i) - B_1(t, t_i) \qquad \text{for} \quad t_{i-1} \leq t \leq t_i, \tag{6.1}$$

$$B_1(t_{i-1}, t_i) \geq 0 \quad \text{and} \quad \lim_{\Delta_0 \to 0} B_1(t_{i-1}, t_i) = 0 \qquad \text{monotonically,} \tag{6.2}$$

$$\bar{C}(t) \geq \bar{C}(t_k) - B_2(t_k, t) \qquad \text{for} \quad t \geq t_k, \tag{6.3}$$

$$B_2(t_k, t) \geq 0 \quad \text{and} \quad \lim_{t_k \to t} B_2(t_k, t) = 0 \qquad \text{monotonically,} \tag{6.4}$$

where $\bar{C}(t)$ is the objective function (2.5). The first property (6.1) can be rephrased slightly so that a portion of the partition can be eliminated from the admissible control zone. Namely, suppose we have for some t_1, t_2, and $t_3 \in [t_1, t_2]$ the following condition

$$\bar{C}(t_3) < \bar{C}(t_2) - B_1(t_1, t_2), \tag{6.5}$$

then the minimum value for $\bar{C}$ cannot occur in the interval $[t_1, t_2]$.

The expressions (6.2) and (6.4) show that as Δ_0 (and thus distance between t_k and t) goes to zero, the functions B_1 and B_2 are monotonically decreasing tozero. Hence, it is possible to get arbitrarly close to the optimal cost by choosing appropriately Δ_0 small enough.

The algorithm of obtaining the ε - optimum requires a preliminary operation. For a given admissible control region $D_0 = [a, \infty)$, we establish an upper bound on t of the admissible control region in this opoeration. There are 3 elements that have to be specified in the preliminary operation.

PRELIMINARY OPERATION

ELEMENT 1. Fix a partition increment Δ_0 and set $t_k = a$ for $k = 0$.

ELEMENT 2. Calculate $\bar{C}(t_k)$ and $\bar{C}(t_k) - B_2(t_k, \infty)$ from Eq. (B.32) given in Appendix B.

ELEMENT 3. Let $\bar{C}^* = min_{i=0,\cdots,k}\bar{C}(t_i)$. If $\bar{C}^* > \bar{C}(t_k) - B_2(t_k,\infty)$, let $t_{k+1} = t_k + \Delta_0$ and set $k = k+1$; go to Element 2; otherwise, t_k is upper bound.

After preliminary operation, the new admissible control region is given by the closed interval $D_1 = [a, b]$.

Now, the algorithm can be applied to a specified problem. To give the general procedure, we assume that the algorithm is currently working with an admissible control region, D_n, and we would like to determine the next, smaller, admissible control region, D_{n+1}.

ALGORITHM

STEP 1. Fix a partition increment Δ_n and determine the patition, t_0, t_1, $\cdots$, t_k such that $t_i - t_{i-1} = \Delta_n$ for $i = 1, \cdots, k-1$ and $t_k - t_{k-1} \leq \Delta_n$.

STEP 2. Calculate $\bar{C}(t_i)$ for $i = 0, \cdots, k$ and $\bar{C}(t_i) - B_1(t_{i-1}, t_i)$ for $i = 1, \cdots, k$ from Eq. (B.21) in Appendix B.

STEP 3. Let $\bar{C}^* = \min_{i=0,\cdots,k} \bar{C}(t_i)$ and $\bar{C}_l^* = \min_{i=1,\cdots,k} \bar{C}(t_i) - B_1(t_{i-1}, t_i)$.

STEP 4. Determine new admissible control zone D_{n+1} deleting all intervals $[t_{i-1}, t_i]$ from D_n such that $\bar{C}(t_i) - B_1(t_{i-1}, t_i) \leq \bar{C}^*$.

STEP 5. If $\bar{C}^* - \bar{C}_l^* > \varepsilon$, fix Δ_{n+1} such that $\Delta_{n+1} < \Delta_n$ and determine new partition t_0, t_1, $\cdots$, t_k such that $t_i - t_{i-1} = \Delta_{n+1}$ for $i = 1, \cdots, k-1$ and $t_k - t_{k-1} \leq \Delta_{n+1}$. Go to Stop 2; otherwise $\bar{C}^*$ is the ε - optimum.

NUMERICAL EXAMPLE

Times to failure for each machine are assumed to be a phase-type distribution with cumulative distribution given by

$$G(x) = P\{X \leq x\} = 1 - \alpha\, e^{Tx}1, \qquad \text{for} \quad x \geq 0; \qquad (6.6)$$

where a = (1,0,0) and

$$\mathbf{T} = \begin{bmatrix} -0.2 & 0.18 & 0 \\ 0 & -0.4 & 0.36 \\ 0 & 0 & -0.5 \end{bmatrix} \qquad \mathbf{T}^{\circ} = \begin{bmatrix} 0.02 \\ 0.04 \\ 0.5 \end{bmatrix},$$

and when a machine fails, it is repaired and the repair times are exponentially distributed with rate μ = 2.0. The cost data set are given by (c_f, c_p, c_d) = (450, 70, 50).

Table 6.1 Establishing an upper bound on t.

t_k	$\bar{C}(t_k)$	$\bar{C}(t_k) - B_2(t_k, \infty)$
.	.	.
4.0	82.70	-773.39
6.0	84.26	-199.02
8.0	88.25	-39.16
10.0	91.91	24.60
12.0	94.75	55.79
14.0	96.81	73.00
16.0	98.26	83.22
18.0	99.26	89.56
.	.	.

Table 6.1. contains values of $\bar{C}(t_k)$ and $\bar{C}(t_k)$ - $B_2(t_k, \infty)$ for various values of t_k. We establish an upper bound of t by increasing t_k until a value $\bar{C}(t_k)$ - $B_2(t_k, \infty)$ reachers a previously computed $\bar{C}(t_k)$ value. Thus new admissible control region, D_1, is [1, 16].

Teble 6.2. Improving the lower and upper boynds on 5.

t_i	$\bar{C}(t_i)$	$\bar{C}(t_i) - B_1(t_{i-1}, t_i)$	t_i	$\bar{C}(t_i)$	$\bar{C}(t_i) - B_1(t_{i-1}, t_i)$
.	.	.	.	.	.
.	.	.	4.41	82.48437	81.10210
3.11	85.29537	82.56325	*4.42	82.48432	81.10850
3.12	85.24293	82.56692	4.43	82.48448	81.11508
3.13	85.19116	82.49114	4.44	82.48456	81.12183
3.14	85.14005	82.45590	.	.	.
3.15	85.08961	82.42119	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	5.44	83.3465	82.4591
4.09	82.6126	81.00078	5.45	83.3550	82.4773
4.10	82.6045	81.00061	5.46	83.3696	82.4955
4.11	82.5967	81.00067	5.47	83.3844	82.5137

Table 6.2 evaluates the values $C(t_i)$ and the lower bound $C(t_i)$ - $B_1(t_{i-1}, t_i)$ over set D_1 when $\triangle_1$ is 0.01. The current best solution is 4.42 with its cost 82.48432.

The minimun of the lower bound of the cost is 81.00061 when t = 4.10 Thus the maximum possible cost error is 1.4837(1.83%).Noe that there is a some range of t where the lower bound exeeds the current best cost (82.48432). Therefore we get the new search region D_2= [3.13, 5.46]

Table 6.3. Finer partition for better solution.

Δ_n	Optimal T^*	$C(T^*)$	Max error (%)
0.01	4.42	82.48432	1.4837 (1.83%)
0.001	4.417	82.48431892	0.06968 (0.084%)
0.0002	4.4174	82.48431867	0.02751 (0.032%)

Table 6.3 shows the sequence of iterations for finer partitons. The current best solution is provided together with its maximum possible error. After 3 iterations for finer partitions, it is possible to get the current best solution T^* = 4.4174 and its optimal cost 82.48431867 with maximum error 0.02751 (0.032%).

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