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Replacement Policies Based on System Age and Random Repair Cost under Imperfect Repair

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ABSTRACT

Replacement policies based on both the system age and the random repair cost are studied. The system is replaced when it reaches age T (Policy A), or when it fails for the first time after age T (Policy B). If the system fails before age T, the repair cost is estimated and repair is then undertaken if the estimated cost is less than a predetermined limit L; otherwise, the system is replaced. After repair, the system is as good as new with probability $(1-p)$ or is as good as old with probability p. The expected cost rate is obtained, its behavior is examined, and way of obtaining optimal T and L is explored.

Introduction

The repair cost limit method has been regarded as a good representation of the way people decide on whether to repair or replace. In the repair cost limit method, when a system fails, its repair cost is estimated by inspection. If the repair cost does not exceed the predetermined cost limit, the system is repaired; otherwise it is replaced. Hastings[1969] considered the repair cost limit problem in the context of a Markov decision problem and applied dynamic progamming techniques for obtaining the repair cost limits at each repair. Nakagawa and Osaki

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[1974] studied a replacement policy with repair time limit. Nguyen and Murthy[1980] showed that the results of Nakagawa and Osaki[1974] are optimal over both deterministic and random repair time limit policies. Kaio and Osaki[1982] discussed a repair limit policy with a cost constraint. In these models, it is assumed that the system is as good as new upon repair. Park [1983, 1985a, 1985b, 1987] proposed cost limit policies under minimal repair. Cleroux et al. [1979], Berg, et al. [1986] and Bai and Yun[1986] studied replacement problems in which the system age and the minimal repair cost are simultaneously considered. A generalization of the perfect and minimal repairs is imperfect repair(see Block et al. [1985], Brown[1983]).

We propose two age replacement policies which considers both the age of the system and the random cost of repair. The system is replaced at age T (Policy A) or the first failure after age T (Policy B). If it fails before age T, the repair cost is estimated and the system is repaired if the estimated cost is less than L, otherwise, it is replaced. After repair, the system is as good as new with probability $(1-p)$ or is minimally repaired with probability p (see Brown et al. [1983], Cleroux [1979], Yun & Bai[1987]). The expected cost rate for each policy is obtained. The proposed models are shown to be a generalization of existing models and way of obtaining optimal policy are explored.

In the sequel, we will use some abbreviations such as i.i.d. (independently and identically distributed), r.v. (random variable), Cdf(cumulative distribution function), Sf(survival function).

Assumptions

1. Repair costs are i.i.d.r.v.'s, and are observable through inspection.
2. Hazard rate of the system is increasing in age and not disturbed by minimal repair.
3. The system can be renewed by replacements and perfect repair.
4. Replacements and repairs take only negligible time.
5. Planning horizon is infinite.

Notation

Y_n : n^{th} failure time ; a r.v.

$F(t), r(t), R(t)$: Cdf, hazard rate, cumulative hazard of Y_1

T : replacement period

L : repair cost limit

$G(x), \bar{G}(x)$: distribution function, survival function of repair cost

Z_n : n^{th} repair cost ; a r.v.

E_L : expected value of repair costs not exceeding L, $\int_0^L xdG(x)/G(L)$

$N(t)$: number of repairs to time t ; a r.v.

$M(t)$: mean residual life, $M(t)=\int_t^{\infty}\exp(-(R(z)-R(t)))dz$

c_0 : replacement cost
F_n : probability that n^{th} failure is the first failure whose repair cost exceeds L
S_c : expected cost of a renewal period
S_d : expected duration of a renewal period
$C(T, L)$: expected cost rate

Policy A

Policy : When the system fails before age T, its repair cost is determined by inspection. If the repair cost exceeds L, then the system is replaced, otherwise it is repaired. In addition, when the age of the system reachs T, it is replaced.

Expected cost of a renewal period

Since the system is minimally repaired at all failures until a renewal point, $N(t)$ is NHPP with mean $R(T)$. The expected cost during a renewal period is given by

$$\begin{aligned} S_c &= \sum_{n=1}^{\infty}[[\bar{G}(L)[c_0+(n-1)E_L]+(1-p)G(L)nE_L]Pr[Y_n \le T] \\ &\quad + \sum_{j=0}^{n-1}[j(E_L+c_0)](1-pG(L))Pr[N(T)=j]][pG(L)]^{n-1} \\ &= \sum_{n=1}^{\infty}\{[[\bar{G}(L)[c_0+(n-1)E_L]+(1-p)G(L)nE_L][1-\sum_{j=0}^{n-1} R(T)^j e^{-R(T)}/j\,!\,] \\ &\quad + \sum_{j=0}^{n-1}(1-pG(L))[j(E_L+c_0)]R(T)^j e^{-R(T)}/j\,!\,]\}[pG(L)] \\ &= [[c_0\bar{G}(L)+G(L)E_L]/(1-pG(L)]+G(L)((1-p)c_0-E_L) \\ &\quad (e^{-R(T)(1-pG(L))}/(1-pG(L))] \end{aligned} \tag{1}$$

See Nakagawa and Kowada[1983].

Expected duration of a renewal period

Conditional expected duration of a renewal period given that the system is renewed first at n th failure is given by

$$S_f = T\ Pr(Y_n > T) + \int_0^T t dPr(Y_m \le t) = \int_0^T [\sum_{j=0}^{n-1} R(t)^j e^{-R(t)/j\,!}]dt \tag{2}$$

Hence, the expected duration of a renewal period is

$$S_d = \sum_{n=1}^{\infty} S_f(1-pG(L))[pG(L)]^{n-1} = \int_0^T e^{-R(t)(1-pG(L))}dt.$$

Analysis

From formulas (1) and (2), the expected cost rate is

$$C(L, T) = S_c/S_d$$
$$= \{[c_0\bar{G}(L) + G(L)E_L]/(1-pG(L)] + G(L)((1-p)c_0 - E_L)e^{-R(T)(1-pG(L))}$$
$$/(1-pG(L))\}[\int_0^T e^{-R(t)(1-pG(L))}dt]^{-1} \quad (3)$$

If we let $q = 1 - pG(L)$, $a = [c_0\bar{G}(L) + G(L)E_L]/q$, and $b = [E_L - (1-p)c_0]/q$, the expected cost rate is simplified as

$$C(T,\ L) = \{a - be^{-qR(T)}\}[\int_0^T e^{-qR(t)}\ dt]^{-1}. \quad (4)$$

Hence, the necessary condition for optimal T is

$$qr(T)\int_0^T e^{-qR(t)}dt + e^{-qR(T)} = a/b. \quad (5)$$

The l.h.s. of (5) is increasing in T. Therefore the value of T satisfying (5) is unique. Now let L be also a decision variable and write $k = L/C_0 (O < k < 1)$. The effect of parameter k to expected cost rate can be examined numerically.

Special Cases

1. When L goes to infinity(no repair cost limit),

$$C_1(T) = [(c_0 + \mu) + (1-p) - \mu)e^{-R(T)(1-p)}]/[(1-p)\int_0^T e^{-R(t)(1-p)}dt],$$

where $\mu = \int_0^\infty G(t)dt$. The optimal T can be obtained from equation [5].

2. When T goes to infinity(no age replacement),

$$C_2(L) = [[c_0\bar{G}(L) + G(L)E_L]/(1-pG(L)][\int_0^\infty e^{-R(t)(1-pG(L))}dt]^{-1},$$

which agrees with Yun and Bai[1987].

3. When $p=0$,

$$C_3(T,L)=[[c_0\bar{G}(L)+G(L)E_L]+G(L)(c_0-E_L)e^{-R(T)}][\int_0^T e^{-R(t)}dt]^{-1}$$

Hence, if $c_0>\int_0^\infty G(x)dx$, the optimal L is 0. Otherwise, the optimal L is infinite.

Hence, the problem of obtaining the optimal T is same as Barlow and Hunter[1960].

4. When $p=$ (minimal repair),

$$G_4(T,L)=[c_0+G(L)E_L/\bar{G}(L)-G(L)E_Le^{-R(T)(1-G(L))}/\bar{G}(L)][\int_0^T e^{-R(t)\bar{G}(L)}dt]^{-1},$$

which agrees with Cleroux et al. [1979].

Policy B

Policy B differs from Policy A in that the system is replaced at the first failure after age T, instead of at age T. The expected cost of a renewal period is the same as that of Policy A. Only the expected duration of a renewal period needs to be derived.

Expected duration of a renewal period

Conditional expected duration of a renewal period given that the system is renewed first at nth failure is

$$S_f=[M(T)+T]Pr\{Y_n>T\}+\int_0^T tdPr\{Y_n\le t\}=M(T)\sum_{j=0}^{n=1}R(T)^j e^{-R(T)}/j\,!$$
$$+\int_0^T[\sum_{j=0}^{n-1}R(t)^j e^{-R(t)}/j\,!]dt.$$

Hence,

$$S_d=\sum_{n=1}^{\infty}S_f(1-pG(L))[pG(L)]^{n-1}=M(T)e^{-R(T)[1-pG(L)]}+\int_0^T e^{-R(t)[1-pG(L)]}\ dt. \qquad (6)$$

Analysis

From (1) and (6), the expected cost rate is given by

$$C(T,L)=[[c_0\bar{G}(L)+G(L)E_L]/(1-pG(L)]+G(L)((1-p)c_0-E_L)(e^{-R(T)(1-pG(L))})$$

$$/(1-pG(L))][M(T)e^{-R(T)(1-pG(L))}+\int_0^T e^{-R(t)(1-pG(L))}\,dt]^{-1} \tag{7}$$

If we let $q=1-pG(L)$, $a=[c_0G(L)+G(L)E_L]/q$, and $b=[E_L-(1-p)c_0]/q$, $C(T,L)$ is simplified as

$$C(T,\ L)=[b-ae^{-qR(T)}][M(T)e^{-qR(T)}+\int_0^T e^{-qR(t)}dt]^{-1} \tag{8}$$

To show that the optimal solution is unique, the following lemmas are needed(for the proof, see Bai & Yun[1986])

Lemma 1. Let $A(T)=e^{-qR(T)}$ and $W(T)=aq\int_0^T A(t)\,dt-b(1-q)M(T)+aA(T)M(T)$. Then $W(T)$ is increasing in T.

Lemma 2. Let $Q(T)=A(T)r(T)W(T)$, where $A(T)$ and $W(T)$ is as in result 1. Then there exists a solution satisfying $Q(T)=0$, it is the unique optimal solution. Otherwise, the optimal solution is $T^*\to\infty$.

From Lemmas 1 and 2, it can be shown that for a given L, the optimal solution of preventive replacement period exists and is unique. Let L be also a decision variable and write $k=L/c_0$. By varying k from 0 to 1, the effect of parameter k to the expected cost rate can be examined.

Special Cases

1. When L goes to infinity(no repair cost limit),

$$C_1(T)=[\mu/(1-p)+((1-p)c_0-\mu)e^{-R(T)(1-p)}/(1-p)][M(T)e^{-R(T)(1-p)}$$
$$+\int_0^T e^{-R(t)(1-p)}dt]^{-1}$$

where $\mu=\int_0^\infty G(t)\,dt$. In this case, the optimal solution can be obtained from Lemma 1 and 2.

2. When T goes to infinity,

$$C_2(L)=[[c_0\bar{G}(L)+G(L)E_L]/(1-pG(L)][\int_0^\infty e^{-R(t)(1-pG(L))}dt]^{-1},$$

which agrees with Yun and Bai[1987].

3. When $p=0$

$$C_3(T,L)=[[c_0\bar{G}(L)+G(L)E_L]+G(L)(c_0-E_L)e^{-R(T)}][M(T)e^{-R(T)}+\int_0^T e^{-R(t)}dt]^{-1}.$$

Hence, if $c_0>\int_0^\infty G(x)dx$, the optimal L is 0. Otherwise, the optimal L is infinite. And the optimal T can be obtained by the method of Muth[1977].

4. When $p=1$(minimal repair),

$$C_4(T,L)=[c_0+G(L)E_L/\bar{G}(L)-G(L)E_Le^{-R(T)(1-G(L))}/\bar{G}(L)][M(T)e^{-R(T)\bar{G}(L)}+\int_0^T e^{-R(t)\bar{G}(L)}dt]^{-1},$$

which agrees with Bai and Yun[1986].

Example 1.

A simple example is considered with

$$F(t)=1-\exp[-t^{1.5}],\ G(x)=1-\exp(-x).$$

Table 1. Optimal expected cost rate

	$p=0.1$			$p=0.3$		
c_0	1.1	1.9	2.7	1.1	1.9	2.7
Policy A	1.21	1.89	2.27	1.31	2.03	2.42
Policy B	1.18	1.82	2.21	1.26	1.95	2.32

Optimal expected cost rate for Policies A and B for selected values of p and c_0 are shown in Table 1. The table indicates that Policy B has the smaller expected cost rate than Policy A.

Discussion

Age replacement policies which considers both the age of the system and the random cost of repair are studied. The system is replaced at age T(Policy A) the first failure after age T (Policy B). If it fails before age T, the repair cost is estimated and the system is minimally repaired if the estimated cost is less than L, otherwise, it is replaced. The optimal policy minimizing expected cost rate is discussed. For a given L, the value of age T minimizing the expected cost is shown to be finite and unique in each policy. Policy B is a generization of the Muth polcy[1977], and has the smaller expected cost rate than replacement policy A. For further studies, we may consider the replacement policy in which the imperfect ratio is a function of t, $p(t)$ (se Berg et al. [1986].

REFERENCES

1. Bai, D.S. and Yun, W.Y. (1986), "*An Age Replacement policy with Minimal Repair Cost Limit*", IEEE Trans. on Reliability, R-35, 452-454.
2. Barlow, R.E. and Hunter, L.C. (1960), "*Optimal Preventive Maintenance Policies*", Oper. Res., 90-100
3. Berg, M., Bienvenu, M. and Cleroux, R. (1986), "*Age Replacement Policy with Age-dependent Minimal Repair*", INFOR, 24, 26-32.
4. Block, H.W., Borges, W.S. and Savits, T.H. (1985), "*Age-dependent Minimal Repair*", J. of Applied Probability, 22, 370-385.
5. Brown, M. and Proschan, F. (1983), "*Imperfect Repair*". J. of Applied Probablity, 20, 851-859.
6. Cleroux, R., Dubuc, S. and Tilquin, C. C. (1979), "*The Age Replacement Problem with Minimal Repair and Random Repair Costs*", Oper. Res., 27, 1158-1167.
7. Hastings, N.A.J. (1969), "*The Repair Limit Method*", Operational Research Quarterly, 20, 337-349.
8. Kaio, N. and Osaki, S. (1982), "*Optimum Repair Limit Policies with a Time Constraint*", International J. of Systems Science, 13, 1345-1350.
9. Muth, E.J. (1977), "*An Optimal Decision Rule for Repair vs. Replacement*", IEEE Trans. on Reliability, R-26, 179-181.
10. Nakagawa, T. and Kowada, M. (1983), "*Analysis of a System with Minimal Repair and Its Application to Replacement Policy*", European J. of Operational Research, 12, 176-182.
11. Nakagawa, T. and Osaki, S. (1974) "*The Optimum Repair Limit Replacement Policies*", Operational Research Quarterly, 25, 311-317.
12. Nguyen, D.G. and Murthy, D.N.P. (1980), "*A Note of the Repair Limit Replacement Policy*", J. of Operational Research Society, 31, 1103-1104.
13. Park, K.S. (1983), "*Cost Limit Replacement Policy under Minimal Repair*", Microelectron. Reliab., 23, 347-349.
14. Park, K.S. (1985a), "*Pseudodynamic Cost Limit Replacement Model under Minimal Repair*", Microelectron. Reliab., 25, 573-579.
15. Park, K.S. (1985b), "*Optimal Number of Major Failures before Replacement*", Microelectron. Reliab., 25, 797-805.
16. Park, K.S. (1987), "*Optimal Number of Minor Failures before Replacement*", International J. of Systems Science, 18, 333-337.
17. Yun, W.Y. and Bai, D.S. (1987), "*Cost Limit Replacement policy Under Imperfect Repair*", Reliability Engineering, 19, 23-28.