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Maximization in Reliability Design when Stress/Strength has Time Dependent Model of Deterministic Cycle Times

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ABSTRACT

This study is to refer to the optimization problems when the stress and strength follow the time dependent model, considering a decision making process in the design methodology from reliability viewpoint. Reliability of a component can be expressed and computed if the probability distributions for the stress and strength in the time dependent case are known. The factors which determine the parameters of the distributions for stress and strength random variables can be controlled in design problems. This leads to the problem of finding the optimal values of these parameters subject to resources and design constraints.

This paper is to present techniques for solving the optimization problems at the design stage like as minimizing the total cost to be spent on controlling the stress and strength parameters for random variables subject to the constraint that the component must have a specified reliability, alternatively, maximizing the component reliability subject to certain constraints on amount of resources available to control the parameters. The derived expressions and computations of reliability in the time dependent case and some optimization models of these cases are discussed. The special structure of these models is exploited to develop the optimization techniques which are illustrated by design examples.

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I. Objectives of the Paper

Let's consider the repeated application of stresses and also the change of the distribution of strength with time and/or cycle. These reliability models are different of a single stress application(static model). The reliability in time/cycle dependent case is changed with passing the time/cycle, and these applications for the design stage are realistically required. The details on the expressions and computations in the time/cycle dependent case models are discussed and developed in section 3.

In design and manufacturing situations, it may be possible to control the values of these parameters and thus determine their values at the design stage. Cost is involved in controlling the values of these parameters. Up to the present, most of the optimization problem have been partially researched by the single stress and strength (static model) case models.

In reality, since the reliability by the stress and strength is decreasing as passing the time, the new optimization methodology by time varying should be exploited and applied.

To control the values of these parameters, certain resources and design constraints must be utilized leading to the following optimization problems in the time/cycle dependent stress and strength models :

(1) Maximize reliability subject to the constraints that a certain amount to resources in available to be invested in parameter control.

The purpose of this paper is to present techniques for solving the above optimization problems when the time/cycle dependent stress and strength follow known distributions such as s-normal. These optimization methodologies are just to produce the determination to predict the optimal values of parameters involved cost at the design stage, that is, the determination of optimal values to set a goal of a given future reliability and/or prediction of the reliability in the future as the passage of time. The limitations of these distributions and other details are given in [Section 3, 4].

II. Nomenclature, Notation and Assumptions

1. Nomenclature

(1) Deterministic Cycle Times : The loads are applied at known times $t_1, t_2, \cdots, t_n$. In this case, the changes may follow a cyclic pattern. These may be seasonal changes, day/night, on-off, up-down cycles etc. The cycle times are known exactly beforehand. The load occurrences may be constant or may vary from cycle to cycle.

(2) Random Cycle Times : In this case, the cycle times are random and independent rather than known. The randomness of the cycle times may be described by exponential or gamma probability density functions. The loading may be deterministic or stochastic in nature.

(3) Random-fixed : "Random" refers to the behavior of the variable at the beginning of time and "fixed" means that the behavior of the variable with respect to time is fixed, or the variable varies in time in a known manner.

(4) Random-independent : The word "Independent" means that the successive values assumed by the variable are statistically independent and thus one value gives us no information about the size of the subsequent values.

2. Notation

$C_1(\mu_y)$, $C_2(\sigma_y)$, $C_3(\mu_x)$, $C_4(\sigma_x)$: Cost function of μ_y, σ_y, μ_x, σ_x

μ_x : mean value of the stress

σ_x : standard deviation of the stress

μ_y : mean value of the strength

σ_y : standard deviation of the strength

λ^0 : Lagrangian multiplier

$f(x)$: the $p, d, f,$ of stress X

$F(x)$: cumulative distribution of X, $F(x)=f(x)dx$

$F(X)$: $1-F(X)$

$F(Y)$: $F(Y)=f(x)dx$

$g(y)$: the $p, d, f,$ of strength Y

$G(x)$: $G(X)=g(x)dx$

$R(t)$: Reliability on time t

$R(n)$: Reliability after the n cycle leads

$\phi(\cdot)$: probability of standardized variable$(\cdot)$

$Z(\cdot)$: the standard variable of the reliability$(\cdot)$

sign * : optimality sign

Z_R : Z_R is only the standard variables of the reliability for the stress/strength interference

$\pi_K(t)$: the probability of K cycles occurring in the time interval $[0, t]$

α : a parameter equal to the mean occurrences per unit time

3. Assumptions

(1) When the stress and strength follow the known distributions such as s-normal these distributions are i, i, d and have the continuous random variables.

(2) The uncertainty about the stress and strength variables in the time/cycle dependent models is classified in "Random-fixed" and "Random-independent".

(3) Repeated stresses are characterized by the time each load is applied and behavior of time interval between the application of loads. The load accurrances are classified as "deterministic cycle times" and "random cycle times".

(4) $\pi_K(t)$ in the random cycle times follows the poisson distribution.

(5) When the coefficient of variation is less then 0.3, the probability of negative values is negligible in the s-normal distribution.

(6) Cost functions of the component are monotonically decreasing or increasing functions.

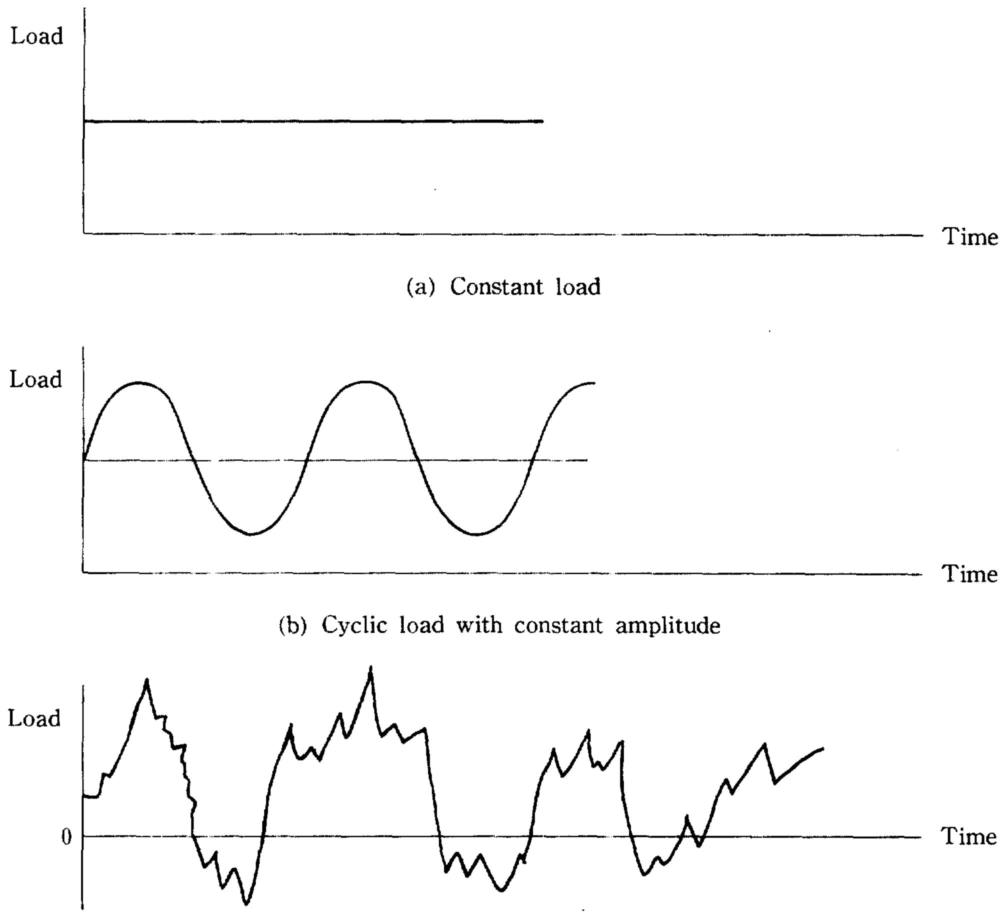
(7) The components presented from this paper follow the time dependent stress and strength model, these characteristics may be parameters of the variables.

III. The Expression and Computation for Reliability with Time

Stress and strength interference models for static case were examined in many literature [1-6, 9, 10, 11, 14, 15, 16, 19].

In this chapter we develop models that consider the repeated application of stresses and also consider the change of the distribution of strength with time/cycle. Such reliability models are generally known as stress and strength time models.

Figure (3-1) shows various patterns of load variation with to time. Cycle load with constant amplitude is shown in Figure (3-1b) This pattern generally occurs in laboratory testing [3, 4, 6, 9, 10, 11, 14, 15]. In general, the components are subjected to a complicated pattern of randomly varying load amplitudes and frequencies. Two examples of this are the load in an aircraft created by atmospheric gusts, and the loads produced in a surface vehicle's suspension components by random irregularity of road surfaces. Figure (3-1c) shows such a random pattern.



(a) Constant load

(b) Cyclic load with constant amplitude

(c) Random-load specturm

Figure 3–1. Patterns of load(stress) variation

Even when different components are subjected to the same fluctuating stress, they fail at different cycles because of nonhomogeneity in material properties, variation in surface conditions, etc. To predict the average life of a component, a number of test specimens are tested at various stress levels until failure. The results are on log-log paper with the stresses on the ordinate and the corresponding lives on the abscissa, as shown in Figure[3-2]. A line representing the average life is fitted, and the horizontal portion of this line indicates that the component will have infinite life if subjected to stresses below this line, and this stress corresponding to this line is called the endurance limit. In practice, several components are tested at a given stress level to estimate the fatigue life. Figure[3-3] shows the scatter in fatigue life at a given stress level.

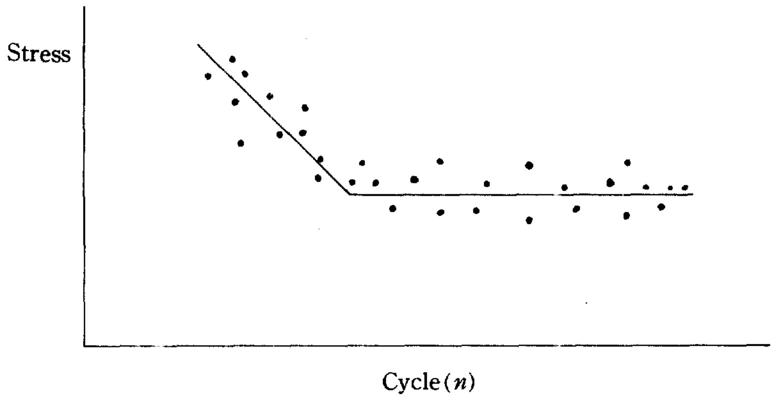


Figure 3–2. Conventional S–N diagram (plotted on a log–log scale)

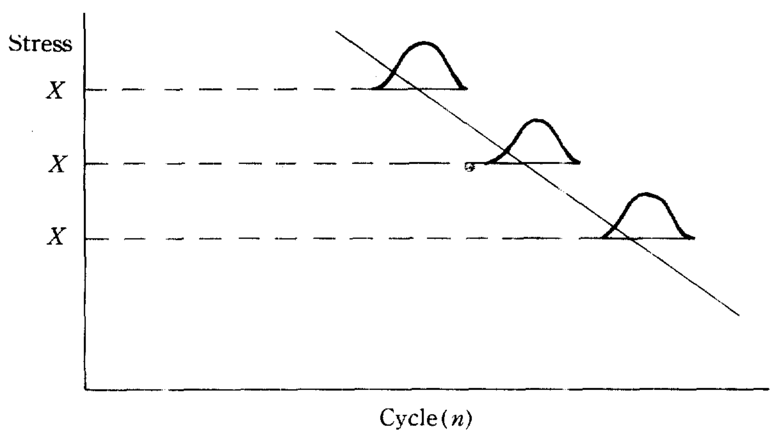


Figure 3–3. Scatter in fatigue life at a given stress (plotted on a log–log scale)

Even when different components are subjected to the same fluctuating stress, they fail at different cycles because of nonhomogeneity in material properties, variation in surface conditions, etc. To predict the average life of a component, a number of test specimens are tested at various stress levels until failure. The results are on log-log paper with the stresses on the ordinate and the corresponding lives on the abscissa, as shown in Figure[3-2] A line representing the average life is fitted, and the horizontal portion of this line indicates that the component will have infinite life if subjected to stresses below this line, and this stress corresponding to this line is called the endurance limit. In practice, several components are tested a given stress level to estimate the fatigue life. Figure[3-3] shows the scatter in fatigue life at a given stress level.

There is uncertainty about the stress and the strength random variables at any instant of time and also about the behavior of the random variables with respect to time and/or cycles.

Section (3-1) develops the expressions and computations for reliability after a applications of loading for all the possible combinations of the stress and strength variables-deterministic cycle times. We now develop individual expressions and determinations for reliability in different cases. We obtain these expressions by combining the two levels of uncertainty classifications (random-fixed and random-independent) for both stress and strength and two types of cycle occurrences (deterministic cycle times).

3-1 Reliability Expressions and Computations for Deterministic Cycle Times

We prefer Rn, the reliability after n cycles (the probability of not having a failure on any one of the n cycles), to $R(E)$ the reliability at time t, the argument t being continuous in Section (3-2).

For example

$$R(t)=Rn \qquad t_n<t\leq t_{n+1},\ \ n=1,2\cdots$$

where t is the instant in time at which the ith cycle occurs.

The corollaries in each following 3 CASES(CASE 1, CASE 2, CASE 3) are exploited to develop and determine the critical factors to solve the reliability optimization problem with time dependent stress and strength model.

CASE 1. Random-independent Stress/Random-fixed Strength

In this case, the successive random stress $X_1,\ X_2,\ \cdots,\ X_n$ are independent and indentically distributed with $p,d,f,f(x)$, the strength Y is a random variable with a known $p,d,f,g(y)$. Thus,

$$R_n=P\ [E_1,\ E_2,\ \cdots,\ E_n]$$

The E_k event means that

$$E_k\sim(Y>X_k)$$

Hence,

$$R_n = P[(X_1 < Y) \cap (X_2 < Y) \cap \cdots \cap (X_n < Y)]$$
$$= P[\max\ (X_1,\ X_2,\ \cdots,\ X_n) < Y]$$

Let $X_{\max} = \max\ (X_1,\ X_2,\ \cdots,\ X_n)$, Then the distribution $F_n(X)$ of $X_{\max}$ is given by

$$F_n(X) = [F(x)]^n$$

Hence

$$R_n = g(y)\ [F(Y)]^n\ dy \tag{3.1}$$

Where

$$F(Y) = \int_0^y f(x)\,dx$$

The proof by another way (using stochastic theory) is in Appendix

Corollary 1. In the deterministic cycle times, if both the stress and the strength are normally distributed and the reliability R_n is $g(y)[F(y)]^n dy$ (after n cycles load), R_n can have inequality as follow :

$$R_n > \exp[-n\ \Phi(-\frac{\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}})] \quad n = 1, 2, \tag{3.2}$$

where $\Phi(\cdot)$ is the probability of initial standard variable$(\cdot)$

<Proof>

$$R_n = \int_0^\infty g(y)\ [F(y)]^n dy = \int_0^\infty g(y)[1 - \bar{F}(y)]^n dy$$

Now, we have the following inequality [64]

$$\prod_{K=0}^{n} (1 - \bar{F}_k(y)) > 1 - \sum_{K=0}^{n} \bar{F}_k(y)$$

If $[1 - \bar{F}_k(y)]$ is identically distributed from $K=0$ to $K=n$

$$(1 - \bar{F}(y)^n > 1 - (n\bar{F}(y))$$

and thus,

$$R_n > 1 - n \int_0^\infty g(y) \bar{F}(y)\, dy \qquad (3.3)$$

If $nF(y) << 1$, then the lower bound given by $E_q(2.3)$ is fairly close to R_n. Now the probability of failure on the stress application is

$$P_f = \int_0^\infty g(y) \bar{F}(y)\, dy$$

Hence, R_n can be expressed as

$$R_n > 1 - nP_f \sim \exp[-nP_f] \qquad (3.4)$$

The relations given by Equation (2.4) are close approximations to R_n when

$$nP_f << 1$$

Therefore,

$$R_n > \exp[-n\{\int_0^\infty g(y)\ (1-F(y))\ dy\}]$$

$$> \exp[-n \int_0^\infty \frac{1}{\sqrt{2\Pi}\sigma_y} e^{-\frac{(y-\mu_y)}{2\sigma_y^2}} (1 - \int_0^y \frac{1}{\sqrt{2\Pi}\ \sigma_y} e^{-\frac{(x-\mu_x)}{2\ \sigma_x^2}}) dy$$

$$> \exp[-n\{1 - \Phi\ (\frac{\mu_y}{\sigma_y}) - \int_{-\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}}^{\infty} \frac{1}{\sqrt{2\Pi}} e^{-\frac{Z^2}{2}} dZ\}]$$

$$> \exp[-n\ \{1 - 0 - (1 - \Phi(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}))\}]$$

$$R_n > \exp[-n\Phi(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}})] \qquad (3.5)$$

From this bound

$$\Phi(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2)}}) = 1 - \Phi\ (\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}) < \frac{-\ln Rn}{n}$$

$$\Phi\ (\frac{\mu_y-\mu_x}{\sqrt{\sigma_x^2+\sigma_y^2)}}) > 1 + \frac{\ln Rn}{n} \qquad (3.6)$$

CASE 2. Random-fixed Stress and Random-independent Strength

The stress X is a random variable with a known $p, d, f, f(x)$. The successive random Strengths $Y_1, Y_2, \cdots, Y_n$ are independent and identically distributed with $p, d, f, g(y)$.

Thus

$$R_n=P[(Y_1>X)\cap(Y_2>X)\cap\cdots\cap(Y_n>X)$$

$$=P[\min(Y_1, Y_2, \cdots, Y_n)>X)$$

The distribution function of a random variable

$$Y_{\min}=\mathrm{Min}(Y_1, Y_2, \cdots, Y_n)$$

By rule of extreme value distribution

$$G_n(Y)=1-[1-G(Y)]^n$$

$$R_n=P[Y_{\min}>X]$$

$$=\int_0^\infty f(x)\ [1-G(X)]^n\ dx$$

Corollary 2. In the deterministic cycle times, if both the stress and the strength are normally distributed and the R_n(after n cycle load) is $f(x)[1-G(X)]^n\ dx$,
R_n can have inequality as follow :

$$R_n>\exp[-n\phi(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}})]\quad n=1,2,\cdots \tag{3.7}$$

Where $\Phi(\cdot)$ is the probability of initial standard variable$(\cdot)$.
This bound has the same resule as CASE 1.

<**Proof**>

$$R_n=\int_0^\infty f(x)[1-G(X)]^n\ dx$$

Now, we have the following inequality

$$(1-G(X))^n>1-nG(x)$$

and thus lower bound on R_n is given by

$$R_n>1-n\int_0^\infty f(x)G(x)dx \tag{3.8}$$

if $nG(x)\leq 1$, then $E_q(2.10)$ is very close to R_n.
Now the probability of failure on the load accurrences

$$P_g = \int_0^\infty f(x)\, G(x)\, dx$$

$$R_n > 1 - nPg \sim \exp[-nPg]$$

where $nPg - 1$.

$$R_n > \exp[-n\int_0^\infty \frac{1}{\sqrt{2\Pi}\sigma_x} e^{-\frac{X-\mu_x}{2\sigma_x^2}} (\int_0^x \frac{1}{\sqrt{2\Pi}\sigma_x} e^{-\frac{y-\mu_y}{2\sigma_y^2}} dy)\, dx]$$

$$> \exp[-n \int_{-\frac{\mu_x-\mu_y}{\sqrt{\sigma_x^2+\sigma_y^2}}}^\infty \frac{1}{\sqrt{2\Pi}} e^{-\frac{Z^2}{2}} dZ$$

$$> \exp[-n\Phi(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}})] \qquad (3.9)$$

From this bound, we get

$$\Phi(\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}})] > 1 + \frac{(\ln R_n)}{n} \qquad (3.10)$$

CASE 1 and CASE 2 are appeared to the same results of reliability with time by the above proof.

CASE 3. Rnadom-independent Stress and Random-independent Strength

Let $f_k(x)$ and $g_k(Y)$ be p, d, f of stress X and strength Y, respectively in cycle $k = 1, 2, \cdots$, Then, since $Xk's$ and Yk's are independent

$$R_n = P[E_n,\ E_{n-1} \cdots\cdots n_1] = P[E_n] \cdot P[E_{n-1}] \cdots P[E_1] \qquad (3.11)$$

$$= \prod_{k=1}^n P[E_k]$$

where $P[E_k] = P(X_k \leq Y_k)$

$$= \int_0^\infty f_k(x) \int_x^\infty g_k(Y)\, dydx$$

In particular, if f and g do not change with time then,

$$\prod_{k=1}^n P\ (E_k) = (P(E_1))^2$$

Corollary 3. In the deterministic cycle times, if both the stress and the strength are normally distributed and the R_n(after n cycle load) is $[\int_0^\infty f(x) \int_x^\infty g(y)\, dydx]^n$

$$R_n \text{ has } [\Phi(\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}})]^n \tag{3.12}$$

where $\Phi(\cdot)$ is the probability of initial standard variable$(\cdot)$.

<**Proof**>

$$R_n=\left[\int_0^\infty \frac{1}{\sqrt{2\Pi}\sigma_x} e^{-\frac{(X-\mu_x)^2}{2\sigma_x^2}} \int_x^\infty \frac{1}{\sqrt{2\Pi}\sigma_r} \cdot e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} dydx\right]^0$$

$$>\left[\int_{-\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}}^\infty \frac{1}{\sqrt{2\Pi}} e^{-\frac{Z^2}{2}} dZ\right]^n$$

$$>\left[1-\Phi\{\frac{-(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}\}\right]^n=\left[\Phi\{\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}\}\right]^n \tag{3.13}$$

So, we may take as this following

$$\Phi\{\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}\}=\exp\{\frac{(\text{In}R_n)}{n}\} \tag{3.14}$$

VI. Optimization in Probabilistic Design with Time/Cycle

First we consider the case in which stress and strength are independent and normally distributed. We know that the reliability depends on the value of the lower limit of the integral. If we want to maximize reliability, the lower limit of the integral should have as low a value as possible.

Let $C_1(\mu_y)$ denote the cost function for the mean strength.

A higher mean value for strength may require that we use better materials, employ different heat treatment processes, or exercise better control on the manufacturing processes for the materials, which would naturally raise the cost.

Thus $C_1(\mu_y)$ is a monotonically increasing function of μ_y.

From the reliability point of view, the lower values of σ_y *for* symmetric distributions are desiable. To reduce σ_y we need to control the factors that introduce variability in the strength such as an unsmooth surface finish, notch effects, or a heterogeneous internal structure. Thus the cost function $C_2(\sigma_y)$ is a monotonically decreasing function of σ_y.

Let $C_3(\mu_x)$ and $C_4(\sigma_x)$ denote the cost functions associated with the mean value and the standard deviation of the stress, respectively. Lower valus of μ_x and σ_x will obviously lead to higher reliability. In order to reduce these values, it may be necessary to increase the dimensions of the component with less dimensional variability, to exercise better control on the operational forces acting on the component. Thus $C_3(\mu_x)$ and $C_4(\sigma_x)$ are both monotonically decreasing functions of μ_x and σ_x respectively.

Now, let us consider the case in which stress and strength with time dependent model are normally distributed and develop the new coupling equpations of each case suggested in previous

section 3 through the following Maximization problem.
maximize the reliability, subject to certain resource constraints ; that is

Objective function of each CASE

1. CASE 1 and CASE 2

$$\max\ R_n = \exp\left[-n\Phi\left(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}\right)\right]$$

$$\max\ \Phi\left(\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}\right)] = \min\ \Phi\left(-\frac{(\mu_y-\mu_x)}{\sqrt{\sigma_y^2+\sigma_x^2}}\right)$$

$$\max\ Z_R = (\mu_y-\mu_x)(\sigma_y^2+\sigma_x^2)^{-1/2}$$

Where Z_R is nothing but the standard variable of the reliability for the stress/strength interference.

2. CASE 3

$$\max\ Rn = \left[\Phi\left(\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}}\right)\right]^n \qquad n=1,2,\cdots$$

hence,

$$\max\ Z^R = (\mu_y-\mu_x)(\sigma_y^2+\sigma_x^2)^{-1/2}$$

subject to

$$C_1(\mu_y)+C_2(\sigma_y)+C_3(\mu_x)+C_4(\sigma_x)\le r_c \tag{4.1}$$

where r denotes the amount of resources available.

$$L(\mu_y,\ \mu_x,\ \sigma_y,\ \sigma_x\ ;\ \lambda^0)$$
$$=(\mu_y-\mu_x)\ (\sigma_y^2+\sigma_x^2)^{-1/2}+\lambda^0[C_y(\mu_1)+C_2(\sigma_y)+C_3(\mu_x)+C_4(\sigma_x)-r_c]$$

To find the locally optimal solutions, we need to solve the following system of equations :

$$\frac{\partial L}{\partial \mu_y} = (\sigma_y^2+\sigma_x^2)^{-1/2}+\lambda^0\frac{\partial C_1(\mu_y)}{\partial \mu_y}=0 \tag{4.2a}$$

$$\frac{\partial L}{\partial \mu_x} = -(\sigma_y^2+\sigma_x^2)^{-1/2}+\lambda^0\frac{\partial C_3(\mu_x)}{\partial \mu_x}=0 \tag{4.2b}$$

$$\frac{\partial L}{\partial \mu_y} = -\sigma_y(\mu_y-\mu_x)\ (\sigma_y^2+\sigma_x^2)^{-3/2}+\frac{\lambda^0\partial C_2(\sigma_y)}{\partial \sigma_y}=0 \tag{4.2c}$$

$$\frac{\partial L}{\partial \sigma_x} = -\sigma_x(\mu_y-\mu_x)\ (\sigma_y^2+\sigma_x^2)^{-3/2}+\frac{\lambda^0\partial C_4(\sigma_x)}{\partial \sigma_x}=0 \tag{4.2d}$$

$$\frac{\partial L}{\partial \lambda^0} = C_1(\mu_y) + C_2(\sigma_y) + C_3(\mu_x) + C_4(\sigma_x) - r_c = 0 \quad (4.2e)$$

From Equations 4.2a and 4.2b we have

$$\frac{\partial C_1(\mu_y)}{\partial(\mu_y)} = -\frac{\partial C_3(\mu_x)}{\partial(\mu_x)} \quad (4.3)$$

and from Equations 4.2c and 4.2d we have

$$\frac{1}{\sigma_y}\frac{\partial C_2(\sigma_y)}{\partial \sigma_y} = \frac{1}{\sigma_x}\frac{C_4(\sigma_x)}{\sigma_x} \quad (4.4)$$

The above relations are used as before to reduce the problem to a search problem in one variable. To insure that any local optimal solution is a global optimal solution, we have to prove that the objective function in Equation 4.1 is a concave function.

This can be proved by studying the Hessian matrix of the objective function, which is

$\frac{(\mu_y-\mu_x)(2\sigma_y^2-\sigma_x^2)}{(\sigma_y^2+\sigma_x^2)^{5/2}}$	$\frac{3\sigma_y\sigma_x(\mu_y-_{\mu}X)}{(\sigma_y^2+\sigma_x^2)^{5/2}}$	$\frac{-\sigma_y}{(\sigma_y^2-\sigma_x^2)^{3/2}}$	$\frac{\sigma_y}{(\sigma_y^2+\sigma_x^2)^{3/2}}$
$\frac{3\sigma_y\sigma_x(\mu_y-\mu_x)}{(\sigma_y^2+\sigma_x^2)^{5/2}}$	$\frac{(\mu_y-\mu_x)(2\sigma_x^2-\sigma_y^2)}{(\sigma_y^2+\sigma_x^2)^{5/2}}$	$\frac{-\sigma_y}{(\sigma_y^2+\sigma_x^2)^{3/2}}$	$\frac{\sigma_x}{(\sigma_y^2+\sigma_x^2)^{3/2}}$
$\frac{-\sigma_y}{(\sigma_y^2+\sigma_x^2)^{3/2}}$	$\frac{-\sigma_x}{(\sigma_y^2+\sigma_x^2)^{3/2}}$	0	0
$\frac{\sigma_y}{(\sigma_y^2+\sigma_x^2)^{3/2}}$	$\frac{\sigma_x}{(\sigma_y^2+\sigma_x^2)^{3/2}}$	0	0

Thus this matrix is negative semidefinite if $(2\sigma_y^2-\sigma_x^2)\geq 0$.

Hence the objective function in Equation 4.1 is concave if we have $\sigma_y/\sigma_x \geq 1/2$, in which case a local optimum will be the global optimum [7, 12].

V. Numerical Example

Maximization Example

The data on the functions for a system are as given below. The units for strength and stress are MPa and the units for cost are \$1,000.

It is often possible to develop analytical expressions for various cost functions that fit the discrete data. The following cost functions represent the above data :

$$C_1(\mu_y) = 0.2\mu_y^{1.5}$$

Table 5—1. Cost data for the values of parameters

μ_y $C_1(\mu_y)$	5 2.23	10 6.32	15 11.62	20 17.89	25 25.00	30 32.86	35 41.41	40 50.60
σ_y $C_2(\sigma_y)$	1 100	2 43.53	3 26.76	4 18.95	5 14.50	6 11.65	7 9.68	8 8.25
μ_x $C_3(\mu_x)$	5 38.07	10 25.12	15 19.69	20 16.57	25 14.50	30 12.99	35 11.85	40 10.93
σ_x $C_4(\sigma_x)$	1 50	2 30.78	3 23.17	4 18.95	5 16.21	6 14.26	7 12.81	8 11.66

$$C_2(\sigma_y) = 100\sigma_y^{-1.2}$$
$$C_3(\mu_x) = 100\mu_x^{-0.5}$$
$$C_4(\sigma_x) = 50\sigma_x^{-0.7}$$

These four cost functions satisfy the convexity requirement because they all have positive second partial derivatives.

$$\max\ Z_k = \left(\frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right) \quad \text{or} \quad \min\ Z_k = \left(-\frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right)$$

CASE 1, 2

$$\max\ R_n = \exp\left[-n\Phi\left(-\frac{(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right)\right]$$

CASE 3

$$\max\ R_n = \left[\Phi\left(\frac{(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right)^n\right]$$

subject to

$$0.2\mu_y^{1.5} + 100\sigma_y^{-1.2} + 100\mu_x^{-0.5} + 50\sigma_x^{-0.7} \le 75 \tag{5.1}$$

The system of equations corresponding to *Eq* 4.2a through 4.2e is

$$\frac{\partial L}{\partial \mu_y} = (\sigma_y^2 + \sigma_x^2)^{-1/2} + \lambda^0 0.3\mu_y^{0.5} = 0 \tag{5.2a}$$

$$\frac{\partial L}{\partial \mu_x} = -(\sigma_y^2 + \sigma_x^2)^{-1/2} - \lambda^0 60\mu_x^{-1.5} = 0 \tag{5.2b}$$

$$\frac{\partial L}{\partial \sigma_y} = -\sigma_y(\mu_y-\mu_x)(\sigma_y^2+\sigma_x^2)^{-3/2}+\lambda^0 \ \ (-120\sigma_y^{-2.2})=0 \tag{5.2c}$$

$$\frac{\partial L}{\partial \sigma_x} = -\sigma_x(\mu_y-\mu_x)(\sigma_y^2+\sigma_x^2)^{-3/2}+\lambda^0 \ \ (-35\sigma_x^{-1.7})=0 \tag{5.2d}$$

$$\frac{\partial L}{\partial \lambda^0} = 0.2\mu_y^{1.5}+100\sigma_y^{-1.2}+100\mu_x^{0.5}+50\sigma_x^{-0.7}-75=0 \tag{5.2e}$$

Using Equation 5.2a and 5.2b we have

$$\mu_x = 27.4248\mu_y^{-0.3125} \tag{5.3}$$

and Equation 5.2c and 5.2d we have

$$\sigma_x = 0.6336\ \sigma_y^{1.185} \tag{5.4}$$

Combining Equation 5.4 and 5.2e yields.

$$100\sigma_y^{-1.2}+68.8183\sigma_y^{-0.8296}=75-[0.2\mu_y^{1.5}+100\mu_x^{-0.6}] \tag{5.5}$$

We use these equations to contruct Table 5-2.

It is clear that once we select a value for μ_y, we can uniquely determine the values of μ_x, σ_y, and σ_x, The dichotomous search technique [7] [12] was used to develop the table using values of μ_y between 20 and 30 MPa.

We may search for the optimal solution by trial and error under the assumption that the function is convex and any local optimal sloution is a global optimal solution.

The optimal solution based on Table 5-2 is :

$\mu_y^* = 24.9380$
$\mu_x^* = 10.0375$
$\sigma_y^* = 6.52508$
$\sigma_x^* = 5.5813$
$Z_R^* = 1.7001$
$R^* = 0.95543$

VI. Conclusion

Up to the present, most of the expressions and computations of reliability for the stress and strength were developed and analyzed by only single stress application (stratic model). However, in the real world, the repeated application of stresses and also the change of the distribution of strength with time varying are considered, since the reliability by the stress and strength is decreasing as passing the time.

Table 5—2. Evaluation of reliability for each parameter

Evaluation Number	μ_y	μ_x	σ_y	σ_x	Z
1	24.5000	10.0937	6.3440	5.6595	1.6946
2	25.5000	10.0406	6.7620	6.1041	1.6970
3	22.5000	10.3654	5.6120	4.8941	1.6296
4	23.5000	10.2255	5.9520	5.2474	1.6729
5	23.3750	10.2426	5.9070	5.2004	1.6687
6	24.3750	10.2094	6.2890	5.6014	1.6838
7	23.9380	10.1667	6.1150	5.4182	1.6858
8	24.9380	10.0375	6.5250	5.8513	1.7001
9	24.2190	10.1297	6.2270	5.5359	1.6909
10	25.2190	10.0024	6.6590	5.9940	1.6984

These above optimal parmeters yield the reliability of each following case with time dependence :

For $R^0=0.95543 (Z=1.7001)$	
CASE 1,2 $n=5$ cycles	$R_n^*=\exp\{-n\ \Phi(-\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}})\}$ $R_5^*=0.80023$
CASE 3 $n=5$ cycles	$R_n^*=[\Phi(-\frac{\mu_y-\mu_x}{\sqrt{\sigma_y^2+\sigma_x^2}})]^n$ $R_5^*=[0.95543]^5=0.79614$

This paper presented the new techniques to solve the optimization problem at the design stage where the stress and strength in the time dependent model have the s-normal and the cost functions for controlling the parameters of these distributions are known. The expressions and computations of reliability and the optimization technique in design reliability discussed here can also be extended to the case when the distributions under consideration are log-normal, exponential, gamma, weibull, and extreme-value cases. The potimization process for log-normal case or exponential case is similar in nature to the normal random variable case. For gamma, weibull, extreme-value cases, closed form expressions for reliability are not possible [17, 18] and the optimization process will involve using a series of tables and numerical methods.

Various search procedures will have to be adapted for this case 7, 8, 12, 13, 20, 21, 23. The problem of reliability computations are complex when we consider correlated stress and strength variables. Further research is being pursued to consider stochastic processes in the design methodology from reliability viewpoint. It is hoped that techniques from stochastic programming will be useful for the optimization problems under consideration.

APPENDIX

Proof by Another Way for CASE 1(Using Stochastic Theory)

Based on the notion of conditional probabilities and conditional densities. Using the compound probability law,

$$R_n = P[E_1,\ E_2,\ \cdots,\ E_n] = P[E_2|E_1,\ \cdots,\ E_{n-1}]$$

or this can be expressed as a recurrence relation

$$\begin{aligned} R_n &= R_{n-1}\ P(E_n|E_1,\ \cdots,\ E_{n-1}) \\ &= P(E_n|E_1,\ E_2,\ \cdots,\ E_{n-1}) \cdot (P(E_1, E_2, E_{n-1}) \end{aligned} \tag{A1}$$

Now,

$$P(E_1) = P(X_1 < Y) = \int_0^\infty g(y)\ \{\int_0^y f(x)\,dx\}\,dy \tag{A2}$$

In order to find $P(E_n|E_1, \cdots, E_{n-1})$, we replace $g(y)$ in Eq. (A2) by appropriate conditional strength density.

Thus

$$\begin{aligned} P = (E_n|E_1, \cdots, E_{n-1}) &= P(X_n < Y|E_1, \cdots, E_{n-1}) \\ &= \int_0^\infty g_n(Y|E_1, \cdots, E_{n-1})\ \{\int_0^y f(x)\,dx\}\ dy \end{aligned} \tag{A3}$$

Where $g_n(Y|E_1, \cdots, E_{n-1})$ is the conditional $p, d, f,$ of random fixed strength Y given nonfailures were observed on each of the first $(n-1)$ cycles. The conditional probability element is written as

$$g_n(Y^*|E_1, \cdots, E_{n-1})\,dy = P[Y^* - \frac{dy}{2} \le Y \le Y^* + \frac{dy}{2}\ |\ E_1, \cdots, E_{n-1}]$$

$$= \frac{P[Y^* - \frac{dy}{2} \le Y \le Y^* + \frac{dy}{2} | E_1, \cdots, E_{n-1}]}{P[E_1, \cdots, E_{n-1}]} \tag{A4}$$

Let event $A = Y^* - (dy/2) \le y \le y^* + (dy/2)$ and event Ei as $Xi < Y^*$, and further, event $Bn = E_1 \cap E_1 \cap \cdots \cap E_{n-1}$. Then $Eq.$ (A4) can be written as

$$g_n(Y^*|E_1, \cdots, E_{n-1})\,dy = \frac{P(A \cap B_n)}{P(B_1)}$$

Again, we have

$$P(A\cap B_n)=P(B_n|A)\ P(A) \tag{A5}$$
$$=P(A)\ P(E_1,\cdots,E_{n-1}|A)$$
$$=P(A)\ P(E_1|A)\ PA(E_2|A\cap E_1)\cdots$$
$$\cdots P(E_{n-1}|A\cap E_1\cap\cdots\cap E_{n-2})$$

Since the stresses are mutually independent, each of the last $(n-1)$ terms in *Eq.* (A 5) is $\int_0^{y*} f(x)\,dx$

Thus,

$$P(A\cap B_n)=P(Y^*+dy/2\leq y\leq Y^*-dy/2[\int_0^{y*} f(x)\,dx]^{n-1}$$
$$g(Y^*)\,dy\ [\int_0^{y*} f(x)\,dy]^{n-1}$$

Thus, in terms of the density function *Eq.* (A 4) may be written as

$$g_n(Y|E_1,\cdots,E_{n-1})=\frac{g(y)[\int_0^{y} f(x)\,dx]^{n-1}}{R_{n-1}} \tag{A6}$$

Substituting *Ea.* (A 6) in *Eq.* (A 3), the conditional probability of success for the nth occurance as

$$P[E_n|E_1,\cdots,E_{n-1}]=\frac{\int_0^{\infty} g(y)\ [\int_0^{y} f(x)\,dx]^n dy}{R_{n-1}} \tag{A7}$$

$$\therefore R_n=\int_0^{\infty} g(y)\ [\int_0^{y} f(x)\,dx]^n dy$$

BIBLIOGRAPHY

1. Aerospace Structural Metals Handbook (1963), Syracuse University Press, Syracuse, N.Y.
2. American Society for Metals (ASM), Metals Handbook(1969), "Properties and Selection," Vol.1, 8th Ed.
3. ASM(1969), Vol. I, "*Properties and Selection of Materials ; Vol, II, Heat Treatment, Cleaning and Finishing ; Vol. III, Machining All Volumes Published by ASM*", Metals Park, Ohio.
4. American Society for Testing Materials

(1951), "*Symposium on Statistical Aspects of Fatigue*", ASTM Special Technical Publication No. 121, June 19.

5. ASTM(1952), "*Symposium on Fatigue with Emphasis on Statistical Approach -II*", ASTM special Technical Publication No. 137, June 24.
6. ASTM(1963) "*A Guide for Fatigue Testing and the Statistical Analysis of Fatigue Data*", ASTM Special Technical Publication No. 91-A, 2nd Ed.
7. Beveridge, G. S. C. and Schechter R. S. (1979), "Optimization Theory and Practice", McGraw Hill Book Co. NY.
8. Cohn, J. C. (1978), "*Multiobjective Programming and Planning*" Academic Press, New York.
9. Disney, R. L., Sheth, N. J. and Lipon, C. (1968), "*The Determination of the Probability of Failure by Stress-Strength INterference Theory*", Proceedings of Amual Symposium on Reliability, pp. 417-422.
10. Haugen, E. B. (1974, 1980), "*Probabilistic Mechanical Design*" John Wiley and Sons Inc. NJ 08873.
11. Hahn, G. J., Nelson, W. (1974), "*Comparison of Methods for Analyzing Censored Life Data to Estimate Relationships between Stress and Product Life*", IEEE Trans.
12. Himmelblau, D. M. (1973), "*Applied Nonlinear Programming*", McGraw-Hill New Youk.
13. Hollander, M. and Wolfe, D. A. (1983), "*Nonparametric Statistical Method*", John Wiley and Sons, NY.
14. Iuculano, G., Zanini, A. (1984), "*Statistical Behaviour of Functional Models Suitable for Controlling Life Step-Stress Test*", Alta Frequenze Vol. 53.
15. Juvenal, R. C. and Lipson, e. (19.., "*Handbook of Stress and Streng...*" Memillan, New York.
16. Kapur, K. C. (1975), "*Reliability Bounds in Probabilistic Design*", IEEE Trans Reliability Vol. R-24, No. 3.
17. Kececioglu, D. and Cormier, D. (1967), "*Designing a Specified Reliability Directly into A Component*", Proceedings of the Third Annual Aerospace Reliability and Maintainability Conference, Washington D. C., pp. 546-564.
18. Lipson, Cl, Sheth, N. J. and Disney, RIL. (1969), "*Reliability Predication Mechanical Stress/Strength Interference*", Rome Air Development Center, Technical Report No. RADC-TR-77-810 1967 and Final Technical Report RADC -TR68-403.
19. Mann, N. R. (1972), "*Design of Over -Stress Life-Test Experiments When Failure Times Have The Two-Parameter Weibull distribution*", Technometrics, Vol. 14, pp. 437-451.
20. Moltoft, J., Jensen, F. (1986), "*Reliability Technology-Theory and applications*", North Holland (Elsevier Science Publishers).
21. Passy, V., Wilde, D. J. (1967), "*Generalized Polynomial Optimization*", SIAM J. Applied Math, Vol. 15, pp. 1344-1356.
22. Saklvia, A. A., bollinger, R. C. (1982), "*On Discrete Hazard Functions*", IEEE Trans, Reliability Vol. R-31, pp. 458-459.
23. Smith, D. J. (1981), "*Reliability and Maintainability IN Perspective*", Halsted Press, John Wiley and Sons.