Kyungpook Mathematical Journal
Volume 30, Number 1, June, 1990

SEMI–CONTINUOUS AND SEMI–WEAKLY CONTINUOUS FUNCTIONS

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1. Introduction

In 1937 regular open sets were introduced. Let (X,T) be a space and let $A,B,C \subset X$. Then A is *regular open*, denoted by $A \in RO(X,T)$, iff $A = Int(\bar{A})$ [14]. In 1937 it was shown that $RO(X,T)$ is a base for a topology T_s on X coarser than T and (X,T_s) was called the *semiregularization space* of (X,T). In 1963 semi open sets and semi–continuous functions were introduced. The subset A is *semi open*, denoted by $A \in SO(X,T)$, iff $A \subset \overline{Int(A)}$ [8]. If (Y,S) is a space and $f:(X,T) \to (Y,S)$ is a function, then f is *semi-continuous* iff for each $V \in S$, $f^{-1}(V) \in SO(X,T)$ [8]. In 1970 semi open sets were used to define semi closed sets, which were used to define the semi closure of a set. The subset B is *semi closed* iff $X - B$ is semi open and the *semi closure* of C, denoted by $sclC$, is the intersection of all semi closed sets containing C [1]. In 1978 the semi closure operator was used to define feebly open sets. The subset A is *feebly open*, denoted by $A \in FO(X,T)$, iff $A \subset scl(Int(A))$[10]. Further investigations of feebly open sets have shown that $FO(X,T)$ is a topology on X and $T \subset FO(X,T) = FO(X,FO(X,T))$[3], $SO(X,T) = SO(X,FO(X,T))$[4], $FO(X,T)_s = T_s$ [5], and $RO(X,T) = \{scl0|0 \in T\}$ [6].

In 1961 weakly continuous functions were introduced. A function $f:(X,T) \to (Y,S)$ is *weakly continuous* iff for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \bar{V}$ [9]. Then in 1985 semi–weakly continuous functions were introduced. A function $f:(X,T) \to (Y,S)$ is *semi-weakly continuous* iff for each $x \in X$ and each open set V containing $f(x)$, there exists a

Received March 12, 1987

semi open set U containing x such that $f(U) \subset sclV$ [11]. In this paper feebly open sets, semi–continuity, and semiregularization spaces are used to further characterize semi–weakly continuous functions and the results are used to further investigate semi–weakly continuous functions and to extend known results for semi–weakly continuous functions.

2. New Characterizations

Theorem 2.1. *Let $f : (Y, S) \to (X, T)$ be a function. Then the following are equivalent:*

(a) $f : (Y, S) \to (X, T)$ *is semi-weakly continuous,*
(b) $f : (Y, FO(Y, S)) \to (X, T)$ *is semi-weakly continuous,*
(c) $f : (Y, S) \to (X, T_s)$ *is semi-continuous,*
(d) $f : (Y, FO(Y, S)) \to (X, T_s)$ *is semi-continuous, and*
(e) $f : (Y, S) \to (X, T_s)$ *is semi-weakly continuous.*

Proof. Since $SO(Y, S) = SO(Y, FO(Y, S))$, then clearly (a) and (b) are equivalent and (c) and (d) are equivalent.

(a) implies (c): Let $y \in Y$ and let $V \in T_s$ such that $f(y) \in V$. Since $RO(X, T) = \{scl0 | 0 \in T\}$ is a base for T_s, then there exists $0 \in T$ such that $f(y) \in scl0 \subset V$. Then $f(y) \in scl0 \in T_s \subset T$ and there exists a semi open set U such that $y \in U$ and $f(U) \subset scl(scl0) = scl0 \subset V$. Since $SO(Y, S)$ is closed under arbitrary unions [8], then $f : (Y, S) \to (X, T_s)$ is semi–continuous.

Since every semi–continuous function is semi–weakly continuous [11], then (c) implies (e).

(e) implies (a): Since $f : (Y, S) \to (X, T_s)$ is semi–weakly continuous, then by the argument above $f : (Y, S) \to (X, (T_s)_s)$ is semi–continuous and since $(T_s)_s = T_s$ [2], then $f : (Y, S) \to (X, T_s)$ is semi–continuous. Let $y \in Y$ and let $V \in T$ such that $f(y) \in V$. Then $sclV \in T_s$ and there exists $U \in SO(Y, S)$ such that $y \in U$ and $f(U) \subset sclV$.

Using the results above and the fact that for a space (X, T), $T_s = FO(X, T)_s$, gives the following additional characterizations of semi–weakly continuous functions.

Corollary 2.1. *Let $f : (Y, S) \to (X, T)$ be a function. Then the following are equivalent:*

(a) $f : (Y, S) \to (X, T)$ *is semi-weakly continuous,*
(b) $f : (Y, S) \to (X, FO(X, T))$ *is semi-weakly continuous,*
(c) $f : (Y, FO(Y, S)) \to (X, FO(X, T))$ *is semi-weakly continuous, and*

(d) $f : (Y, FO(Y,S)) \to (X, T_s)$ *is semi-weakly continuous.*

3. New Properties and Extensions of Known Properties

Theorem 3.1. *Let* $f : (Y,S) \to (X,T)$ *be semi-weakly continuous and let* $U \in S$. *Then* $f/U : (U, S_U) \to (X,T)$ *is semi-weakly continuous.*

Proof. Since $f : (Y,S) \to (X,T_s)$ is semi-continuous and $U \in S$, then by Theorem 3 in [12], $f/U : (U, S_U) \to (X,T_s)$ is semi-continuous, which implies $f/U : (U,S_U) \to (x,T)$ is semi-weakly continuous.

Theorem 3.2. *Let* $f : (Y,S) \to (X,T)$ *be a function and let* $\{A_\alpha | \alpha \in A\}$ *be a cover of* Y *by semi open sets such that* $f/A_\alpha : (A_\alpha, S_{A\alpha}) \to (X,T)$ *is semi-weakly continuous for each* $\alpha \in A$. *Then* $f : (Y,S) \to (X,T)$ *is semi-weakly continuous.*

The proof is straightforward using results above and Theorem 4 in [12] and is omitted.

Theorem 3.3. *For each* $\alpha \in A$ *let* (Y_α, S_α) *and* (X_α, T_α) *be spaces and let* $f_\alpha : Y_\alpha \to X_\alpha$ *be function, let* $Y = \Pi_{\alpha \in A} Y_\alpha$, *let* S *denote the usual product topology on* Y, *let* $X = \Pi_{\alpha \in A} X_\alpha$, *and let* T *denote the usual product topology on* X. *Then the function* $f : (Y,S) \to (X,T)$ *defined by* $f((y_\alpha)_{\alpha \in A}) = (f_\alpha(y_\alpha))_{\alpha \in A})$ *is semi-weakly continuous iff* $f_\alpha : (Y_\alpha, S_\alpha) \to (X_\alpha, T_\alpha)$ *is semi-weakly continuous for each* $\alpha \in A$.

Proof. Let W denote the product topology on X determined by $\{(X_\alpha, (T_\alpha)_s | \alpha \in A\}$. Then $T_s = W$[7].

Suppose $f_\alpha : (Y_\alpha, S_\alpha) \to (X_\alpha, T_\alpha)$ is semi-weakly continuous for each $\alpha \in A$. Then $f_\alpha : (Y_\alpha, S_\alpha) \to (X_\alpha, (T_\alpha)_s)$ is semi-continuous for each $\alpha \in A$ and by Theorem 5 in [12], $f : (Y,S) \to (X,W) = (X,T_s)$ is semi-continuous which implies $f : (Y,S) \to (X,T)$ is semi-weakly continuous

The proof of the converse statement is straightforward using Theorem 5 in [12] and results above and is omitted.

Theorem 3.4. *For each* $\alpha \in A$ *let* (X_α, T_α) *be a space, let* X, T, *and* W *be as in the statement and proof of Theorem 3.3, let* (Y,S) *be a space, let* $f : (Y,S) \to (X,T)$ *be semi-weakly continuous, and let* $p_\beta : X \to X_\beta$ *be the projection function for each* $\beta \in A$. *Then* $p_\beta \circ f : (Y,S) \to (X_\beta, T_\beta)$ *is semi-weakly continuous for each* $\beta \in A$.

Proof. Let $\beta \in A$. Since $f : (Y,S) \to (X,T_s) = (X,W)$ is semi-continuous and $P_\beta : (X,W) \to (X_\beta, (T_\beta)_s)$ is the projection function, then by Theorem 6 in [12], $p_\beta \circ f : (Y, S) \to (X_\beta, (T_\beta)_s)$ is semi-continuous, which implies $p_\beta \circ f : (Y, S) \to (X_\beta, T_\beta)$ is semi-weakly continuous.

Example 11 in [8], which was used to show that the composition of two semi-continuous functions need not be semi-continuous, also shows that the composition of two semi-weakly continuous functions need not be semi-weakly continuous.

Theorem 3.5. *Let (X_α, T_α) be a space for each $\alpha \in A$, let X, T, and W be as in the statement and proof of Theorem 3.3, let $B \subset A$, and let P denote the usual product topology on X determined by $\{(X_\alpha, T_\alpha) | \alpha \in A - B\} \cup \{(X_\alpha, (T_\alpha)_s) | \alpha \in B\}$. Then $P_s = T_s$ and $SO(X, P) \subset SO(X, T)$.*

Proof. Since P_s is the usual product topology on X determined by $\{(X_\alpha (T_\alpha)_s) | \alpha \in A - B\} \cup \{(X_\alpha, ((T_\alpha)_s)_s) | \alpha \in B\}$ and $(T_\alpha)_s = ((T_\alpha)_s)_s$ for each $\alpha \in B$, then P_s is the usual product topology on X determined by $\{(X_\alpha, (T_\alpha)_s) | \alpha \in A\}$, which implies $P_s = W = T_s$. Let $\mathcal{O} \in SO(X, P)$. Since $P \subset T$, then $Int_p(\mathcal{O}) \subset Int_T(\mathcal{O})$ and since $Int_p(\mathcal{O}) \in P$, then $\overline{Int_p(\mathcal{O})}_p = \overline{Int_p(\mathcal{O})}_{p_s}[2] = \overline{Int_p(\mathcal{O})}_{T_s} \subset \overline{Int_T(\mathcal{O})}_{T_s} = \overline{Int_T(\mathcal{O})}_T$. Since $\mathcal{O} \in SO(X, P)$, then $\mathcal{O} \subset \overline{Int_p(\mathcal{O})}_p \subset \overline{Int_T(\mathcal{O})}_T$, which implies $\mathcal{O} \in SO(X, T)$.

Theorem 3.6. *Let (X_i, T_i) be a space for each $i \in A = \{1, 2\}$, let X and T be as in the statement of Theorem 3.3, let $f : X_1 \to X_2$ be a function, and let $g : X_1 \to X$, defined by $g(x) = (x, f(x))$ for each $x \in X_1$, denote the graph function of f. Then $f : (X_1, T_1) \to (X_2, T_2)$ is semi-weakly continuous iff $g : (X_1, T_1) \to (X, T)$ is semi-weakly continuous.*

Proof. Suppose $f : (X_1, T_1) \to (X_2, T_2)$ is semi-weakly continuous. Let P be the usual product topology on X determined by $\{(X_1, T_1), (X_2, (T_2)_s)\}$. Since $f : (X_1, T_1) \to (X_2, (T_2)_s)$ is semi-continuous, then $g : (X_1, T_1) \to (X, P)$ is semi-continuous [13], which implies $g : (X_1, T_1) \to (X, P_s) = (X, T_s)$ is semi-continuous and $g : (X_1, T_1) \to (X, T)$ is semi-weakly continuous. The converse statement is Theorem 2 in [11].

Let Y and X be sets and let $f : Y \to X$ be a function. Then $\{(y, f(y)) | y \in Y\}$ is called the graph of f and is denoted by $G(f)$.

Theorem 3.7. *Let (X_i, T_i) be a space for each $i \in \{1, 2\}$, where (X_2, T_2) is Hausdorff, let X and P be as in the statement and proof of Theorem 3.6, and let $f : (X_1, T_1) \to (X_2, T_2)$ be semi-weakly continuous. Then $G(f)$ is semi closed in (X, P).*

Proof. Since $f : (X_1, T_1) \to (X_2, T_2)$ is semi-weakly continuous, then the graph function $g : (X_1, T_1) \to (X, T_s)$ is semi-continuous and since (X_2, T_2) is Hausdorff, then $(X_2, (T_2)_s)$ is Hausdorff [2], which implies $G(f)$ is semi closed in (X, P) [13].

Combining the results above gives the next result, which was given as a theorem in [11].

Corollary 3.1. *Let (X_i, T_i) be a space for each $i \in \{1,2\}$, where (X_2, T_2) is Hausdorff, let X and T be as in the statement of Theorem 3.6, and let $f : (X_1, T_1) \to (X_2, T_2)$ be semi-weakly continuous. Then $G(f)$ is semi closed in (X, T).*

A space (X, T) is *S-connected* iff X can not be written as a union of two nonempty disjoint semi open sets [15].

Combining Theorem 6 in [11] with results above give the next result.

Corollary 3.2. *Let (X_i, T_i) be a space for each $i \in \{1,2\}$, where (X_1, T_1) is S-connected, let X and P be as in Theorem 3.7, and let $f : (X_1, T_1) \to (X_2, T_2)$ be semi-weakly continuous such that $G(f)$ is closed in (X, P). Then f is constant.*

The following example shows that the statement "$G(f)$ is closed in (X, P)" in Corollary 3.2 cannot be replaced by the statement "$G(f)$ is closed in (X, T)", where T is the usual product topology determined by $\{(X_1, T_1), (X_2, T_2)\}$.

Example 3.1. Let N denote the positive integers, let E denote the even positive integers, let 0 denote the odd positive integers, let A denote the negative integers, let $X_1 = N \cup A$, let $X_2 = A \cup \{0, 1, 2\}$, and let

$$f : X_1 \to X_2 \text{ defined by } f(x) = \begin{cases} x & \text{if } x \in A \\ 1 & \text{if } x \in O \\ 2 & \text{if } x \in E \end{cases}.$$

Then $T_1 = \{\emptyset\} \cup \{B \subset X_1 | A - B \text{ is finite }\}$ is a topology on X_1 and $\{0 \cup \{p\} | p \in X_2\}$ is a base for a topology T_2 on X_2. Then (X_1, T_1) is S-connected, $f : (X_1, T_1) \to (X_2, T_2)$ is semi-weakly continuous, $G(f)$ is closed in the usual product space determined by $\{(X_1, T_1), (X_2, T_2)\}$, and f is not constant.

References

[1] N. Biswas, "On Characterizations of Semi-Continuous Functions," Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8, 48 (4) (1970), 399-402.

[2] N. Bourbaki, Gerneral Topology, Addison-Wesley Publishing Co., 1966.

[3] C. Dorsett, "Feeble Separation Axioms, the Feebly Induced Topology, and Separation Axioms and the Feebly Induced Topology," Karadeniz Univ. Math. J., 8(1985), 43-54.

[4] _______, " New Characterizations of Certain Functions and Connected and Separable Spaces, and the Feebly Induced Topology," submitted.

[5] _______, " New Characterizations of Topological Properties Using Regular Open Sets and r-Topological Properties," submitted.

[6] _______, "Properties of Topological Spaces and the Semiregularization Topology," submitted.

[7] L. Herrington, "Properties of Nearly-Compact Spaces," Proc. A.M.S., 45 no. 3 (1974), 431-436.

[8] N. Levine, "Semi-open Sets and Semi-Continuity in Topological Spaces," Amer. Math. Monthly, 70 (1963), 36-41.

[9] _______, " A Decomposition of Continuity in Topological Spaces," Amer. Math. Monthly, 68 (1961), 44-46.

[10] S. Maheshwari and U. Tapi, " Note on Some Applications of Feebly Open Sets," Madhya Bharati, J. Un. Saugar, 1978-1979.

[11] T. Noiri and B. Ahmad, " On Semi-Weakly Continuous Mappings," Kyungpook Math. J., 25 no. 2 (1985), 123-126.

[12] T. Noiri, " On Semi-continuous Mappings," Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8, 54(1973), 210-214.

[13] _______, " A Note On Semi-continuous Mappings," Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8, 55(1973), 400-403.

[14] M. Stone, "Applications of the Theory of Boolean Rings to General Topology," Trans. A.M.S., 41 (1937), 374-481.

[15] T. Thompson, " Characterizations of Irreducible Spaces," Kyungpook Math. J., 21(1978), 191-194.

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