Kyungpook Mathematical Journal
Volume 30, Number 1, June, 1990

ON PAIRWISE FEEBLY R_0-SPACES

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Introduction

In a topological space X a set A is *semiopen* [3] if for some open set O, $O \subset A \subset clO$, where clO denotes the closure of O in X. Complement of a semi open set is *semiclosed*. The intersection of all the semi closed sets containing a set A is called the semiclosure [1] of A and denote it by $sclA$. In a topological space X a set A is termed *feebly open* [4] if for some open set O, $O \subset A \subset sclO$. Every open set is feebly open and every feebly open set is semiopen but the converses may be false [4]. Any union of feebly open sets is feebly open [4]. The complement of a feebly open set is *feebly closed* [4]. The intersection of all the feebly closed sets containing a set A is the *feebly closure* [4] of A. Denote it by $fclA$. It is always feebly closed. Further [4] A is feebly closed iff $A = fclA$; $A \subset fclA \subset clA$; $A \subset B$ implies $fclA \subset fclB$; and $p \in fclA$ iff each feebly open set containing p meets A.

The theory of bitopological spaces was first developed by Kelly [2] in 1963. A bitopological space (X, P_1, P_2) is a nonempty set X equipped with two topologies P_1 and P_2. The axioms of pairwise feebly T_0 and pairwise feebly T_1 (stated below) are strictly weaker than pairwise T_0 and pairwise T_1 respectively [6].

Definition [5]. A bitopological space (X, P_1, P_2) is *pairwise feebly* T_0 (resp. *pairwise feebly* T_1) if for each pair of distinct points x, y of X there exists a P_1-feebly open set containing x but not y or (resp. and) a P_2-feebly open set containing y but not x.

The aim of this paper is to investigate and study an axiom which is independent of both these axioms and that pairwise feebly T_0 implies

Received June 28, 1989

pairwise feebly T_1. At the same time it also presents the role of feebly open sets in topology.

Throughout the paper $X\backslash A$ denotes the complement of A in X and $i, j = 1, 2$ such that $i \neq j$.

Definition 1. A bitopological space (X, P_1, P_2) is *pairwise feebly* R_0, if for every P_i–feebly open set G, $x \in G$ implies that P_j–$fcl\{x\} \subset G$.

Remark 1. Pairwise feebly R_0 is independent of both pairwise feebly T_1 and pairwise feebly T_0, as shown by the following examples;

Example 1. Let $X = \{a, b, c\}, P_1 = \{\emptyset, \{a\}, X\}$ and $P_2 =$ the discrete topology on X. Then (X, P_1, P_2) is pairwise feebly T_1 and consequently pairwise feebly T_0 also but it is not pairwise feebly R_0.

Example 2. Let $X = \{a, b, c\}$ and $P_1 = P_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then (X, P_1, P_2) is pairwise feebly R_0 but it is not pairwise feebly T_0 and so it is not pairwise feebly T_1.

Theorem 1. *Every pairwise feebly T_0 pairwise feebly R_0 space (X, P_1, P_2) is pairwise feebly T_1.*

Proof. Let $x, y \in X$ and $x \neq y$. Since X is pairwise feebly T_0, there is a set which is either P_1–feebly open or P_2–feebly open containing one of the points but not the other. Let G be P_1–feebly open and $x \in G$ but $y \notin G$. Since X is pairwise feebly R_0, $P_2 - fcl\{x\} \subset G$. Then $X\backslash P_2$–$fcl\{x\}$ is a P_2–feebly open set containing the point y but not x. Consequently X is pairwise feebly T_1.

Definition 2. In a bitopological space (X, P_1, P_2), for any $x \in X$, *bi–fcl* $\{x\} = P_1$–$fcl\{x\} \cap P_2$–$fcl\{x\}$ and *bi–fker* $\{x\} = P_1$–$fker\{x\} \cap P_2$–$fker\{x\}$.

The following theorem gives several characterizations of pairwise feebly R_0–spaces.

Theorem 2. *In a bitopological space (X, P_1, P_2) the following conditions are equivalent:*

(a) *(X, P_1, P_2) is pairwise feebly R_0.*
(b) *If $x \in X, P_i$-$fcl\{x\} \subset P_j$-$fker\{x\}$.*
(c) *If $x, y \in X, y \in P_i$-$fker\{x\}$ if and only if $x \in P_j$-$fker\{y\}$.*
(d) *If $x, y \in X, y \in P_i$-$fcl\{x\}$ if and only if $x \in P_j$ -$fcl\{y\}$.*
(e) *If F is P_i-feebly closed and $x \notin F$ then there exists a P_j-feebly open*

set G such that $x \notin G$ and $F \subset G$.

(f) *If F is P_i-feebly closed, then $F = \cap\{G|G$ is P_j-feebly open and $F \subset G\}$.*

(g) *If G is P_i-feebly open then $G = \cup\{F|F$ is P_j-feebly closed, $F \subset G\}$.*

(h) *If F is P_i-feebly closed and $x \notin F$ then P_j-$fcl\{x\} \cap F = \emptyset$.*

Proof. (a)$\Longrightarrow$ (b). For any $x \in X$, we have P_j–$fker\{x\} = \cap$ $\{P_j$–feebly open set $G|x \in G\}$. By (a), each P_j–feebly open set G containing x contains P_i–$fcl\{x\}$. Hence, P_i–$fcl\{x\} \subset P_j$–$fker\{x\}$.

(b)$\Longrightarrow$ (c). For any $x, y \in X$, if $y \in P_i$–$fker\{x\}$ then $x \in P_i$–$fcl\{y\}$. Now by (b), $x \in P_j$–$fker\{y\}$. Similarly if $x \in P_j$–$fker\{y\}$, then $y \in P_i$–$fker\{x\}$.

(c)$\Longrightarrow$ (d). For any $x, y \in X$, if $y \in P_i$–$fcl\{x\}$ then $x \in P_i$–$fker\{y\}$. By (c), $x \in P_j$–$fcl\{y\}$. Similarly if $x \in P_j$–$fcl\{y\}$, then $y \in P_i$–$fcl\{x\}$.

(d)$\Longrightarrow$(e). Let F be a P_i–feebly closed set and $x \notin F$. Then for any point $y \in F$ implies P_i–$fcl\{y\} \subset F$ implies $x \notin P_i$–$fcl\{y\}$. Now by (d), $x \notin P_i$–$fcl\{y\}$ implies $y \notin P_j$–$fcl\{x\}$. That is there exists a P_j–feebly open set G_y such that $y \in G_y$ and $x \notin G_y$. Let, $G = \cup_{y \in F}\{G_y|G_y$ is P_j–feebly open, $y \in G_y$ and $x \notin G_y\}$. Then G is P_j–feebly open such that $F \subset G$ and $x \notin G$.

(e)$\Longrightarrow$(f). Let F be a P_i–feebly closed set and suppose that $H = \cap\{P_j$–feebly open set $G|F \subset G\}$. Clearly, $F \subset H$. Let $x \notin F$ then by (e) there exists a P_j–feebly open set G such that $x \notin G$ and $F \subset G$. Hence, $x \notin H$. And so, $F = H$.

(f)$\Longrightarrow$(g). Evident.

(g)$\Longrightarrow$(h). Let F be a P_i–feebly closed set and $x \notin F$. Then $X \backslash F = G$ (say) is a P_i–feebly open set containing x. By (g), there exists a P_j–feebly closed set H such that $x \in H \subset G$. Therefore, P_j–$fcl\{x\} \subset G$. Hence P_j–$fcl\{x\} \cap F = \emptyset$.

(h)$\Longrightarrow$(a). Let G be P_i–feebly open and $x \in G$. Then $x \notin X \backslash G$ which is P_i–feebly closed. By (h), $P_j - fcl\{x\} \cap (X \backslash G) = \emptyset$. This implies that P_j–$fcl\{x\} \subset G$. Thus (a) holds.

Theorem 3. *If (X, P_1, P_2) is pairwise feebly R_0 and $x, y \in X$, then either bi-$fcl\{x\}$ equals bi-$fcl\{y\}$ or they are disjoint.*

Proof. Suppose that, $(bi$–$fcl\{x\}) \cap (bi$–$fcl\{y\}) \neq \emptyset$. Let $p \in P_1$—$fcl\{x\} \cap P_2$–$fcl\{x\} \cap P_1$–$fcl\{y\} \cap P_2$–$fcl\{y\}$. Then P_1–$fcl\{p\} \subset P_1$–$fcl\{x\} \cap P_1$–$fcl\{y\}$ and P_2–$fcl\{p\} \subset P_2$–$fcl\{x\} \cap P_2$–$fcl\{y\}$. Now by (d) in Theorem 2, we get $p \in P_1$–$fcl\{x\} \Rightarrow x \in P_2$–$fcl\{p\} \Rightarrow P_2 - fcl\{x\} \subset P_2$–$fcl\{p\}$ $\subset P_2$–$fcl\{y\}$. And so, $p \in P_1$–$fcl\{x\} \Rightarrow P_2$–$fcl\{x\} \subset P_2$–$fcl\{y\}$.

Similarly, $p \in P_2\text{-}fcl\{x\} \Rightarrow P_1\text{-}fcl\{x\} \subset P_1\text{-}fcl\{y\}$. $p \in P_1\text{-}fcl\{y\} \Rightarrow P_2\text{-}fcl\{y\} \subset P_2\text{-}fcl\{x\}$ and $p \in P_2\text{-}fcl\{y\} \Rightarrow P_1\text{-}fcl\{y\} \subset P_1\text{-}fcl\{x\}$. Consequently, $bi\text{-}fcl\{x\} = bi\text{-}fcl\{y\}$.

Theorem 4. *If (X, P_1, P_2) is pairwise feebly R_0 and $x, y \in X$ then either $bi - fker\{x\}$ equals $bi - fker\{y\}$ or they are disjoint.*

In view of (c) in Theorem 2, this follows as Theorem 3.

Theorem 5. *Every biopen subspace of a pairwise feebly R_0 space (X, P_1, P_2) is pairwise feebly R_0.*

The proof requires the following lemmas :

Lemma 1 [4]. *Let Y be a subspace of a topological space. If O is feebly open in Y and Y is open in X then O is feebly open in X.*

Lemma 2 [4]. *If U is open and V feebly open in a topological space X then $U \cap V$ is feebly open in U.*

Proof. Let (Y, T_1, T_2) be a biopen subspace of (X, P_1, P_2). Let A be T_i–feebly closed and $x \in Y, x \notin A$. Then $Y \backslash A$ is T_i–feebly open. So $Y \backslash A$ is P_i–feebly open by Lemma 1. Now $X \backslash (Y \backslash A) = (X \backslash Y) \cup A$, is P_i–feebly closed and x does not belong to it. Therefore, by (e) in Theorem 2 there is a P_j–feebly open set G such that $x \notin G$, and $(X \backslash Y) \cup A \subset G$. Since Y is P_j–open and G is P_j–feebly open, it follows by Lemma 2. that $Y \cap G$ is T_j–feebly open and it contains A and $x \notin Y \cap G$. Hence by (e) of Theorem 2, it results that (Y, T_1, T_2) is pairwise feebly R_0.

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