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INITIAL VALUE PROBLEM OF HIGHER ORDER INTEGRO–DIFFERENTIAL EQUATIONS

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1. Introduction

Let X be a Banach space, $f(t)$ be a continuous function in X, and $(A_j(t), t \in I, j = 1, 2, \cdots, k)$ be a family of bounded linear operators defined on X, such that for $h(t) \in X$ we have the relation

$$\|A_j(t)h(t)\| \leq N_j \|h(t)\| \tag{1.1}$$

where N_j, $j = 1, 2, \cdots, k$ are positive constants. Consider now the higher order differential equation

$$D^k x(t) = \sum_{j=1}^{k} A_j(t) D^{k-j} x(t) + f(t) \tag{1.2}$$

with the initial data

$$D^j x(t)|_{t=0} = g_j, j = 0, 1, 2, \cdots k-1 \tag{1.3}$$

where $D = d/dt$. The initial value problem of different forms of higher order differential equations has been considered in [1], [2], [4] and others. Herein the initial value problems (1.2) and (1.3) are considered in X, the existence, uniqueness and smoothness of the solution are proved and the application of higher order integro–differential equations is given.

2. Solution of the problem.

By using the same argument as in [1] and [2], the following lemma can easly proved.

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Lemma 2.1. *Let $v_j(t) = D^{k-j}x(t)$, then the initial value problem (1.2) and (1.3) can be transformed to the one*

$$\frac{dV(t)}{dt} = A^*(t)V(t) + F(t) \tag{2.1}$$

and

$$V_0 = (v_1, v_2, \cdots, v_k) = (g_{k-1}, g_{k-2}, \cdots, g_0) \tag{2.2}$$

where $V(t) = (v_1(t), \cdots, v_k(t))$, $F(t) = (f(t), 0, 0, \cdots, 0)$ and

$$A^*(t) = \begin{bmatrix} A_1(t) & \cdots & \cdots & \cdots & \cdots & A_k(t) \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{bmatrix} \tag{2.3}$$

is a $k \times k$ matrix, and denotes the transpose of the matrix. It is clear that $A^*(t)$ for each $t \in I$ is a bounded linear operator defined on the Banach space X^* of column vectors V, therefore [3] we have the following theorem.

Theorem 2.1. *If $g_j \in X, j = 0, 1, 2, \cdots, k-1$, then there exists one and only one solution*

$$x(t) \in X \quad and \quad D^k x(t) \in X$$

of the initial value problem (1.2) and (1.3).

Proof. From the properties of $A^*(t), F(t)$ and V_0, we can deduce that ([3] and [5]) there exists one and only one solution of (2.1) and (2.2), this solution is given by

$$V(t) = U(t,0)V_0 + \int_0^t U(t,s)F(s)ds \tag{2.4}$$

and satisfies

$$\|V(t)\| \le e^{at}\|V_0\| + \int_0^t e^{a(t-s)}\|F(s)\|ds \tag{2.5}$$

where $\{U(t,s)\}$ is the semigroup of linear bounded operators generated by A^* in X^*, and a is a positive constant. Now from (2.4) and (2.5) we deduce that $V(t) \in X^*$ from which we get $x(t) = v_k(t) \in X$. Differentiating (2.4) we get

$$\frac{dV(t)}{dt} = A^*(t)U(t,0)V_0 + F(t) + \int_0^t A^*(t)U(t,s)F(s)ds$$

which proves that $DV(t) \in X^*$, from which we deduce that $D^k x(t) = Dv_1(t) \in X$.

3. Integro–differential equations

The results of the previous section apply to Volterra and Fredholm equations as well.

Example 1. Consider the equation

$$D^k x(t) = \sum_{j=1}^{k} \int_0^t B_j(s) D^{k-j} x(s) ds + f(t) \tag{3.1}$$

with the initial data (1.3), where $(B_j(t), t \in I, j = 1, 2, \cdots, k)$ is a family of bounded linear operators defined on $C(I)$. Let

$$A_j(t)x(t) = \int_0^t B_j(s)x(s)ds \tag{3.2}$$

then we get

$$\|A_j(t)x(t)\| \leq T\|B_j(t)\|\|x\| \leq N_j\|x\| \tag{3.3}$$

where $\|x\| = \max_{t\in I} |x(t)|$. Therefore from theorem (2.1) the initial value problem (3.1) and (1.3) has a unique solution $x(t) \in C(I)$ and $D^k x(t) \in C(I)$.

Example 2. Consider the equation

$$D^k x(t) = \sum_{j=1}^{k} \int_a^b K_j(t,s) D^{k-j} x(s) ds + f(t) \tag{3.4}$$

with the initial data (1.3), where $K_j(t,s) \in L_2((a,b)x(a,b))$. Let

$$A_j(t)x(t) = \int_a^b K_j(t,s)x(s)ds \tag{3.5}$$

then we get

$$\begin{aligned} \|A_j(t)x(t)\|_2 &\leq \|x\|_2 \int_a^b \int_a^b |K_j(t,s)|^2 ds dt \\ &\leq N_j\|x\|_2 \end{aligned} \tag{3.6}$$

where $\|x\|_2 = \int_a^b |x(t)|^2 dt$ (3.7). Therefore from theorem (2.1) the initial value problem (3.4) and (1.3) has a unique solution $x(t) \in L_2(a,b)$ and $D^k x(t) \in L_2(a,b)$.

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