J. Korean Math. Soc. 27(1990), No. 2, pp. 255-280

ASYMPTOTIC PROPERTIES OF LÉVY FLOWS

HIROSHI KUNITA AND JAE-PILL OH*

0. Introduction

In the past, we met the stochastic differential equations of the jump type processes and the Lévy processes in diffeomorphisms group, and many other properties of Lévy flows in [1] and [3]. As we have known, we can guess that the Lévy processes are similar as the Brownian motions in many sides. Thus, perhaps, it is reasonable to think that some properties of Brownian motions may be preserved in Lévy processes, and we summarized mainly the asymptotic properties of Lévy flows by compairing with that of Brownian flows which will be appeared in [4]. Of course, we can get the similar results as the Brownian flows cases if we pay attention to the jump parts of Lévy processes, and it is the important and a little hard side when we handle.

On the other hand, in [1] and [3], we already met the existence theorems and the representation theorems of Lévy flows in case of the square integrable Lévy processes. But we will weaken the square integrable condition, and then think mainly the infinitesimal generators of Lévy flows.

Section I is the preliminary part, and we define the stationary Lévy process with its characteristics and the stochastic integral of the Lévy process.

In Section II, we get the representation theorem of Lévy flow without the square integrable condition. Thus our result is more general than that of [1] and [3]. We also meet some results about the C^m-valued Lévy process, the G^m_+-valued Lévy process and the G^m-valued Lévy process. But they are also in [1] and [3]. Therefore,

Received December 30, 1989.
Revised May 30, 1990.
* This research was partially supported by KOSEF, 1988.

we will skip their proofs.

In Section Ⅲ, we get some results of our title. We divide it into three subsections again. In subsection §1, we treat mainly the determinant of Jacobian matrix of Lévy flow and the asymptotic behaviors of Lévy flows in case of measure preserving. In subsection §2 and §3, we treat the asymptotic properties of Lévy flows in case of not measure preserving. They are distinguished as the case of the Lévy flow and the inverse flow of it, respectively.

I. Preliminaries

Let m be a nonnegative integer. Denote by $C^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)$ the set of all maps $f:\boldsymbol{R}^d\to\boldsymbol{R}^d$ which are m-times continuously differentiable. In case $m=0$, it is often denoted by $C(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)$. Define the differential operator D^k by

$$D^k=\left(\frac{\partial}{\partial x^1}\right)^{k^1}\left(\frac{\partial}{\partial x^2}\right)^{k^2}\cdots\left(\frac{\partial}{\partial x^d}\right)^{k^d},$$

where $|k|=k^1+k^2+\cdots+k^d$, and define the norm as

$$\|f\|_m=\sup_{x\in R^d}\frac{|f(x)|}{(1+|x|)}+\sum_{1\le|k|\le m}\sup_{x\in R^d}|D^kf(x)|.$$

Denote by $C_b^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)$ the set $\{f\in C^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)\,;\,\|f\|_m<\infty\}$. Then it is a separable Banach space with the norm $\|\cdot\|_m$.

Denote by $\tilde{C}^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)$ the set of all $\boldsymbol{R}^d$-valued functions $g(x,y)$ $x,\ y\in\boldsymbol{R}^d$ which are m-times continuously differentiable with respect to each variables x and y. For $g\in\tilde{C}^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)$, define a norm as

$$\|g\|_m^{\sim}=\sup_{x,y\in R^d}\frac{|g(x,y)|}{(1+|x|)(1+|y|)}+\sum_{1\le|k|\le m}\sup_{x,y\in R^d}|D_x^kD_y^kg(x,y)|,$$

and set $\tilde{C}_b^m(\boldsymbol{R}^d\,;\,\boldsymbol{R}^d)=\{g\in\tilde{C}^m\,;\,\|g\|_m^{\sim}<\infty\}$.

Let $X_t\equiv X(t,\ \omega),\ t\in[0,\ t]$ be a C^m-valued Lévy process defined on $(\Omega,\mathcal{F},P)$. We always assume that X_t is stationary, and $X_0=0$. Define the Poisson random measure $N_p((0,\ t]\times A)$ over $[0,\ T]\times C^m$ associated to X_t by

$$N_p((0,\ t]\times A)=\#\{s\in(0,\ t]\,;\,\Delta X_s\in A\},\quad \Delta X_s=X_s-X_{s-},$$

where A is a Borel subset of C^m excluding 0, and the intensity measure ν_p of N_p by

$$\nu_p((0,\ t]\times A)=E[N_p((0,\ t]\times A)].$$

Then since X_t is stationary, ν_p is the product measure: $\nu_p(dt,\ df)=dt\nu(df)$. The measure ν is called the *characteristic measure* of X_t.

Let $X_t(x)$ be the C^m-valued Lévy process. Then for n-points $x^{(n)} \equiv (x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^{nd}$ where each $x_i \in \boldsymbol{R}^d$, the n-point process $X_t(x^{(n)}) \equiv (X_t(x_1), X_t(x_2), \cdots, X_t(x_n))$ is an nd-dimensional Lévy process. Lévy process X_t has independent increment, thus its *characteristic function* is represented by the Lévy-Khinchin's formula;

$$\begin{aligned} &E\left[\exp\left\{i\sum_{k=1}^{n}(\alpha_k, X_t(x_k))\right\}\right] \\ &=\exp t\left\{i\sum_{k=1}^{n}(\alpha_k, b(x_k)) - \frac{1}{2}\sum_{k,l=1}^{n}\alpha_k a(x_k, x_l)\alpha_l\right. \\ &\quad + \int_U [\exp(i\sum_{k=1}^{n}(\alpha_k, f(x_k))) - 1 - i\sum_{k=1}^{n}(\alpha_k, f(x_k))]\nu(df) \\ &\quad \left. + \int_{U^c}[\exp(i\sum_{k=1}^{n}(\alpha_k, f(x_k)) - 1]\nu(df)\right\}, \end{aligned} \tag{I.1}$$

where

(I.2) $b(x)$ is an m-times continuously differentiable $\boldsymbol{R}^d$-valued function,

(I.3) $a(x, y)(=a^{ij}(x, y))$ is an m-times continuously differentiable $d\times d$-matrix valued function such that $a^{kl}(x, y)=a^{lk}(y, x)$ for any $k, l=1, 2, \cdots, d$ and $x, y \in \boldsymbol{R}^d$, and is nonnegative definite;

$$\sum_{i,j=1}^{n}\alpha_i a(x_i, x_j)\alpha_j \geq 0 \text{ for any } x_i, \alpha_j \in \boldsymbol{R}^d, \text{ and } i, j=1, 2, \cdots, n.$$

(I.4) ν is an σ-finite measure on C^m such that $\nu(\{0\})=0$, and there is an open neighborhood U of 0 in C^m which satisfies $\nu(U^c)<\infty$ and for any $x\in \boldsymbol{R}^d$ and $|k|\leq m$,

$$\int_U |D^k f(x)|^2 \nu(df) < \infty.$$

Note that the law of $X_t(\omega)$ is uniquely determined by the system (a, b, ν, U), and we call it as the *characterirtics* of the C^m-valued Lévy process X_t.

Let $X_t(x)$, $t\in[0, T]$, $x\in \boldsymbol{R}^d$ be a Lévy process with characteristics (a, b, ν, U), and $\mathcal{F}_{s,t}$ be the least sub-σ-field of $\mathcal{F}$ for which $X_u - X_v$; $s\leq v\leq u\leq t$ are measurable. Since $X_t(x)$ is a semimartingale, there is the set $U\subset C^m$ such that $X_t(x)$ is decomposed to the sum of the process of bounded variation $B_t(x)$ and the L^2-martingale process $Y_t(x)$ where $Y_t(x)=X_t(x)-B_t(x)$ and

$$B_t(x) \equiv b(x)t + \int_{U^c} f(x) N_p((0, t], df). \tag{I.5}$$

Thus we can define the stochastic integral of the Lévy process X_t as

$$\int_s^t X(\phi_{r-}, dr) \equiv \int_s^t Y(\phi_{r-}, dr) + \int_s^t b(\phi_{r-})dr + \int_s^t \int_{U^c} f(\phi_{r-}) N_p(dr, df), \tag{I.6}$$

where ϕ_t is an $\mathcal{F}_{s,t}$-adapted, right continuous with left hand limit process. If we think the stochastic differential equation of the Lévy process X_t, which is denoted as

$$d\varphi_t = X(\varphi_{t-}, dt), \tag{I.7}$$

then the solution is defined by the right continuous with left hand limits $\mathcal{F}_{s,t}$-adapted process φ_t which satisfies

$$\varphi_{s,t}(x) = x + \int_s^t X(\varphi_{r-}(x), dr). \tag{I.8}$$

II. Infinitesimal generators of Lévy flows

In this section, we will meet the C^m-valued Lévy process, the G_m^+-valued Lévy process, the G^m-valued Lévy process, and the infinitesimal generator of Lévy flow. But we will omit the proofs of Propositions, because we can get them from [1] and [3]. Suppose we are given a system (a, b, ν, U) satisfying (I.2), (I.3) and (I.4). We will introduce the following hypothesis.

(C^m, I) $b(x) = (b^i(x))$ belongs to C_b^m.

(C^m, II) $a(x, y) = (a^{ij}(x, y))$ belongs to $\tilde{C}_b^m$.

(C^m, III) The measure ν is supported by C_b^m. Further, there is a set $U \subset C_b^m$ such that $\nu(U^c) < \infty$ and

$$\sup_x \int_U |D^k f(x)|^2 \nu(df) < \infty \text{ for all } |k| \leq m.$$

Then we can get the following results.

PROPOSITION II.1. *Assume that the characteristics (a, b, ν, U) satisfy the above conditions (C^m, I), (C^m, II) and (C^m, III). Then there is an C^{m-1}-valued Lévy process X_t with characteristics (a, b, ν, U).*

Define the metric ρ_m as

$$\rho_m(f, g) = \sum_{|k| \leq m} \rho(D^k f, D^k g), \quad f, g \in C^m, \tag{II.1}$$

where ρ is the compact uniform metric

$$\rho(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\sup_{|x| \leq N} |f(x) - g(x)|}{1 + \sup_{|x| \leq N} |f(x) - g(x)|}, \tag{II.2}$$

then the space (C^m, ρ_m) is a Fréchet space with the metric ρ_m.

Let us define the product of two elements f and g of $C(\boldsymbol{R}^d; \boldsymbol{R}^d)$ by the composition $f\circ g$ of the maps. Then $C(\boldsymbol{R}^d; \boldsymbol{R}^d)$ becomes a topological semigroup by the topology ρ. We denote the semigroup by G_+. Further, denote by G_m^+ the sub-semigroup of G_+ consisting of C^m-maps. Then it is again a topological semigroup by the metric ρ_m. We will call the G_+^m-valued Levy process as the *Lévy flow* of C^m-maps.

PROPOSITION II.2. *Assume that the characteristics* (a, b, ν, U) *of the* C^m*-valued Levy process* X_t *satisfy* (C^m, I), (C^m, II) *and* (C^m, III). *Then the solution* φ_t *of the equation* (I.8) *of the* C^m*-valued Lévy process* X_t *defines a Lévy flow* φ_t *of* C^{m-1}*-maps* (G_+^{m-1}*-valued Lévy process*).

From the above proposition, the associated Lévy process X_t is called the *infinitesimal generator* of the Lévy flow φ_t, and the flow φ_t is said to be generated by the C^m-valued Lévy process X_t.

Denote by G the totality of the maps $f\in G_+$, which are homeomorphisms $f: \boldsymbol{R}^d\to\boldsymbol{R}^d$, define the metric

$$d(f, g)=\rho(f, g)+\rho(f^{-1}, g^{-1}), \qquad (\text{II}.3)$$

then the space (G, d) is a topological subgroup of G_+. The G-valued Lévy process is called the Lévy flow of homeomorphisms. If we define

$$d_m(f, g)=\rho_m(f, g)+\rho_m(f^{-1}, g^{-1}), \qquad (\text{II}.4)$$

then the subspace (G^m, d_m) of (G, d) is also a topological group. Thus we can define the G^m-valued Lévy process similarly as the G_+^m-valued Lévy process.

Now, for Proposition II.3 below, we will define the followings. The *correction term* $C(x, t)$ of X_t is a continuous function such that

$$C(x, t)=\frac{1}{2}tc(x), \qquad (\text{II}.5)$$

where $c(x)=(c^j(x))$ is m-times continuously differentiable and

$$c^j(x)=\sum_{i=1}^{d}\frac{\partial}{\partial y^i}a^{ij}(x, y)\,|_{y=x}. \qquad (\text{II}.6)$$

The following condition also will be used.

(C^m, IV) ν is supported by $\{f; f+id\in G^m\}$. Further, set $\hat{f}=id-(f+id)^{-1}$ and $\hat{U}=\{f; \|\hat{f}\|_m\leq 1\}$, then $\nu[(\hat{U})^c]<\infty$ and for $x\in\boldsymbol{R}^d$,

$$\int_{\hat{U}}|D^k\hat{f}(x)|^2\nu(df)<\infty,$$

$$\int_{U\wedge\hat{U}}|D^kf(x)-D^k\hat{f}(x)|\nu(df)<\infty.$$

PROPOSITION II.3. *Assume that the characteristics* (a, b, ν, U) *of* X_t *satisfy* (C^m, I), (C^m, II), (C^m, III) *and* (C^m, IV). *Further assume that the correction term* C *is a* C^m*-function such that* $\|C\|_m<\infty$. *Then the solution of the equation* (I.8) *defines a* G^{m-1}*-valued Lévy process.*

Let $\varphi_{s,t}$ be a G^m-valued Lévy process with starting time s, and $\varphi_{s,t}^{-1}$ be a inverse map of $\varphi_{s,t}$. Then it has also the independent increments and the multiplicative property to the backward direction;

$$\varphi_{s,u}^{-1}=\varphi_{s,t}^{-1}\circ\varphi_{t,u}^{-1}, \text{ for } s\le t\le u. \qquad (\mathrm{II}.7)$$

Hence $\varphi_{s,t}^{-1}$ can be regard as a G^m-valued Lévy process to the backward direction. Thus we will call $\varphi_{s,t}^{-1}$ as the *backward Lévy flow*, and there exists a C^m-valued Lévy process $\hat{X}_t$ such that

$$\varphi_{s,t}^{-1}(x)=x-\int_s^t\hat{X}(\varphi_{r,t}^{-1}(x),\ d\hat{r}), \qquad (\mathrm{II}.8)$$

where the right hand side is the backward stochastic integral. The $\hat{X}_t$ is called the infinitesimal generator of the inverse flow $\varphi_{s,t}^{-1}$, and sometimes called as the *conjugate* of X_t.

Now, we will think the main subject of this section. Let $\varphi_{s,t}(x)$, $0\le s\le t\le T$ be a Lévy flow of homeomorphism (G-valued Lévy process). Set $\varphi_{s,t}(x^{(n)})=(\varphi_{s,t}(x_1),\ \varphi_{s,t}(x_2),\cdots,\varphi_{s,t}(x_n))$. Then for each fixed s and $x^{(n)}$, the n-point motion $\varphi_{s,t}(x^{(n)})$, $t\in[s,\ T]$ is a stochastic process starting at $x^{(n)}$ at time s, and has a Markov property with respect to filtration $\{\mathcal{F}_{s,t}\ ;0\le s\le t\le T\}$.

Let $C_0^{(n)}\equiv C_0(\boldsymbol{R}^{nd}\ ;\boldsymbol{R})$ be the set of all bounded continuous functions F on $\boldsymbol{R}^{nd}$ such that $F(x)\to 0$ as $|x|\to\infty$. Then it is a Banach space with the supremum norm. For each $s\le t$ and $F\in C_0^{(n)}$, define the linear operator

$$T_{s,t}^{(n)}F(x^{(n)})=E[F(\varphi_{s,t}(x^{(n)}))],$$

then it is a bounded continuous function on $\boldsymbol{R}^{nd}$, and the family of linear operators $\{T_{s,t}^{(n)}\ ;0\le s\le t\le T\}$ satisfies the semigroup property; For any $s\le t\le u$ and $F\in C_0^{(n)}$, $T_{s,u}^{(n)}F=T_{t,u}^{(n)}\circ T_{s,t}^{(n)}F$, by the Markov property of the n-point motion. Thus the semigroup $T_t\equiv T_{0,t}^{(1)}$, $t\in[0,\ T]$ has the Féller property.

Let $\varphi_{s,t}$, $0\le s\le t\le T$ be a G^m-valued Lévy process and $J_t(x^{(n)})$ be the Jacobian matrix of the flow $\varphi_{0,t}(x^{(n)})\equiv\varphi_t(x^{(n)})$. Then the flow

$$(\varphi_t(x^{(n)}),\ D^k\varphi_t(x^{(n)})x^{<k>}\ ;\ |k|\le m)$$
$$\equiv(\varphi_t(x^{(n)}),\ J_t(x^{(n)})y_1^{(n)},\ J_t(x^{(n)})y_2^{(n)},\cdots,J_t(x^{(n)})y_d^{(n)},$$

$$J_t^2(x^{(n)})z_1^{(n)}, \cdots, J_t^2(x^{(n)})z_{d\times d}^{(n)}, \cdots, J_t^m(x^{(n)}))w_{d^m}^{(n)})$$

is a Markov process on $\boldsymbol{R}^M$. Hence for a fixed $(x^{(n)},\ x^{<k>}\ ;\ |k|\leq m) \equiv (x^{(n)},\ y_1^{(n)}\cdots,\ y_d^{(n)},\ z_1^{(n)},\ \cdots,\ z_{d\times d}^{(n)},\ \cdots,\ w_{d^m}^{(n)})$, it is a Féller process if we define the semigroup of it by

$$T_t^{m,(n)}F(x^{(n)},\ x^{<k>}\ ;\ |k|\leq m)=E[F(\varphi_t(x^{(n)}), D^k\varphi_t(x^{(n)})x^{<k>}\ ;\ |k|\leq m)].$$

Thus we can introduce the following conditions.

Condition 1. For a compact subset $K\subset\boldsymbol{R}^M$, define the subset $C_K(\boldsymbol{R}^M)$ of $C_0(\boldsymbol{R}^M\ ;\ \boldsymbol{R})$. Then for any $F\in C_K(\boldsymbol{R}^M)$, there is a linear operator $\mathcal{L}^m$ such that

$$\lim_{t\downarrow 0}\frac{T_t^m F-F}{t}=\mathcal{L}^m F.$$

Further, $\mathcal{L}^m F$ is Hölder continuous.

Condition 2. Set $F_c(x)\equiv F(x+c)$. Then $\{\mathcal{L}^m F_c\}$ and $\{\mathcal{L}^m F_c^2\}$ are uniformly bounded with respect to c.

On the other hand, on $C^m=C^m(\boldsymbol{R}^d\ ;\ \boldsymbol{R}^d)$, define the point process associated with the Lévy flow $\varphi_{s,t}$ by

$$N_p(t,\ E)=\sum_{s\leq t}\chi_E(\varDelta\varphi_{s-,s}(x)),$$

where $\varDelta\varphi_{s-,s}(x)=\varphi_{s-,s}(x)-x$ and E is a Borel subset of C^m. Then it is a time homogeneous Poisson point process. Thus set the intensity measure $\nu_p(1,\ E)=E[N_p(1,\ E)]$ as the characteristic measure $\nu(E)$, i.e., put $\nu(E)\equiv\nu_p(1,\ E)$.

Lemma II.4. *Assume Condition* 1 *and Condition* 2, *then* $\nu(\{f\ ;\ \|f\|_m\leq\varepsilon\}^c)<\infty$ *and*

$$\sup_x\int_{\|f\|_m\leq\varepsilon}|D^k f(x)|^2\nu(df)<\infty.$$

Proof. First, think the case of $m=0$. For $F\in C_K$, set

$$M(x,\ t)=F(\varphi_t(x))-F(x)-\int_0^t \mathcal{L}F(\varphi_{s-}(x))ds, \qquad (\text{II}.9)$$

then $M(x,t)$ is a L^2-martingale. It is also bounded martingale and satisfies

$$\langle M(x,t)\rangle=\int_0^t(\mathcal{L}F^2-2F\mathcal{L}F)(\varphi_{s-}(x))ds.$$

Let $M(x,\ t)=M^c(x,\ t)+M^d(x,\ t)$ be Mayer's decomposition, where

$$M^d(x, t)=\int_U\int_0^t\{F[\varphi_{s-}(x)+f(\varphi_{s-}(x))]-F(\varphi_{s-}(x))\}\tilde{N}_p(ds, df).$$

and $M^c(x, t)=M(x, t)-M^d(x, t)$. From the facts $\langle M^d(x, t)\rangle\leq\langle M(x, t)\rangle$ and

$$\langle M^d(x, t)\rangle=\int_U\int_0^t\{F[\varphi_{s-}(x)+f(\varphi_{s-}(x))]-F(\varphi_{s-}(x))\}^2 ds\nu(df),$$

we get the inequality

$$\int[F(x+f(x))-F(x)]^2\nu(df)\leq(\mathcal{L}F^2-2F\mathcal{L}F)(x)$$
$$<\infty \text{ for any } F\in C_K. \qquad (\text{II}.10)$$

By the mean value theorem, the left hand side of (II.10) is

$$\int[F(x+f(x))-F(x)]^2\nu(df)=\int|\sum_{i=1}^d\frac{\partial F}{\partial x^i}(x+\theta(f(x)-x))f^i(x)|^2\nu(df)$$

for some θ. For fixed x_0, there is a function F of C_K such that

$$C_1\int_{\|f\|_m\leq\varepsilon}|f(x_0)|^2\nu(df)\leq\int|\sum_{i=1}^d\frac{\partial F}{\partial x^i}(x_0+\theta(f(x_0)-x_0))f^i(x_0)|^2\nu(df)$$

holds for some $C_1>0$. Thus we get that for each x,

$$\int_{\|f\|_m\leq\varepsilon}|D^kf(x)|^2\nu(df)<\infty.$$

Similarly as above, we can show that for some constant C_2 and some x_0,

$$C_2\cdot\sup_c\int_{\|f\|_m\leq\varepsilon}|f(x_0+c)|^2\nu(df)\leq\sup_c(\mathcal{L}F_c^2-2F_c\mathcal{L}F_c)(x_0)<\infty.$$

This proves the result.

On the other hand, the stochastic integral

$$N(x, t)=\int_0^t M(\varphi_{r-}^{-1}(x), dr)$$

is well defined. It is a C-valued martingale and a C-valued Lévy process. Thus it is right continuous with left hand limits with respect to the strong topology of C. Hence $\#\{s\,;\,\|\Delta\varphi_{s-,s}\|\geq\varepsilon\}$ is finite. This implies that $\nu(\{f\,;\,\|f\|\leq\varepsilon\}^c)<\infty$.

Second, we will think the case of $m\geq1$. Let $F(x, x^{<k>}\,;\,|k|\leq m)$ be a C_K^2-function. Set

$$M(x, x^{<k>}, t\,;\,|k|\leq m)=F(\varphi_t(x), D^k\varphi_t(x)x^{<k>}\,;\,|k|\leq m)$$
$$-F(x, x^{<k>}\,;\,|k|\leq m)-\int_0^t\mathcal{L}^mF(\varphi_r(x), D^k\varphi_r(x)x^{<k>}\,;\,|k|\leq m)dr.$$

Then M is a square integrable martingale. Thus we get the assertion similarly as above.

LEMMA II.5. *Set* $U_\varepsilon=\{f\in C_b^m\,;\,\|f\|_m\leq\varepsilon\}$. *Then for any* x,

$$\sum_{s\leq t}(\Delta\varphi_{s-,s}(x))^2<\infty \ a.s.$$

if and only if for each x *and* $|k|\leq m$, $\nu(U_\varepsilon^c)<\infty$ *and*

$$\sup_x\int_{U_\varepsilon}|D^kf(x)|^2\nu(df)<\infty. \qquad (\text{II}.11)$$

Proof. We will show only the case of $m=0$. Suppose (II.11) is satisfied. Then from the fact, we see that for $0<\varepsilon'<\varepsilon$,

$$\sum_{0<\varepsilon'\leq|\Delta\varphi_{s-,s}(x)|<\varepsilon}|\Delta\varphi_{s-,s}(x)|^2=\int_{U_\varepsilon-U_{\varepsilon'}}f(x)^2N_p(t,\ df). \qquad (\text{II}.12)$$

On the other hand,

$$\int_{U_\varepsilon-U_{\varepsilon'}}f(x)^2N_p(t,\ df)$$

is finite-valued localmartingale. Then there is an increasing sequence $\tau_n\uparrow\infty$ such that

$$\int_{U_\varepsilon-U_{\varepsilon'}}|f(x)|^2N_p(\tau_n,\ df)<\infty \text{ for any } n,$$

because of finiteness and $\varphi_{s,t}(x)$ is cádlág. Then because $f(x)$ is martingale, we get that

$$\int_{U_\varepsilon-U_{\varepsilon'}}f(x)^2N_p(\tau_n\wedge t,\ df)$$

is martingale with mean 0. If we take the expectation, then

$$E[\int_{U_\varepsilon-U_{\varepsilon'}}f(x)^2N_p(\tau_n\wedge t,\ df)]=E[(\tau_n\wedge t)\nu(U_\varepsilon-U_{\varepsilon'})].$$

If we tend to infinite and zero, i.e. $n\to\infty$ and $\varepsilon'\to0$, then

$$E[\int_{U_\varepsilon}f(x)^2N_p(t,\ df)]=E[t\nu(U_\varepsilon)]$$
$$=t\nu(U_\varepsilon)<\infty$$

by (II.11). Thus we get from the relation (II.12),

$$\sum_{s\leq t,\ |\Delta\varphi_{s-,s}(x)|<\varepsilon}|\Delta\varphi_{s-,s}(x)|^2<\infty. \text{ a.s. for any } x.$$

Further, suppose that $\nu(U_\varepsilon^c)<\infty$. Then from the fact

$$E[N_p(t,\ U_\varepsilon^c)]=t\nu(U_\varepsilon^c)<\infty,$$

we get that

$$N_p(t,\ U_\varepsilon^c)<\infty, \text{ a.s..}$$

This means that $\{s\ ;\ |\Delta\varphi_{s-,s}(x)|\geq\varepsilon\}$ is a finite set. Therefore $\sum_{s\leq t}(\Delta\varphi_{s-,s}(x))^2$ exists a.s.. Thus we get

$$\sum_{s\leq t}|\Delta\varphi_{s-,s}(x)|^2<\infty, \text{ a.s..}$$

Conversely suppose that

$$\sum_{s\leq t}(\Delta\varphi_{s-,s}(x))^2<\infty \text{ a. s. for any } x.$$

Then there exists a sequence of stopping times $\{\tau_n\}$ such that $\tau_n\uparrow T$ and

$$\sum_{s\leq\tau_n, |\Delta\varphi_{s-,s}(x)|<\varepsilon}|\Delta\varphi_{s-,s}(x)|^2$$

is bounded a. s. Thus it is integrable and we get for any x,

$$E[\int_{U_\varepsilon}|f(x)|^2\nu(df)\tau_n]=E[\int_{U_\varepsilon}f(x)^2N_p(\tau, df)]<\infty.$$

Therefore, we get the iequality (II.11). On the other hand, since $\{s ; |\Delta\varphi_{s-,s}(x)|\geq\varepsilon\}$ is a finite set, we get

$$t\nu(U_\varepsilon^c)=E[N_p(t, U_\varepsilon^c)]<\infty,$$

and thus $N_p(t, U_\varepsilon^c)$ is finite. This implies that $\nu(U_\varepsilon^c)<\infty$.

Let $\{h_n(x)\}$ be a sequence of the functions of $C_K(\boldsymbol{R})^M$ such that

$$h_n(x)=x \text{ for } |x|\leq n.$$

Then by the sense of bounded convergence, the following limits

$$A_n^{ij}(x, y)=\lim_{t\downarrow 0}\frac{1}{t}E[(h_n^i(\varphi_t(x))-h_n^i(x))(h_n^j(\varphi_t(y))-h_n^j(y))],$$

$$b_n(x)=\lim_{t\downarrow 0}\frac{1}{t}E[h_n(\varphi_t(x))-h_n(x)]$$

exist. If we put $\varphi_n(x, t)=h_n(\varphi_t(x))$, then $\varphi_n(x, t)$ is a bounded semimartingale. Thus it is decomposed as

$$\varphi_n(x, t)=x+M_n(x, t)+\int_0^t b_n(\varphi_{r-}(x))dr,$$

where $M(x, t)$ is a martingale such that

$$\langle M_n^i(x, t), M_n^j(y, t)\rangle=\int_0^t A_n^{ij}(\varphi_{r-}(x), \varphi_{r-}(y))dr.$$

LEMMA II.6. *If we define the function* $a_n^{ij}(x, y)$ *as*

$$a_n^{ij}(x, y)=A_n^{ij}(x, y)-\int_{C^*}[h_n^i(x+f(x))-h_n^i(x)][h_n^j(y+f(y)) -h_n^j(y)]\nu(df),$$

then it does not depend on n.

Proof. If $n\geq m$, then we get

$$h_n(h_m(x))=h_n(x)=x \text{ if } |x|\leq m.$$

Thus $h_n(\varphi_m(x, t))=\varphi_n(x, t)$. If we put $\sigma_m=\inf\{t>0 ; |\varphi_t|\geq m\}$, then we get $\varphi_m(x, t)=\varphi_n(x, t)$ if $t<\sigma_m$. Therefore, $M_n^c(x, t)=M_m^c(x, t)$ if $t<\sigma_m$, and thus $a_n^{ij}=a_m^{ij}$. Thus $a^{ij}(x, y)$ does not depend on n if $|x|, |y|\leq n$.

Let $M_n(x, t)=M_n^c(x, t)+M_n^d(x, t)$ be the orthogonal decomposition of the martingale part, where

$$M_n^d(x, t)=\int_{C^m}\int_0^t [h_n(\varphi_{r-}+f(\varphi_{r-}))-h_n(\varphi_{r-})]N_p(dr, df).$$

Then there exists the nonnegative definite C^m-function $a(x, y)$ such that $\|a\|_m^\sim<\infty$ and

$$\langle M_n^{ci}(x, t), M_n^{cj}(y, t)\rangle=\int_0^t a^{ij}(\varphi_{r-}(x), \varphi_{r-}(y))\, dr.$$

Also, from the $M_n^d(x, t)$,

$$\begin{aligned}\langle M_n^{di}(x, t), M_n^{dj}(y, t)\rangle=&\int_{C^m}\int_0^t [h_n^i(\varphi_{s-}(x)+f(\varphi_{s-}(x)))-h_n^i(\varphi_{s-}(x))]\\ &\cdot[h_n^j(\varphi_{s-}(y)+f(\varphi_{s-}(y)))-h_n^j(\varphi_{s-}(y))]N_p(ds, df).\end{aligned}$$

THEOREM II.7. *Assume that Condition* 1 *and Condition* 2 *hold, and* $\varphi_{s,t}(x)$ *is a* G^m*-valued Lévy process. Then there is a* C^{m-1}*-valued Lévy process* X_t *such that*

$$\varphi_{s,t}(x)=x+\int_s^t X(\varphi_{s,r-}(x), dr).$$

Proof. We will show only the case of $m=1$. Let $F(x)$ be a C^2-function with compact support. If we set $\varphi_n(x, t)=h_n(\varphi(x, t))$, then $F(\varphi_t(x))=F(\varphi_n(x, t))$ if n is large. Since $\varphi_n(x, t)$ is a semimartingale, if D^i denotes the derivate of i-th coordinate, we have by Itô formula

$$\begin{aligned}F(\varphi_t)=&F(x)+\sum_{i=1}^d\int_0^t D^iF(\varphi_{r-}(x))d\varphi_n^i(x, r)\\ &+\frac{1}{2}\int_0^t\sum_{i,j=1}^d D^iD^jF(\varphi_{r-}(x))d\langle\varphi_n^{ci}(x, r), \varphi_n^{cj}(x, r)\rangle\\ &+\sum_{s\le t}[F(\varphi_s(x))-F(\varphi_{s-}(x))-\sum_{i=1}^d D^iF(\varphi_{s-}(x))\Delta\varphi_n^i(x, s)],\end{aligned}$$

and $F(\varphi_t(x))$ is a semimartingale, where $\Delta\varphi_n^i(x, s)=h_n^i(\varphi_s)-h_n^i(\varphi_{s-})$ and

$$\langle\varphi_n^{ci}(x, t), \varphi_n^{cj}(y, t)\rangle=\int_0^t a^{ij}(\varphi_{r-}(x), \varphi_{r-}(y))dr$$

by Lemma II.6. If we set

$$Y(x, t)\equiv F(\varphi_t(x))-F(x),$$
$$Y_n(x, t)\equiv h_n(\varphi_t(x))-h_n(x),$$

then they arc C-valued semimartingales, and we get

$$\begin{aligned}\int_0^t Y(\varphi_{r-}^{-1}(x), dr)=&\sum_{i=1}^d\int_0^t D^iF(x)Y_n^i(\varphi_{r-}^{-1}(x), dr)+\frac{1}{2}\sum_{i,j=1}^d D^iD^jF(x)a^{ij}(x, x)t\\ &+\sum_{s\le t}[F(\varphi_{s-,s}(x))-F(x)-\sum_{i=1}^d D^iF(x)(h_n^i(\varphi_{s-,s}(x))-h_n^i(x))].\end{aligned}$$

(Ⅱ.13)

Since $F(h_n(x))=F(x)$, we have $F(\varphi_{s-,s}(x))=F(h_n(\varphi_{s-,s}(x)))$ for large n. Thus by the mean value theorem, the larst term of (Ⅱ.13) is dominated. Indeed, by Lemma Ⅱ.5,

$$|\sum_{s\le t} D^iD^jF(x,\ t)[h_n^i(\varphi_{s-,s}(x))-h_n^i(x)][h_n^j(\varphi_{s-,s}(x))-h_n^j(\varphi_{s-,s}(x))]|$$
$$\le L\sum_{s\le t}|\varphi_{s-,s}(x)-x|^2,$$

and $\sum|\varphi_{s-,s}(x)-x|^2<\infty$, because $\{h_n\}$ is uniform Lipschitz continuous i. e.

$$|h_n(x)-h_n(y)|\le L|x-y|,$$

whewe L is a Lipschitz constant. Therefore the last member of (Ⅱ.13) converges to

$$\sum_{s\le t}[F(\varphi_{s-,s}(x))-F(x)-\sum_{i=1}^d D^iF(x)(\varphi^i_{s-,s}(x)-x^i)].$$

Consequently,

$$\lim_{n\to\infty}\int_0^t D^iF(x)\,Y_n^i(\varphi_{r-}^{-1}(x),dr)$$

exists. Since this is valid for any C^2-function F of compact support, we get that there exists $X(x,\ t)$ such that

$$\lim_{n\to\infty}\int_0^t Y_n(\varphi_{r-}^{-1}(x),\ dr)=X(x,\ t).$$

It is a semimartingale process with independent increments. Therefore it satisfies

$$\int_0^t Y(\varphi_{r-}^{-1}(x),\ dr)=\sum_{i=1}^d\int_0^t D^iF(x)X^i(x,\ dr)+\frac{1}{2}\sum_{i,j=1}^d D^iD^jF(x)a^{ij}(x,\ x)t$$
$$+\sum_{s\le t}[F(\varphi_{s-,s}(x))-F(x)-\sum_{i=1}^d D^iF(x)(h_n^i(\varphi_{s-,s}(x))-h_n^i(x))].$$

(Ⅱ.14)

The left hand side, and the second and third terms of the right hand side of (Ⅱ.14) are continuous in x. Therefore $X(x,\ t)$ is continuous in x. Hence $X(x,\ t)$ is a C-valued Lévy process. Further, from (Ⅱ.14),

$$F(\varphi_t(x))-F(x)=\sum_{i=1}^d\int_0^t D^iF(\varphi_{r-}(x))X^i(\varphi_{r-}(x),dr)$$
$$+\frac{1}{2}\sum_{i,j=1}^d\int_0^t D^iD^jF(\varphi_{r-})a^{ij}(\varphi_{r-}(x),\ \varphi_{r-}(x))dr$$
$$+\sum_{s\le t}[F(\varphi_s)-F(\varphi_{s-})-\sum_{i=1}^d D^iF(\varphi_{s-})\Delta\varphi_s^i].$$

This show that $X(x,\ t)$ is the infinitesimal generator of $\varphi_t(x)$.

Corollary II.8. *The $X(x, t)$ of Theorem II.7. is represented as*

$$X(x, t)=M^c(x, t)+b(x)t+\iint_0^t k(f(x))\tilde{N}_p(ds, df)$$
$$+\iint_0^t (i-k)(f(x))N_p(ds, df),$$

where $k(x)$ is a continuous smooth function with compact support such that $k(x)=x$ if $|x|\leq\varepsilon$, i is an identity function and $M^c(x, t)$ is a Brownian motion with covariance $a^{ij}(x, y)$.

Proof. Set

$$X_n(x, t)=\int_0^t Y_n(\varphi_r^{-1}(x), dr).$$

Then by the above theorem, it is a C-valued Lévy process. Thus it is represented as

$$X_n(x, t)=M_n^c(x, t)+b_n(x)t+\iint_0^t [h_n(x+f(x))-h_n(x)]\tilde{N}_p(dr, df)$$
$$+\iint_0^t [h_n(x+f(x))-h_n(x)]N_p(dr, df).$$

Since $\langle M_n^c(x, t), M_n^c(y, t)\rangle=a(x, y)t$, $M^c(x, t)$ does not depend on n. It is a Brownian motion. Therefore $M_n^c(x, t)=M^c(x, t)$. Further the small jump part of the equation;

$$\iint_0^t k[h_n(x+f(x))-h_n(x)]\tilde{N}_p(dr, df)$$

converges to

$$\iint_0^t k[f(x)]\tilde{N}_p(ds, df),$$

because of

$$\int[k(x+f(x))-k(x)]^2\nu(df)\leq L^2\int|f(x)|^2\nu(df)$$

and $\nu(\{f\,;\,|f|<2\varepsilon\})<\infty$. The big jump part;

$$\iint_0^t (i-k)[h_n(x+f(x))-h_n(x)]N_p(dr, df)$$

converges to

$$\iint_0^t (i-k)(f(x))N_p(dr, df).$$

Since $X_n(x, t)$ converges, also $\{\tilde{b}_n(x)\}$ converges to $b(x)$. Therefore, we get the assertion of Coroally.

III. Asymptotic properties of Lévy flows

§1. Measure preserving Levy flows

Assume that the Lévy flow $\varphi_{s,t}$ is temporally homogeneous. Thus $a(x, y)$, $b(x)$ do not depend on t and the semigroup of the n-point motion satisfies $T_{s,t}^{(n)}=T_{0,t-s}^{(n)}(\equiv T_{t-s}^{(n)})$.

Let Π be a Borel measure on $\boldsymbol{R}^d$. For a continuous function F, set

$$\Pi(F)=\int F(x)\pi(dx),$$

if the integral is well defined. Let $\mathcal{M}^1(\boldsymbol{R}^d)$ be the set of all probability measure on $\boldsymbol{R}^d$. Then it is a complete metric space by the weak topology. Next, let $\mathcal{M}(\boldsymbol{R}^d)$ be the set of all Borel measure on $\boldsymbol{R}^d$. Then it is a complete metric space by the vague topology. A sequence $\{\Pi_n\}$ of $\mathcal{M}(\boldsymbol{R}^d)$ is said to converge to Π vaguely if the sequence $\{\Pi_n(F)\}$ converge to $\Pi(F)$ for any $F\in C(\boldsymbol{R}^d)$, where $C(\boldsymbol{R}^d)$ is the space of all continuous functions over $\boldsymbol{R}^d$ with compact supports equipped with the uniform topology.

Now let Π be an element of $\mathcal{M}(\boldsymbol{R}^d)$. For each t and ω, we define the *image measure* of Π by the maps $\varphi_t(\cdot)(\omega)$ and $\varphi_t^{-1}(\cdot)(\omega)$ by

$$\varphi_t^{-1}(\Pi)(A)\equiv\Pi(\varphi_t(A)),\quad \varphi_t(\Pi)(A)\equiv\Pi(\varphi_t^{-1}(A)),\quad A\in\mathcal{B}(\boldsymbol{R}^d),$$

respectively. Then these can be regard as the stochastic processes with values in $\mathcal{M}(\boldsymbol{R}^d)$. We want to study the asymptotic properties of the process $\varphi_t^{-1}(\Pi)$ and $\varphi_t(\Pi)$ as $t\to\infty$.

The flow φ_t is called *Π-expanding* if $\varphi_t^{-1}(\Pi)(A)$ increases with t a. s. for any A of $\mathcal{F}$. The flow is called *Π-shrinking* if $\varphi_t^{-1}(\Pi)(A)$ is decreases with t a. s. for any A of $\mathcal{F}$. If it is Π-expanding and Π-schrinking, then the flow is called *Π-preserving.* In particular, it is called *incompressible* if it preserves the Lebesgue measure.

In the following, we will assume that the Lévy process satisfies $(C^m$, I$)$, $(C^m$, II$)$ and $(C^m$, III$)$. Let $J_t(x)$ be the Jacobian matrix of $\varphi_t(x)$. Then by the formula of the change of variables, we have

$$\int_{\varphi_t(A)}F(y)dy=\int_A F(\varphi_t(x))\,|\det(J_t(x))|\,dx \text{ a. s.},$$

for any continuous function F. Therefore, if the measure Π has a strictly positive density function $\pi(x)$, the image measure $\varphi_t^{-1}(\Pi)$ satisfies

$$\begin{aligned}\varphi_t^{-1}(\Pi)(A) &\equiv \Pi(\varphi_t(A)) \\ &= \int_{\varphi_t(A)} \pi(y)\,dy \\ &= \int_A \pi(\varphi_t(x))\,|\det(J_t(x))|\,dx \quad \text{a. s.} \\ &= \int_A \pi(\varphi_t(x))\,|\det(J_t(x))|\,\frac{1}{\pi(x)}\pi(dx).\end{aligned}$$

Set

$$\alpha(x,\ t) = \pi(\varphi_t(x))\,|\det(J_t(x))|\,\frac{1}{\pi(x)}.$$

Then $\alpha(x,\ t)$ is called a *Radon-Nikodym density* of the measure $\varphi_t^{-1}(\Pi)$ with respect to Π.

If $\varphi_t(x)$ is a Lévy flow, then there is a C^m-valued Lévy process $X(x,\ t)$ satisfying (C^m, I), (C^m, II) and (C^m, III) such that the flow φ_t is governed by the Itô's stochastic differential equation based on $X(x,\ t)$:

$$\varphi_t(x) = x + \int_0^t X(\varphi_{s-}(x),\ ds),$$

where $\tilde{N}_p = N_p - N_p$, in our case, $N_p((0,\ t],\ df) = t\nu(df)$, and

$$X_t(x) = X_t^c(x) + \int_U f(x)\tilde{N}_p((0,\ t],\ df) + \int_{U^c} f(x) N_p((0,\ t],\ df).$$

If we put

$$\mathring{X}_t^c(x) \equiv X_t^c(x) - C_t(x),$$

where C is the correction term of X^c, then we can think the corresponding Stratonovich's stochastic differential equation based on $\mathring{X}^c(x, t)$, and thus we get the Lévy flow of the form;

$$\begin{aligned}\varphi_t(x) = x &+ \int_0^t \mathring{X}^c(\varphi_s,\ \circ ds) + \int_0^t\int_U f(\varphi_{s-})\tilde{N}_p(ds,\ df) \\ &+ \int_0^t\int_{U^c} f(\varphi_{s-}) N_p(ds,\ df).\end{aligned}$$

We denote by

$$\operatorname{div}(\pi\mathring{X}^c)(x,\ t) = \sum_{i=1}^d \frac{\partial(\pi\mathring{X}^{ci})}{\partial x^i}(x,\ t),$$

then we get the followings.

LEMMA III. 1. 1. *The determinant of Jacobian matrix* $\det(J_t)$ *of the Lévy flow* $\varphi_t(x)$ *is given by*

$$\det(J_t) = 1 + \int_0^t \det(J_s)\operatorname{div}(\mathring{X}^c)(\varphi_s,\ \circ ds)$$

$$+\int_0^t\int_U \det(J_{s-})\{det(Jf+I)(\varphi_{s-})-1\}\tilde{N}_p(ds, df)$$
$$+\int_0^t\int_{U^c} \det(J_{s-})\{''\}N_p(ds, df).$$
$$+\int_0^t\int_U \det(J_{s-})\{det(Jf+I)(\varphi_{s-})-1-\text{div}(f)(\varphi_{s-})\}ds\nu(df), \tag{III.1}$$

where Jf is the Jacobian matrix of the map f, and I is the identity matrix.

Proof. From the Lévy flow:

$$\varphi_t^i(x)=x^i+\int_0^t \mathring{X}^{ci}(\varphi_{s-}(x), \circ ds)$$
$$+\int_0^t\int_U f^i(\varphi_{s-}(x))\tilde{N}_p(ds, df)+\int_0^t\int_{U^c} f^i(\varphi_{s-}(x))N_p(ds, df),$$

we get the following by the diffeomorphisms,

$$\frac{\partial\varphi_t^i}{\partial x^j}=\delta_{ij}+\sum_{l=1}^d\int_0^t \frac{\partial\mathring{X}^{ci}}{\partial x^l}(\varphi_{s-}, \circ ds)\frac{\partial\varphi_s^l}{\partial x^j}$$
$$+\sum_{l=1}^d\int_0^t\int_U \frac{\partial f^i}{\partial x^l}(\varphi_{s-})\frac{\partial\varphi_{s-}^l}{\partial x^j}\tilde{N}_p(ds, df)$$
$$+\sum_{l=1}^d\int_0^t\int_{U^c} \frac{\partial f^i}{\partial x^l}(\varphi_{s-})\frac{\partial\varphi_{s-}^l}{\partial x^j}N_p(ds, df).$$

Apply the Itô formula for $F(x^1, x^2, \cdots, x^d)=x^1\cdot x^2\cdots\cdot x^d$ (product) and $X_t=\left(\frac{\partial\varphi_t^1}{\partial x^{i1}}, \cdots, \frac{\partial\varphi_t^d}{\partial x^{id}}\right))$, then

$$\prod_{j=1}^d\frac{\partial\varphi_t^j}{\partial x^{ij}}=\prod_{j=1}^d\delta_{ij}$$
$$+\sum_{l=1}^d\sum_{j=1}^d\int_0^t\frac{\partial\varphi_s^1}{\partial x^{i1}}\cdots\frac{\partial\varphi_s^{j-1}}{\partial x^{i(j-1)}}\frac{\partial\mathring{X}^{cj}}{\partial x^l}(\varphi_s, \circ ds)\frac{\partial\varphi_s^l}{\partial x^{ij}}\frac{\partial\varphi_s^{j+1}}{\partial x^{i(j+1)}}\cdots\frac{\partial\varphi_s^d}{\partial x^{id}}$$
$$+\int_0^t\int_U\left\{\prod_{j=1}^d\left[\frac{\partial\varphi_{s-}^j}{\partial x^{ij}}+\sum_{l=1}^d\frac{\partial f^j}{\partial x^l}(\varphi_{s-})\frac{\partial\varphi_{s-}^l}{\partial x^{ij}}\right]-\prod_{j=1}^d\frac{\partial\varphi_{s-}^j}{\partial x^{ij}}\right\}\tilde{N}_p(ds, df)$$
$$+\int_0^t\int_{U^c}\left\{\,''\,\right\}N_p(ds, df)$$
$$+\int_0^t\int_U\left\{\prod_{j=1}^d\left[\frac{\partial\varphi_{s-}^j}{\partial x^{ij}}+\sum_{l=1}^d\frac{\partial f^j}{\partial x^l}(\varphi_{s-})\frac{\partial\varphi_{s-}^l}{\partial x^{ij}}-\prod_{j=1}^d\frac{\partial\varphi_{s-}^j}{\partial x^{ij}}\right.\right.$$
$$\left.-\sum_{l=1}^d\sum_{j=1}^d\frac{\partial\varphi_{s-}^1}{\partial x^{i1}}\cdots\frac{\partial\varphi_{s-}^{j-1}}{\partial x^{i(j-1)}}\frac{\partial f^j}{\partial x^l}(\varphi_{s-})\frac{\partial\varphi_{s-}^l}{\partial x^{ij}}\frac{\partial\varphi_{s-}^{j+1}}{\partial x^{i(j+1)}}\cdots\frac{\partial\varphi_{s-}^d}{\partial x^{id}}\right\}ds\nu(df).$$

But the determinant of the Jacobian matrix is defined as

$$\det(J_t)=\sum_{(i^1,i^2,\cdots,i^d)}\varepsilon\begin{pmatrix}1, 2, 3, \cdots, d\\ i^1, i^2, \cdots, i^d\end{pmatrix}\frac{\partial\varphi_t^1}{\partial x^{i1}}\frac{\partial\varphi_t^2}{\partial x^{i2}}\cdots\frac{\partial\varphi_t^d}{\partial x^{id}},$$

thus multiply $\varepsilon(\sigma)$ to each term of the above and sum up for all i^1, $i^2, \cdots, i^d$. Then we obtain

$$\begin{aligned}\det(J_t)=1&+\int_0^t\det(J_s)\operatorname{div}(\mathring{X}^c)(\varphi_s, \circ ds_s)\\&+\int_0^t\int_U\{\det([(I+Jf)(\varphi_-)]J_{s-})-\det(J_{s-})\}\tilde{N}_p(ds, df)\\&+\int_0^t\int_{U^c}\{\,''\,\}N_p(ds, df)\\&+\int_0^t\int_U\{\det([(I+Jf)(\varphi_{s-})]J_{s-})-\det(J_{s-})\\&\qquad-\operatorname{div}(f)(\varphi_{s-})\det(J_{s-})\}ds\nu(df).\end{aligned}$$

From this, we get the result.

THEOREM Ⅲ.1.2. *The density $\alpha(x, t)$ is represented by*

$$\begin{aligned}\alpha(x, t)=\exp\Big[&\int_0^t\frac{1}{\pi(\varphi_s)}\operatorname{div}(\pi\mathring{X}^c)(\varphi_s, \circ ds)\\&-\int_0^t\int_U\frac{1}{\pi(\varphi_{s-})}\operatorname{div}(\pi f)(\varphi_{s-})ds\nu(df)\\&+\int_0^t\int_U\left\{\frac{\pi(\varphi_{s-}+f(\varphi_{s-}))}{\pi(\varphi_{s-})}\det(Jf+I)(\varphi_{s-})-1\right\}ds\nu(df)\\&+\int_0^t\int_U\{\,''\,\}\tilde{N}_p(ds, df)\\&+\int_0^t\int_{U^c}\{\,''\,\}N_p(ds, df)\Big]\\&\times\prod_{s\le t}\frac{\pi(\varphi_s)}{\pi(\varphi_{s-})}\det J(\varphi_{s-,s})(\varphi_{s-})\exp\Big[1-\frac{\pi(\varphi_s)}{\pi(\varphi_{s-})}\det J(\varphi_{s-,s})(\varphi_{s-})\Big].\end{aligned}\qquad(\text{Ⅲ}.2)$$

Proof. From the Lévy flow $\varphi_t(x)$, we get from the Itô formula,

$$\begin{aligned}\pi(\varphi_t)=\pi(x)&+\sum_{i=1}^d\int_0^t\frac{\partial\pi}{\partial x_i}(\varphi_s)\mathring{X}^{ci}(\varphi_s, \circ ds)\\&+\int_0^t\int_U\{\pi(\varphi_{s-}+f(\varphi_{s-}))-\pi(\varphi_{s-})\}\tilde{N}_p(ds, df)\\&+\int_0^t\int_{U^c}\{\,''\,\}N_p(ds, df)\\&+\int_0^t\int_U\{\pi(\varphi_{s-}+f(\varphi_{s-}))-\pi(\varphi_{s-})-\sum_{i=1}^d f^i(\varphi_{s-})\frac{\partial\pi}{\partial x_i}(\varphi_{s-})\}ds\nu(df).\end{aligned}\qquad(\text{Ⅲ}.3)$$

On the other hand, by Itô formula, we have

$$\pi(\varphi_t)\det(J_t)=\pi(x)+\int_0^t \det(J_{s-})\,d\pi(\varphi_s)+\int_0^t \pi(\varphi_{s-})\,d[\det(J_s)]$$
$$+\langle\pi(\varphi_t)^c,\ \det(J_t)^c\rangle+\sum_{s\le t}\Delta\pi(\varphi_t)\Delta\det(J_t).$$

Substitute (Ⅲ.1) and (Ⅲ.3) to the above. Then we get

$$\begin{aligned}
&\pi(\varphi_t)\det(J_t)=\pi(x)\\
&+\int_0^t \det(J_s)\sum_{i=1}^d \frac{\partial\pi}{\partial x^i}(\varphi_s)\mathring{X}^{ci}(\varphi_s,\ \circ ds)+\int_0^t \det(J_s)\pi(\varphi_s)\operatorname{div}(\mathring{X}^c)(\varphi_s,\ \circ ds)\\
&+\int_0^t\int_U \det(J_{s-})\{\pi(\varphi_{s-}+f(\varphi_{s-}))-\pi(\varphi_{s-})\\
&\qquad\qquad +\pi(\varphi_{s-})[\det(I+Jf)(\varphi_{s-})-1]\}\tilde{N}_p(ds,\ df)\\
&+\int_0^t\int_{U^c} \det(J_{s-})\{\ {''}\ \}N_p(ds,\ df)\\
&+\int_0^t\int_U \det(J_{s-})\{\pi(\varphi_{s-}+f(\varphi_{s-}))-\pi(\varphi_{s-})-\sum_{i=1}^d f^i(\varphi_{s-})\frac{\partial\pi}{\partial x^i}(\varphi_{s-})\\
&\quad +\pi(\varphi_{s-})\det(Jf+I)(\varphi_{s-})-\pi(\varphi_{s-})-\pi(\varphi_{s-})\operatorname{div}(f)(\varphi_{s-})\}ds\nu(df)\\
&+\int_0^t\int_{U\cup U^c}[\pi(\varphi_{s-}+f(\varphi_{s-}))-\pi(\varphi_{s-})]\\
&\qquad\qquad \cdot[\det(J_{s-})(\det(Jf+I)(\varphi_{s-})-1)]N_p(ds,\ df)\\
&=\pi(x)+\int_0^t \pi(\varphi_{s-})\det(J_{s-})[\frac{1}{\pi(\varphi_s)}\operatorname{div}(\pi\mathring{X}^c)(\varphi_s,\circ ds)\\
&\quad+\int_U\{\frac{\pi(\varphi_{s-}+f(\varphi_{s-}))}{\pi(\varphi_{s-})}\det(Jf+I)(\varphi_{s-})-1\}\tilde{N}_p(ds,\ df)\\
&\quad+\int_{U^c}\{\ {''}\ \}N_p(ds,\ df)\\
&\quad+\int_U\{\frac{\pi(\varphi_{s-})+f(\varphi_{s-}))}{\pi(\varphi_{s-})}\det(Jf+I)(\varphi_{s-})-1\\
&\qquad\qquad -\frac{1}{\pi(\varphi_{s-})}\operatorname{div}(\pi f)(\varphi_{s-})\}ds\nu(df)].
\end{aligned}$$

The above equation can be regarded as a linear integral equation for $\pi(\varphi_t)\det(J_t)$. The solution is then represented as Theorem Ⅲ.1.2 by the Doléans-Dade exponential formula.

If we put

$$l(x)=\frac{1}{\pi(x)}\{\operatorname{div}[\pi(b-c)](x)-\int_U \operatorname{div}(\pi f)(x)\nu(df)\},$$

then we get the following results.

THEOREM Ⅲ.1.3. *Let $\pi(x)$ be a strictely positive C^3-function such* *that* $\Pi(dx)=\pi(x)dx$. *Then the following statements are equivalent.*

(*i*) $\varphi_t(\Pi)$ *is* Π-*preserving.*

(*ii*) $div\{\pi\mathring{X}^c\}(x)-\int_U div(\pi f)(x)\nu(df)=0$, *and*

$\pi(x+f(x))det(I+Jf)(x)=\pi(x)$, *for all* $f\in Supp(\nu)$.

(*iii*) $\sum_i \frac{\partial}{\partial x^i}\{\pi(x)a^{ij}(x,\ y)\}=0$, $l(x)=0$, *and*

$\pi(x+f(x))det(I+Jf)(x)=\pi(x)$, *for all* $f\in Supp(\nu)$.

(*iv*) *Let* $\Pi^{(n)}$ *be the n-product of measure* Π, *and* $\hat{T}_t^{(n)}$ *be the linear operator of the n-point motion of the backward Lévy flow* $\varphi_t^{-1}(x)$. *Then for every n,* $\{\hat{T}_t^{(n)}\}$ *is an adjoint semigroup of* $\{T_t^{(n)}\}$ *with respect to the measure* $\Pi^{(n)}$, *i.e.*

$$\int(T_t^{(n)}F)Gd\Pi^{(n)}=\int(\hat{T}_t^{(n)}G)Fd\Pi^{(n)}$$

holds for any bounded measurable functions F and G on R^d *with compact supports.*

(*v*) Π *is an invarient measure of the one-point motion and* $\Pi\otimes\Pi$ *is an invarient measure of the two-point motion.*

Proof. (i)→(ii). Note that φ_t is Π-preserving if and only if $\alpha(x,t)\equiv 1$ holds for any t a.s. for every x. Therefore from Theorem Ⅲ. 1.2, (i) and (ii) are equivalent.

(ii)→(iii). From the conditions (C^m, Ⅰ) and (C^m, Ⅱ), if we compute the infinitesimal covariance of $\mathrm{div}(\pi\mathring{X})$ and $\mathring{X}$, and the infinitesimal mean of $\mathrm{div}(\pi\mathring{X})$, then we get the relation of (iii).

(iii)→(iv). First we think the one-point motion. If we put $y=\varphi_t(x)$, then we get $x=\varphi_t^{-1}(y)$ and $dy=d[\varphi_t(x)]=\det(J_t(x))dx$. If $\varphi_t(x)$ is π-preserving, then from the Radon-Nikodym density, we get

$$\pi(x)\alpha(x,\ t)=\pi(\varphi_t(x))\det(J_t(x)),$$
$$\pi(x)=\pi(\varphi_t(x))\det(J_t(x)),$$

because of $\alpha(x,\ t)=1$. Thus we get

$$\int f(\varphi_t(x))g(x)\pi(x)=dx\int f(\varphi_t(x))g(x)\pi(\varphi_t(x))\det(J_t(x))dx$$
$$=\int f(y)g(\varphi_t^{-1}(y))\pi(y)dy.$$

Taking the expactation, then for any bounded measurable function f and g on $\boldsymbol{R}^d$, we get

$$\int[T_t^{(1)}f(x)]g(x)\Pi(dx)=\int[\hat{T}_t^{(1)}g(y)]f(y)\Pi(dy).$$

By the Mathematical induction, we get the n-poit motion;

$$\int\cdots\int f_1(\varphi_t(x_1))g_1(x_1)\cdots f_n(\varphi_t(x_n))g_n(x_n)\pi(x_1)\cdots\pi(x_n)dx_1\cdots dx_n$$
$$=\int\cdots\int f_1(y_1)g_1(\varphi_t^{-1}(y_1))\cdots f_n(y_n)g_n(\varphi_t^{-1}(y_n))\pi(y_1)\cdots$$
$$\cdots\pi(y_n)dy_1\cdots dy_n.$$

Taking the expactation, then we get

$$\int\cdots\int T_t^{(n)}(f_1\otimes f_2\otimes\cdots\otimes f_n)(x_1\cdots x_n)(g_1\otimes g_2\otimes\cdots\otimes g_n)(x_1\cdots x_n)$$
$$\cdot\pi^{\otimes n}(dx_1\cdots dx_n)$$
$$=\int\cdots\int \hat{T}_t^{(n)}(g_1\otimes\cdots\otimes g_n)(y_1\cdots y_n)(f_1\otimes f_2\otimes\cdots\otimes f_n)(y_1\cdots y_n)$$
$$\cdot\pi^{\otimes n}(dy_1\cdots dy_n).$$

Thus we get the result.

(iv)→(v) is trivial. For the proof of (v)→(i), suppose that Π and $\Pi\otimes\Pi$ are invarient measure of $\{T_t^{(1)}\}$ and $\{T_t^{(2)}\}$, respectively. Then for every continuous function g *on* $\boldsymbol{R}^d$ with compact support, we have by the same method of Browaian flow case that for every $t>0$,

$$E\left[\left(\int_{R^d}\{g(\varphi_t(x))-g(x)\}\pi(dx)\right)^2\right]=0.$$

This implies that

$$\int g(\varphi_t(x))\Pi(dx)=\int g(x)\Pi(dx)\quad a.\ s.$$

Therefore $\varphi_t(x)$ is Π-preserving.

§2. Asymptotic behaviors of $\varphi_t^{-1}(\Pi)$

We will consider the asymptotic behaviors of $\varphi_t^{-1}(\Pi)$ in case that $\varphi_t(x)$ is not π-perserving. We assume that the transition probability $P_t(x,\ \cdot)$ has a strictely positive continuous density function $P_t(x,\ y)$. Therefore we can apply the recurrence-transience dichotomy. Thus if we put

$$m(f)(x)=\frac{\pi(x+f(x))}{\pi(x)}\det(Jf+I)(x)-1,$$

then we get the following result.

THEOREM III. 2. 1. *Let $\pi(x)$ be a strictely positive C^3-function such that $\Pi(dx)=\pi(x)dx$. Then we get;*

(i) (a) $\varphi_t(x)$ is Π-expanding if and only if $\sum_i\frac{\partial}{\partial x^i}\{\pi(x)a^{ij}(x,\ y)\}$

$=0$, $l(x)\geq 0$ *and*

$$m(f)(x)\geq 0 \text{ and } \int_U m(f)(x)(\nu)(df)<\infty, \text{ for all } x \text{ and } f\in Supp(\nu).$$

(*b*) $\varphi_t(x)$ *is* Π*-schrinking if and only if* $\sum_i \frac{\partial}{\partial x^i}\{\pi(x)a^{ij}(x, y)\}=0$, $l(x)\leq 0$ *and*

$$m(f)(x)\leq 0 \text{ and } \int_U |m(f)(x)|\nu(df)<\infty, \text{ for all } x \text{ and } f\in Supp(\nu).$$

(*ii*) *Assume that the one-point motion is recurrent in the sense of Harris. Then we get;*

(*a*) *If* $\varphi_t(x)$ *is* Π*-expanding but is not* Π*-preserving, then* $\varphi_t^{-1}(\Pi)(A)$ *increases to infinity a. s. as* $t\uparrow\infty$ *if* $\Pi(A)>0$.

(*b*) *If* $\varphi_t(x)$ *is* Π*-schrinking but is not* Π*-preserving, then* $\varphi_t^{-1}(\Pi)(A)$ *decreases to* 0 *a. s. as* $t\uparrow\infty$ *if* $\Pi(A)<\infty$.

(*iii*) *Assume shat the one-point motion is transient. Then we get;*

(*a*) *If* $\varphi_t(x)$ *is* Π*-expanding and*

$$\underline{lim}_{x\to\infty}[l(x)+\int_U m(f)(x)\nu(df)]>0,$$

then $\varphi_t^{-1}(\Pi)(A)$ *increase to infinity a. s. as* $t\uparrow\infty$ *if* $\Pi(A)>0$.

(*b*) *If* $\varphi_t(x)$ *is* Π*-expanding and*

$$l(x)+\int_{C^m} m(f)(x)\nu(df)$$

is of compact support, then $\varphi_t^{-1}(\Pi)(A)$ *increases to a finite valued random variable a. s. as* $t\uparrow\infty$ *if* $\Pi(A)<\infty$.

(*c*) *If* $\varphi_t(x)$ *is* Π*-schrinking and*

$$\overline{lim}_{x\to\infty}[l(x)+\int_U m(f)(x)\nu(df)]<0,$$

then $\varphi_t^{-1}(\Pi)(A)$ *decrease to* 0 *a. s. if* $\Pi(A)<\infty$.

(*d*) *If* $\varphi_t(x)$ *is* Π*-schrking and*

$$l(x)+\int_{C^m} m(f)(x)\nu(df)$$

is of compact support, then $\varphi_t^{-1}(\Pi)(A)$ *decreases to a strictely positive random variable a. s. as* $t\to\infty$ *if* $\Pi(A)>0$.

Proof. (i) (a) By Theorem Ⅲ. 1. 2, φ_t is Π-expanding if and only if

$$\mathrm{div}(\pi\mathring{X}^c)(x, t)+t\int_U[m(f)(x)-\mathrm{div}(\pi f)(x)]\nu(df)$$
$$+\int_U m(f)(x)\tilde{N}_p(t, df)+\int_{U^c} m(f)(x)N_p(t, df)$$

is an increasing process. Thus it is equivalent to (a).

(b) A similar fact is valid for Π-schrinking flow.

(ii) (a) If $\varphi_t(x)$ is Π-expanding but not Π-preserving, then $l(x)$ is not identically 0 or $\int_U m(f)(x)\nu(df)$ is not identically 0 for some bounded set U. Then we have

$$\int_0^\infty l(\varphi_s(x))ds=\infty \text{ a. s. for any } x$$

if $l(x)$ is not identically 0, or

$$\int_0^\infty\int_U m(f)(\varphi_s(x))\nu(df)ds=\infty \text{ a. s. for any } x$$

if $\int_U m(f)(x)\nu(df)$ is not identically 0. Since

$$\begin{aligned}\alpha(x,\ t)\geq&\exp\int_0^t l(\varphi_{s-}(x))ds+\int_0^t\int_{C^m} m(f)(\varphi_{s-}(x))N_p(ds,\ df)\}\\ &\times\prod_{s\leq t}[m(\Delta\varphi_{s-,s})(\varphi_{s-}(x))+1]\exp[-m(\Delta\varphi_{s-,s})(\varphi_{s-}(x))]\\ \geq&\exp\{\int_0^t l(\varphi_{s-}(x))ds\}\prod_{s\leq t}[m(\Delta\varphi_{s-s})(\varphi_{s-}(x))+1],\end{aligned}$$

where $\Delta\varphi_{s-,s}(x)=\varphi_{s-,s}(x)-x$, we get that

$$\lim_{t\downarrow\infty}\alpha(x,\ t)=\infty.$$

Thus $\varphi_t^{-1}(\Pi)(A)$ increases to infinity for any x as $t\uparrow\infty$.

(b) The assertion can be proved similarly as (a).

(iii) Consider the class that of she one-point motion is transient.

(a) If

$$\varliminf_{x\to\infty}[l(x)+\int_U m(f)(x)\nu(df)]>0,$$

then we get

$$\int_0^\infty[l(\varphi_s)+\int_U m(f)(\varphi_{s-})\nu(df)]ds=\infty \text{ a. s.}$$

Thus we can get the result similarly as (ii).

(b) If

$$l(x)+\int_{C^m} m(f)(x)\nu(df)$$

is of compact support, then

$$\int_0^\infty[l(\varphi_s)+\int_U m(f)(\varphi_{s-})(\nu(df)]ds<\infty \text{ a. s.}$$

Thus, we get the resultsi milarly as above.

(c) and (d) is the apposite case of (a) and (b). Thus we can guess that it is hold.

Second, we will discuss the asymptotic behavior of $\varphi_t^{-1}(\Pi)$ in case that φ_t is neither Π-expanding nor Π-schrinking. The asymptotic behavior of $\varphi_t^{-1}(\Pi)$ is concerned with the invarient measure of the Feller semigroup $\{\hat{T}_t^{(1)}\}$, i. e. the asymptotic property depends crucially on the ergodic property of the one-point motion. We will first consider the case that $\{\hat{T}_t^{(1)}\}$ has an invariant property.

THEOREM III. 2. 2. *Assume that the semigroup* $\{\hat{T}_t^{(1)}\}$ *has an invarient probability* Λ. *Then we get;*

(i) The $\mathcal{M}^1(\boldsymbol{R}^d)$*-valued stochastic process* $\varphi_t^{-1}(\Lambda)$ *converges to an* $\mathcal{M}^1(\boldsymbol{R}^d)$*-valued random variable* Λ_∞ *a. s. as* $t\to\infty$. *It satisfies*

$$E[\Lambda_\infty(A)]=\Lambda(A), \text{ for every } A\in\mathcal{B}(\boldsymbol{R}^d).$$

(ii) If $\sum_i \frac{\partial}{\partial x^i}\{\pi(x)a^{ij}(x, y)\}\neq 0$ *or* $\int_U [m(f)(x)]^2\nu(df)\neq 0$, *then the measure* Λ_∞ *is singular to* Λ *a. s.*.

Proof. (i) From the properties of the flow, we see that

$$\varphi_{0,t}=\varphi_{s,t}\circ\varphi_{0,s}, \quad \varphi_{0,t}^{-1}=\varphi_{0,s}^{-1}\circ\varphi_{s,t}^{-1}.$$

If Λ is $\hat{T}_t^{(1)}$-invarient, then for any bounded continuous function f on $\boldsymbol{R}^d$, we have that

$$\begin{aligned}\varphi_t^{-1}(\Lambda)(f)&=\int f(\varphi_{0,t}^{-1}(x))\Lambda(dx)\\&=\int f(\varphi_{0,s}^{-1}\circ\varphi_{s,t}^{-1}(x))\Lambda(dx)\\&=\int[f\circ\varphi_{0,s}^{-1}](\varphi_{s,t}^{-1}(x))\Lambda(dx),\end{aligned}$$

because $\varphi_{s,t}^{-1}$ is independent to $\mathcal{F}_{0,s}$ and $f\circ\varphi_{0,s}^{-1}$ is $\mathcal{F}_{0,s}$-measurable. Thus if we take the expectation, then

$$\begin{aligned}E[\varphi_t^{-1}(\Lambda)(f)\,|\,\mathcal{F}_{0,s}]&=\int E[f\circ\varphi_{0,s}^{-1}](\varphi_{s,t}^{-1}(x))\Lambda(dx)\\&=\int\hat{T}_{s,t}^{(1)}(f\circ\varphi_{0,s}^{-1})(x)\Lambda(dx)\\&=\int(f\circ\varphi_{0,s}^{-1})(x)\Lambda(dx)=\varphi_s^{-1}(\Lambda)(f),\end{aligned}$$

becase Λ is $\hat{T}_t^{(1)}$-invariant. Therefore $\varphi_t^{-1}(\Lambda)(f)$ is a martingale. From the martingale convergence theorem,

$$\Lambda_\infty(f)\equiv\lim_{x\to\infty}\varphi_t^{-1}(\Lambda)(f)$$

exists a. s.. By the same terminology of the case of Brownian flow, Λ_∞ is the $\mathcal{M}(\boldsymbol{R}^d)$-valued random variable such that

$$\Lambda_\infty(f)=\int f(x)\Lambda_\infty(dx),$$

and we get

$$E[\Lambda_\infty(f)]=E[\varphi_t^{-1}(\Lambda)(f)]=\Lambda(f),$$

where $\Lambda(f)=\int f\Lambda(dx)$. Thus $E[\Lambda_\infty(\boldsymbol{R}^d)]=1$, and therefore, Λ_∞ is an $\mathfrak{M}^1(\boldsymbol{R}^d)$-valued random variable.

(ii) If Λ is an $\hat{T}_t^{(1)}$-invarient, then by the same terminology of the case of Brownian flow, the Radon-Nikodym density $\alpha(x, t)$ of $\varphi_t^{-1}(\Lambda)$ with respect to Λ is a localmartingale. Then

$$\alpha(x, t)=1+\int_0^t \alpha(x, s-)M(x, ds),$$

where

$$M(x, t)=\int_0^t \frac{1}{\pi(\varphi_{s-})}\operatorname{div}(\pi\mathring{Y})(\varphi_s, ds)+\int_0^t\int_{C^m} m(f)(\varphi_{s-})\tilde{N}_p(ds, df)$$

is a local martingale and $\mathring{Y}(x, t)$ is the martingale part of $\mathring{X}^c(x, t)$. But it is known that if $[M, M]_\infty=\infty$ a. s., then $\alpha(x, \infty)=0$ for any x. See D. Lepingle, J. Mémin [5]. Now, the condition implies that

$$\begin{aligned}\langle M(x, t)\rangle&=\langle\int_0^t \frac{1}{\pi(\varphi_{s-})}\operatorname{div}(\pi\mathring{Y})(\varphi_s, ds)+\int_0^t\int_{C^m} m(f)(\varphi_{s-})\tilde{N}_p(ds, df)\rangle\\&=\int_0^t\left\{\frac{1}{[\pi(\varphi_{s-})]^2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x^i\partial y^j}a^{ij}(x, y)\,|_{x=y=\varphi_s}\right\}ds\\&\quad+\int_0^t\int_{C^m}[m(f)(\varphi_{s-})]^2 ds\nu(df)\end{aligned}$$

tends to infinity as $t\to\infty$ a. s.. Thus we get $\langle M, M\rangle_\infty=\infty$ a. s. and $[M, M]_\infty=\infty$ a. s.. Therefore $\alpha(x, \infty)=0$ a. s. and Λ_∞ is singular to Λ.

§ 3. Asymptotic behaviors of $\varphi_t(\Pi)$

In this section, we will consider the asymptotic behaviors of $\varphi_t(\Pi)$ as $t\to\infty$ in case of when φ_t^{-1} is not Π-preserving. The discussion will be divided into two cases. First, we will consider in case that the one-point motion has an invariant probability.

THEOREM III. 3. 1. *Assume that the matrix $\{a^{ij}(x, y)\}$ is positive definite. If its one-point motion has an invariant probability Λ, then the following assertions hold.*

(*i*) *If Π is a probability, then for any Borel set $A\in\mathcal{B}(\boldsymbol{R}^d)$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi_s(\Pi)(A)ds=\Lambda(A) \quad a.s.$$

(ii) If Π is an infinite measure, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi_s(\Pi)(A)ds=\infty \quad a.s.$$

holds for any Borel set $A\in\mathcal{B}(\boldsymbol{R}^d)$ such that $\Lambda(A)>0$.

Proof. (i) Note the relation

$$\begin{aligned}\frac{1}{T}\int_0^T\varphi_s(\Pi)(A)ds&=\frac{1}{T}\int_0^T\Pi[\varphi_s^{-1}(A)]ds\\&=\frac{1}{T}\int_0^T\left\{\int_{\varphi^{-1}(A)}\pi(y)\right\}dy\\&=\frac{1}{T}\int_0^T\left\{\int_{R^d}\chi_A(\varphi_s(y))\pi(y)dy\right\}ds\\&=\int_{R^d}\left\{\frac{1}{T}\int_0^T\chi_A(\varphi_s(y))ds\right\}\Pi(dy).\end{aligned}$$

If the one-point motion is recurrent, then the integrand of the above converges to $\Lambda(A)$ a. s. as $T\to\infty$ for any y by an ergodic theorem, i. e.

$$\frac{1}{T}\int_0^T\chi_A(\varphi_s(y))ds\to\Lambda(A) \quad \text{a. s. as } T\to\infty.$$

If Π is a probability, then by the Fubini's theorem we get

$$\int_{R^d}\left\{\frac{1}{T}\int_0^T\chi_A(\varphi_s(y))ds\right\}\Pi(dy)\to\Lambda(A) \quad \text{a. s. as } T\to\infty.$$

(ii) Suppose next $\Pi(\boldsymbol{R}^d)=\infty$. Let K be any compact subset of $\boldsymbol{R}^d$. Then we have

$$\frac{1}{T}\int_0^T\Pi[\varphi_s^{-1}(A)\cap K]ds=\int_K\left\{\frac{1}{T}\int_0^T\chi_A(\varphi_s(y))ds\right\}\Pi(ds).$$

If Π is a measure, then the above integrand converges to $\Lambda(A)\Pi(K)$ a. s. But since $\Pi(K)\uparrow\infty$ as $K\uparrow\boldsymbol{R}^d$, we obtain the second assertion.

Next, we will consider the case that the one-point motion does not have an invarient probability. In this case we may think that the one-point motion is transient. It is not assumed also that the conditions (C^m, I), (C^m, II) and (C^m, III) hold, and that the condition $\{a^{ij}(x, y)\}$ is positive definite.

THEOREM III. 3. 2. *Assume that one-point motion does not have an invarient probability. If Π is a probability, then*

$$\lim_{t\to\infty}\varphi_t(\Pi)(A)=0 \quad in\ L^1(P)$$

holds for any bounded Borel set $A \subset \boldsymbol{R}^d$.

Proof. Note that the relation

$$\varphi_t(\Pi)(A) = \Pi[\varphi_t^{-1}(A)] = \int_{\varphi_t^{-1}(A)} \Pi(dy) = \int_{R^d} \chi_A(\varphi_s(y))\, \Pi(dy).$$

Because the one-point motion does not have the invarient probability,

$$\chi_A(\varphi_t(x)) \to 0 \text{ in } L^1(P) \text{ for any } x \text{ as } t \to \infty.$$

Thus we have the result.

References

1. T. Fujiwara and H. Kunita, *Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group*, Journal of Math. of Kyoto Univ. Vol. **25**, No. 1 (1985), 71–106.
2. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processess*, North-Holland /Kodansha, 2nd ed (1989).
3. H. Kunita, *Convergence of stochastic flows with jumps and Lévy processes in diffeomorphisms group*, Ann. Inst. Henri Poincare Vol. **22**, No. 3 (1986), 287–321.
4. H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, a forthcoming book.
5. D. Lepingle and J. Mémin, *Sur l'intégrabilité uniforme des martingales exponentielles*, Z. W., **42** (1978), 175–203.

Kyushu University
Fukuoka 812, Japan
and
Kangweon National University
Chuncheon 200–701, Korea