

RECURRENCE IN DYNAMICAL SYSTEMS*

SUNG KYU CHOI, CHINKU CHU AND JONG SUH PARK

The classical notions of nonwandering, Poisson stable and recurrent motions were studied extensively by Birkhoff to qualitatively classify motions defined by certain autonomous systems of differential equations.

An elegant necessary and sufficient condition for a flow on a locally compact phase space to be recurrent was appeared in [4]. For a certain transformation group, Knight [5] characterized recurrence and stability in terms of standard dynamical relations. That relations are the prolongational, prolongational limit, orbit, orbit closure and limit relations which are denoted by D, J, C, K and L , respectively. Elaydi [3] obtained criteria for regionally recurrent flows by introducing prolongational techniques which are widely used in dynamical systems theory. Also, Knight [5] characterized weakly regionally recurrent flows of characteristic 0.

In this paper, we examine the concept of almost recurrence which is a slightly weaker type of recurrence, characterize strong almost recurrent points in a locally compact metric phase space and then obtain relationship between recurrence, almost recurrence and strong almost recurrence.

1. Basic definitions and notations

The real numbers, nonnegative real numbers and Euclidean plane will be denoted by \mathbf{R}, \mathbf{R}^+ and \mathbf{R}^2 , respectively. Let X be a metric space with a metric d . For a point $x \in X$ and a number $\varepsilon > 0$, we denote $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. For a set $M \subset X$, \bar{M} denotes its closure. A pair (X, π) consisting of X and a continuous mapping $\pi :$

Received December 26, 1989.

Revised February 17, 1990.

* The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1988.

$X \times \mathbf{R} \rightarrow X$ is called a *dynamical system* or (*continuous*) *flow* provided $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t+s)$ for each $x \in X$ and $t, s \in \mathbf{R}$.

We shall write xt in place of $\pi(x, t)$. In line with this notation, the set $\{xt : x \in M \in X, t \in T \subset \mathbf{R}\}$ is denoted by MT .

For a point $x \in X$, the mapping $\pi(x, t) = xt$, $t \in \mathbf{R}^+$, is called the *positive motion* through x . The *positive orbit* $C^+(x)$ through $x \in X$ is the range of the positive motion through x . Thus $C^+(x) = x\mathbf{R}^+$. $K^+(x) = \overline{C^+(x)}$ is called the *positive orbit closure*. A set $M \subset X$ is said to be *positively invariant* if for each $x \in M$ we have $C^+(x) \subset M$. Clearly, $C^+(x)$ and $K^+(x)$ are positively invariant for every $x \in X$. The negative versions of these concepts are similarly defined for the negative time.

For a point $x \in X$, the *positive limit set* $L^+(x)$, the *positive prolongation set* $D^+(x)$ and the *positive prolongational limit set* $J^+(x)$ are defined as follows: $L^+(x) = \{y \in X : xt_i \rightarrow y \text{ for some sequence } t_i \rightarrow \infty\}$, $D^+(x) = \{y \in X : x_i t_i \rightarrow y \text{ for some sequences } x_i \rightarrow x \text{ and } t_i \in \mathbf{R}^+\}$,

$J^+(x) = \{y \in X : x_i t_i \rightarrow y \text{ for some sequences } x_i \rightarrow x \text{ and } t_i \rightarrow \infty\}$, where $t_i \rightarrow \infty$ means that the sequence (t_i) has no convergent subsequences. The negative versions $L^-(x)$, $D^-(x)$ and $J^-(x)$ of the above sets are similarly defined for $t_i \rightarrow -\infty$. Also, we define $C(x) = C^+(x) \cup C^-(x)$, $K(x) = K^+(x) \cup K^-(x)$ and $L(x) = L^+(x) \cup L^-(x)$, etc. For the definitions of the limit and prolongation sets for a transformation group, see [1].

A flow (x, π) is said to be *minimal* if we have $K(x) = x$ for every $x \in X$. A point $x \in X$ or a motion $\pi(x, t)$ is called *recurrent* if for every $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \in \mathbf{R}$, $C(x) \subset B_\varepsilon(x [t, t+T])$. It is well-known that $x \in X$ is recurrent if and only if $K(x)$ is a compact minimal set [7].

A point $x \in X$ or a motion $\pi(x, t)$ is called *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \varepsilon$ for every $t \in \mathbf{R}$, and a flow (X, π) is called *Lyapunov stable* if every point of X is Lyapunov stable.

2. Almost recurrent points

We say that a point x in X is *almost recurrent* if for every $\varepsilon > 0$ there exists a $T > 0$ such that $xt[0, T] \cap B_\varepsilon(x) \neq \emptyset$ for all $t \in \mathbf{R}$.

Bender [2] introduced the concept of almost recurrent motions in the

name of pseudo-recurrent motions. He showed that the motion $\pi(f, t)$ in the dynamical system $\pi : C \times \mathbf{R} \rightarrow C$, where (C, d) is the function space of continuous functions from $\mathbf{R}^n \times \mathbf{R}$ into \mathbf{R}^n with a sup metric, is recurrent if and only if f is recurrent, i. e., for any $\varepsilon > 0$ and K a compact set in \mathbf{R}^n , there exists an $L = L(\varepsilon, K) > 0$ such that for any interval I of length L there is an $s \in I$ such that $|f(x, t) - f(x, s)| < \varepsilon$ for all $(x, t) \in K \times \mathbf{R}$.

Clearly, every recurrent point is almost recurrent. Its converse holds if the orbit closure of almost recurrent point is compact [6]. Moreover, $K(x)$ is minimal if x is almost recurrent [7].

THEOREM 2.1. *If a point $x \in X$ is almost recurrent, then every point $y \in C(x)$ is also almost recurrent.*

Proof. Let $\varepsilon > 0$ be given and $y = xs$ for some $s \in \mathbf{R}$. By the continuity of π , there exists a $\delta > 0$ such that $B_\delta(x)s \subset B_\varepsilon(xs)$. Also, there exists a $T > 0$ such that $xt[0, T] \cap B_\delta(x) \neq \phi$ for all $t \in \mathbf{R}$. Therefore we have $\phi \neq (xt[0, T] \cap B_\delta(x))s \subset xt[s, s+T] \cap B_\varepsilon(xs)$. It follows that y is almostrecurrent.

LEMMA 2.2. $\cap \{K(y) : y \in K(x)\}$ is the intersection of all the closed invariant subsets of $K(x)$.

Proof. Let \mathcal{A} be the set of all the closed invariant subsets of $K(x)$. It is clear that $\cap \{A : A \in \mathcal{A}\} \subset \cap \{K(y) : y \in K(x)\}$.

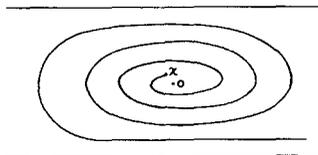
Let $A \in \mathcal{A}$. Then there is a point $z \in A \subset K(x)$. Thus we have $K(z) \subset A$ and hence $\cap \{K(y) : y \in K(x)\} \subset \cap \{A : A \in \mathcal{A}\}$.

We consider a set

$$R(x) = \{y \in X : \text{for every } \varepsilon > 0, \text{ there exists a } T > 0 \text{ such that } xt[0, T] \cap B_\varepsilon(y) \neq \phi \text{ for all } t \in \mathbf{R}\},$$

that is, it is a set of points at which the point x recurs almost.

From the definition of $R(x)$, $R(x) \subset L(x)$ is immediate. However the two sets need not coincide. For instance, let us consider the flow which has the constant speed as shown in the below figure.



The point $\{0\}$ is a rest point. Clearly, this flow has the property that $L(x)$ consists of the two straight lines, but $R(x)$ is the empty set.

THEOREM 2.3. *$R(x)$ is a closed invariant subset of $L(x)$ if $R(x) \neq \phi$. Thus $R(x) = \cap \{K(y) : y \in K(x)\}$.*

Proof. To show that $R(x)$ is closed, let $y \in \overline{R(x)}$. Then, for every $\varepsilon > 0$, we have $B_{\varepsilon/2}(y) \cap R(x) \neq \phi$. Let $z \in B_{\varepsilon/2}(y) \cap R(x)$. Then there exists a $T > 0$ such that for all $t \in \mathbf{R}$, $x(t+\tau) \in B_{\varepsilon/2}(z)$ for some $\tau \in [0, T]$. It follows that

$$d(x(t+\tau), y) \leq d(x(t+\tau), z) + d(z, y) < \varepsilon.$$

This implies that $y \in R(x)$ and hence $R(x) = \overline{R(x)}$.

Let $y \in R(x)$ and $\tau \in \mathbf{R}$. We claim that $y\tau \in R(x)$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{\delta}(y)\tau \subset B_{\varepsilon}(y\tau)$ by the continuity of π . Since $y \in R(x)$, there exists a $T > 0$ such that $xt[0, T] \cap B_{\delta}(y) \neq \phi$ for all $t \in \mathbf{R}$. Then we have $x(t+s) \in B_{\delta}(y)$ for some $s \in [0, T]$. It follows that $x(t+s)\tau \in B_{\delta}(y)\tau \subset B_{\varepsilon}(y\tau)$ for all $t \in \mathbf{R}$. If we take $T' = T + \tau > 0$, then we have $xt[0, T'] \cap B_{\varepsilon}(y\tau) \neq \phi$ for all $t \in \mathbf{R}$. This means that $R(x)$ is invariant.

Finally, in view of Lemma 2, it suffices to show that $R(x) \subset \cap \{K(y) : y \in K(x)\}$ since $R(x)$ is closed invariant.

Let $z \in R(x)$ and $y \in K(x)$. Then there is a sequence (y_n) in $C(x)$ such that $y_n \rightarrow y$ and for any $\varepsilon > 0$, there is a $T > 0$ such that for all $w \in C(x)$, $d(z, w\tau) < \varepsilon/2$ for some $\tau \in [0, T]$. Also, we have for each n , $d(z, y_n\tau_n) < \varepsilon/2$ for some $\tau_n \in [0, T]$. If we assume that $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$, then $d(y\tau, y_m\tau_m) < \varepsilon/2$ for some integer m because $y_n\tau_n \rightarrow y\tau$ as $n \rightarrow \infty$. Therefore we have $d(z, y\tau) < \varepsilon$ and hence $z \in K(y)$. This completes the proof.

THEOREM 2.4. *$x \in R(x)$ if and only if $K(x) = R(x)$. Thus we have $R(x) = C(x) = K(x)$ if the point x is periodic.*

Proof. The "if" part is trivial. For the "only if" part, it suffices to show that $K(x) \subset R(x)$ because $R(x) \subset L(x) \subset K(x)$ is immediate. Let $y \in K(x)$. Then for any $\varepsilon > 0$, $d(y, x\tau) < \varepsilon/2$ for some $\tau \in \mathbf{R}$. Also, there exists a $\delta > 0$ such that $B_{\delta}(x)\tau \subset B_{\varepsilon}(y)$. Since $x \in R(x)$, there is a $T > 0$ such that $xt[0, T] \cap B_{\delta}(x) \neq \phi$ for all $t \in \mathbf{R}$, i. e., for all $t \in \mathbf{R}$, $x(t+s) \in B_{\delta}(x)$ for some $s \in [0, T]$. Then it follows that $x(t+s)\tau \in B_{\delta}(x)\tau \subset B_{\varepsilon}(y)$, so we have for all $t \in \mathbf{R}$, $xt[0, T'] \cap B_{\varepsilon}(y) \neq \phi$

for some $T' = T + \tau > 0$. Therefore $y \in R(x)$.

In general, for a point $y \in L(x)$, $R(y)$ need not be equal to $R(x)$. However, for a point $y \in R(x)$, we have $R(y) = R(x)$ if the point x is Lyapunov stable.

THEOREM 2.5. *Suppose that a point x in X is Lyapunov stable. Then we have $R(y) = R(x)$ whenever $y \in R(x)$.*

Proof. It suffices to show that $R(x) \subset R(y)$. Let $z \in R(x)$ and $\varepsilon > 0$ be given. Since $\pi(x, t)$ is Lyapunov stable, there exists a $\delta > 0$ such that $w \in B_\delta(y)$ implies that $wt \in B_{\varepsilon/2}(yt)$ for all $t \in \mathbf{R}$. Moreover, we have $B_\delta(y) \cap C(x) \neq \emptyset$ because $y \in R(x) \subset L(x) \subset K(x)$. Thus $xs \in B_\delta(y)$ for some $s \in \mathbf{R}$. Also, there exists a $T > s$ such that $xt[0, T] \cap B_{\varepsilon/2}(z) \neq \emptyset$ for all $t \in \mathbf{R}$, in other words, $x(t+r) \in B_{\varepsilon/2}(z)$ for some $r \in [0, T]$. Putting $T' = 2T$, we have $x(t+T+r') \in B_{\varepsilon/2}(z)$ for some $r' \in [0, T]$ since $t+T \in \mathbf{R}$ for all $t \in \mathbf{R}$. Now, since $d(xs, y) < \delta$ and $t+T-s+r \in \mathbf{R}$, we have

$$d(xs(t+T-s+r), y(t+T-s+r)) = d(x(t+T+r), y(t+T-s+r)) < \varepsilon/2.$$

It follows that $y(t+T-s+r) \in B_\varepsilon(z)$. Note that $0 < T-s+r < 2T = T'$. Therefore $z \in R(y)$.

3. Strong almost recurrent points

Throughout this section, we consider flows (X, π) on a locally compact metric space (X, d) .

A point $x \in X$ is called *strong almost recurrent* if for any $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$ such that $y \in B_\delta(x)$ and $t \in \mathbf{R}$ implies $y[t, t+T] \cap B_\varepsilon(x) \neq \emptyset$.

For instance, every point in an irrational flow on a torus [7] is strong almost recurrent.

First, we characterize strong almost recurrent points as the following.

THEOREM 3.1. *A point $x \in X$ is strong almost recurrent if and only if for every neighborhood U of x there exists a neighborhood V of x such that \overline{VR} is a compact subset of UR .*

Proof. (\Rightarrow) Let U be any neighborhood of x . Then $\overline{B_\varepsilon(x)} \subset U$ is compact for some $\varepsilon > 0$ and since x is strong almost recurrent, there

exist numbers $\delta, T > 0$ such that $y \in B_\delta(x)$ and $t \in \mathbf{R}$ implies

$$y[t-T, t] \cap B_\varepsilon(x) \neq \phi.$$

It follows that $yt \in B_\varepsilon(x)[0, T]$. Taking a neighborhood $V = B_\delta(x)$ of x , we have $\overline{VR} \subset \overline{B_\varepsilon(x)}[0, T] \subset UR$ and \overline{VR} is compact.

(\Leftarrow) Suppose the contrary. Then there exists an $\varepsilon > 0$ such that for each $n, x_n[t_n, t_n + T_n] \cap B_\varepsilon(x) = \phi$ for some $x_n \in B_{1/n}(x)$ and $t_n \in \mathbf{R}$. If we assume that the sequence $(t_n + T_n/2)$ is bounded, then we have $t_n + T_n/2 \rightarrow t$. Thus $x_n(t_n + T_n/2) \rightarrow xt$. Also, there exists a $\delta > 0$ such that $d(y, xt) < \delta$ implies $d(y(-t), x) < \varepsilon$. Moreover, we have $x_m(t_m + T_m/2) \in B_\delta(xt)$ for some $T_m > 2|t|$. Then

$$d\left(x_m\left(t_m + \frac{T_m}{2} - t\right), x\right) < \varepsilon.$$

Since $t_m < t_m + \frac{T_m}{2} - t < t_m + T_m$, we have

$$x_m[t_m, t_m + T_m] \cap B_\varepsilon(x) \neq \phi$$

which is a contradiction. Therefore the sequence $(t_n + T_n/2)$ must be unbounded and so $t_n + \frac{T_n}{2} \rightarrow \infty$ or $t_n + \frac{T_n}{2} \rightarrow -\infty$.

Assume that $t_n + \frac{T_n}{2} \rightarrow \infty$. There exists a neighborhood U of x such that \overline{UR} is compact and it is contained in $B_\varepsilon(x)R$. We may assume $x_n\left(t_n + \frac{T_n}{2}\right) \rightarrow y$ because $x_n(t_n + T_n/2) \in \overline{UR}$. Then we have $yt \in B_\varepsilon(x)$ for some $t \in \mathbf{R}$ since $y \in J(x) \subset B_\varepsilon(x)R$. Putting $\delta = \varepsilon - d(x, yt) > 0$, there is an $\rho > 0$ such that $d(y, z) < \rho$ implies $d(yt, zt) < \delta$. Furthermore we have $x_m\left(t_m + \frac{T_m}{2}\right) \in B_\rho(y)$ for some $T_m > 2|t|$. Then $d\left(x_m\left(t_m + \frac{T_m}{2} + t\right), yt\right) < \delta$. Therefore

$$\begin{aligned} d\left(x_m\left(t_m + \frac{T_m}{2} + t\right), x\right) &\leq d\left(x_m\left(t_m + \frac{T_m}{2} + t\right), yt\right) + d(yt, x) \\ &< \delta + d(yt, x) = \varepsilon. \end{aligned}$$

Since $t_m < t_m + \frac{T_m}{2} + t < t_m + T_m$,

$$x_m[t_m, t_m + T_m] \cap B_\varepsilon(x) \neq \phi,$$

a contradiction. This completes the proof.

In view of the above characterization of strong almost recurrent

points, we obtain the following.

COROLLARY 3.2. *If a point $x \in X$ is strong almost recurrent, then it is recurrent.*

Proof. It is clear that x is almost recurrent. Thus $\overline{x\mathbf{R}}$ is minimal [7]. By Theorem 3.1, there exists a neighborhood U of x such that $\overline{U\mathbf{R}}$ is compact. Therefore $\overline{x\mathbf{R}}$ is compact and hence x is recurrent.

EXAMPLE 3.3. The following shows that the converse of Corollary 3.2 does not hold:

Let $X = \{(r, \theta) \in \mathbf{R}^2 : 0 \leq r \leq 1\}$ and $\pi : X \times \mathbf{R} \rightarrow X$ be defined by

$$\pi((r, \theta), t) = \left(\frac{r}{r + (1-r)e^t}, 2\pi t + \theta \right).$$

If we consider a point $x = (1, 0) \in X$, then it is recurrent but not strong almost recurrent. For, let $\varepsilon = 1/2$ and for any δ with $0 < \delta < 1/2$ and $t > 0$, we take $y = (r, 0)$, where $r = 1 - \frac{\delta}{2}$. Then we may choose $n > \log \frac{r}{1-r}$. Clearly, $y \in B_\delta(x)$. Since

$$yn = \left(\frac{r}{r + (1-r)e^n}, 2n\pi \right) = \left(\frac{r}{r + (1-r)e^n}, 0 \right),$$

we have $y[n, n+t] \cap B_\varepsilon(x) = \emptyset$ and so x is not strong almost recurrent.

THEOREM 3.4. *If $x \in X$ is almost recurrent and Lyapunov stable, then it is strong almost recurrent.*

Proof. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \varepsilon/2$ for all $t \in \mathbf{R}$ by the Lyapunov stability of x . Also, there is a $T > 0$ such that $x[t, t+T] \cap B_{\varepsilon/2}(x) \neq \emptyset$ for all $t \in \mathbf{R}$ by the assumption. Thus $xs \in B_{\varepsilon/2}(x)$ for some $s \in [t, t+T]$. Then we have $d(ys, x) \leq d(ys, xs) + d(xs, x) < \varepsilon$. It follows that $y[t, t+T] \cap B_\varepsilon(x) \neq \emptyset$.

THEOREM 3.5. *If $x \in X$ is strong almost recurrent, then xt is also strong almost recurrent for all $t \in \mathbf{R}$.*

Proof. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \varepsilon$. Also, there exist numbers $\rho, T > 0$ such that $y[s, s+T] \cap B_\delta(x) \neq \emptyset$ whenever $y \in B_\rho(x)$ and $s \in \mathbf{R}$. Now, there is an $\alpha > 0$ satisfying $d(z(-t), x) < \rho$ if $d(z, xt) < \alpha$. Thus for any $y \in$

$B_\alpha(xt)$ and $s \in \mathbf{R}$, $y(-t) \in B_\rho(x)$ implies $y(-t)[s, s+T] \cap B_\delta(x) \neq \phi$. Therefore we have $y(r-t) \in B_\delta(x)$ for some $r \in [s, s+T]$. Since $d(xt, yr) < \varepsilon$, we have $y[s, s+T] \cap B_\varepsilon(xt) \neq \phi$ which means that xt is strong almost recurrent.

By the same manner in the proof of Theorem 3.1, we obtain the following.

THEOREM 3.6. *If $x \in X$ is strong almost recurrent and $y \in L(x)$, then y is also strong almost recurrent.*

Proof. Suppose that y is not strong almost recurrent. Then there is an $\varepsilon > 0$ such that for each n , there are points $y_n \in B_{1/n}(y)$ and $t_n \in \mathbf{R}$ with the property $y_n[t_n, t_n+T_n] \cap B_\varepsilon(y) = \phi$. By the same method in the proof of Theorem 3.1, we have a contradiction if we assume the sequence $(t_n + T_n/2)$ is bounded. Hence we may suppose that $t_n + T_n/2 \rightarrow \infty$ as $n \rightarrow \infty$. If we can proceed as the proof of Theorem 3.1, then we obtain an inequality

$$\begin{aligned} d\left(y_{n_m}\left(t_{n_m} + \frac{T_{n_m}}{2} + s\right), y\right) &\leq d\left(y_{n_m}\left(T_{n_m} + \frac{T_{n_m}}{2} + s, zs\right) + d(zs, y) \\ &< \rho + d(zs, y) = \varepsilon, \end{aligned}$$

where $y_{n_m}, T_{n_m} > T_{n_m-1}$, is in a neighborhood U_m of y , $y_{n_m}\left(t_{n_m} + \frac{T_{n_m}}{2}\right) \rightarrow z, s \in \mathbf{R}$ with $zs \in B_\varepsilon(y)$ and $\rho = \varepsilon - d(y, zs) > 0$. Furthermore, since $t_{n_m} < t_{n_m} + \frac{T_{n_m}}{2} + s < t_{n_m} + T_{n_m}$ we have

$$y_{n_m}[t_{n_m}, t_{n_m} + T_{n_m}] \cap B_\varepsilon(y) \neq \phi,$$

a contradiction.

COROLLARY 3.7. *If $x \in X$ is strong almost recurrent, then for every $y \in \overline{x\mathbf{R}}$, y is also strong almost recurrent.*

References

1. J.S. Bae, S.K. Choi and J.S. Park, *Limit sets and prolongations in topological dynamics*, J. Differential Equations, **64**(1986), 336-339.
2. P.R. Bender, *Recurrent solutions to system of ordinary differential equations*, J. Differential Equations, **5**(1969), 271-282.
3. S. Elaydi, *Criteria for regionally recurrent flows*, Proc. Amer. Math. Soc., **85**(1982), 461-464.

4. R. A. Knight, *Recurrent and Poisson stable flows*, Proc. Amer. Math. Soc., **83**(1981). 49–53.
5. R. A. Knight, *Characterization of recurrent and stable transformation groups*, Bull. Inst. Math. Academia Sinica, **10**(1982), 185–196.
6. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton University Press, New Jersey, 1960.
7. K. S. Sibirsky, *Introduction to Topological Dynamics*, Noordhoff, Leyden, 1985.

Chungnam National University
Taejon 305-764, Korea