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A MATRIX REPRESENTATION OF POSETS AND ITS APPLICATIONS

MIN SURP RHEE

1. Introduction

Let $X=\{x_1, x_2, \cdots, x_n\}$ be a finite partially ordered set (a poset, for short) with $|X|=n$, and let $\mathcal{L}(X)=\{L_1, L_2, \cdots, L_e\}$ denote the set of all linear extensions of X with $|\mathcal{L}(X)|=e$. If $S_n=\{1, 2, \cdots, n\}$ is a poset with a natural order, then L_i defines a bijective map $l_i : X \to S_n$, via $l_i(x_j)=k$ if the level of x_j in L_i is k. Now, let $p(x_j|k)=\frac{1}{e}|\{l_i : l_i(x_j)=k\}|$ and $p(k|x_j)=\frac{1}{e}|\{l_i : l_i(x_j)=k\}|$. Then $p(x_j|k)=p(k|x_j)$. In this fashion we can associate a finite poset X with an $n\times n$ matrix $D(X)=(d_{jk})$, where $d_{jk}=p(x_j|k)=p(k|x_j)$. Then it follows from $\sum_{j=1}^{n} p(x_j|k)=\sum_{k=1}^{n} p(k|x_j)=1$ that the matrix $D(X)$ is a doubly-stochastic matrix. In this case we say that the matrix $D(X)$ is the *doubly-stochastic matrix representation of* X. In this paper we will study some properties and applications of this representation on finite posets. Specially, we will show that every series-parallel poset is singular.

In general we use standard notations. We denote by $X\oplus Y$ and $X+Y$ the *ordinal sum* and the *disjoint sum* of X and Y, respectively. Also, we denote by C_n and $\underline{n}$ a *chain* and an *antichain with n vertices, respectively*. Throughout this paper we assume that every poset is finite and nonempty.

2. Definitions and well-known results

In this section we will give some definitions and properties which will be used later.

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Definition 2.1. A poset X is said to be *N-free* if it contains no cover preserving subposet isomorphic to the poset with Hasse diagram 

Definition 2.2. A poset is said to be *series–parallel* if it can be decomposed into singletons using ordinal sum and disjoint sum.

Proposition 2.3. *A finite poset is series–parallel if and only if it contains no subposet isomorphic to the poset with Hasse diagram* 

Proof. The proof can be found in [9].

From above definitions and Proposition 2.3. we know that every series–parallel poset is N-free.

Definition 2.4. Let x and y be vertices of a poset X. Then we define the following numerical functions:

$f_1(x) = |\{y : x<y\}|$, the number of descendants of x,

$f_2(x) = |\{y : x>y\}|$, the number of ancestors of x.

A poset X is said to be *a family* if both $f_1(x)>f_1(y)$ and $f_2(x)<f_2(y)$ implies $x<y$ for any vertices x and y in X.

Proposition 2.5. *If X is a family, then it is series–parallel.*

Proof. It follows from [7].

Definition 2.6. A poset X is said to be a *P–graph* if it can be decomposed into antichains using only ordinal sum, and a poset X is said to be a *P–series* if it can be decomposed into P–graph using only disjoint sum.

Proposition 2.7. *Every P–graph is a family and every P–series is series–parallel.*

Proof. The proof follows from [7].

3. Symmetric posets and singular posets

Definition 3.1. A poset X is said to be *symmetric* if $D(X)$ is symmetric for some relebelling of vertices of X.

THEOREM 3.2. *If X and Y are symmetric, then so is $X \oplus Y$.*

Proof. It can be easily shown from the definition of ordinal sum.

EXAMPLE 3.3. (1) Every chain and every antichain are symmetric.
(2) A poset N with Hasse diagram  is symmetric.
(3) Every P-graph is symmetric.
(4) A poset of the form $N \oplus \cdots \oplus N$ is symmetric.

Now we conjecture the following:
A poset X is symmetric if and only if it is either a P-graph or a graph of the form $N \oplus \cdots \oplus N$

DEFINITION 3.4. A poset X is said to be *singular* if $D(X)$ is singular. Otherwise, it is called *nonsingular.*

EXAMPLE 3.5. (1) Every nonempty chain is nonsingular.
(2) Every antichain n is singular if $n \geq 2$.

THEOREM 3.6. *Let X and Y be posets. Then*
(1) *If X and Y are nonsingular, then so is $X \oplus Y$.*
(2) *If either X or Y is singular, then so is $X \oplus Y$.*

Proof. The proof can be easily shown from the definition of ordinal sum.

COROLLARY 3.7. *If X is a P-graph which is not a chain, then X is singular.*

Proof. It follows from Example 3.5 and Theorem 3.6.

PROPOSITION 3.8. *The poset $C_{n_1}+C_{n_2}$ is singular.*

Proof. Let $C_{n_1}=\{x_1, \cdots, x_{n_1}\}$ and $C_{n_2}=\{x_{n_1+1}, \cdots, x_{n_1+n_2}\}$ be chains. Then $D(C_{n_1}+C_{n_2})=(p_{ij})$ is an $(n_1+n_2)\times(n_1+n_2)$ matrix. Note that $\sum_{i=1}^{n_1} p_{ij}=c$ and $\sum_{i=n_1+1}^{n_1+n_2} p_{ij}=1-C$ for all j, where $0<c<1$. Hence $D(C_{n_1}+C_{n_2})$ is singular, and so $C_{n_1}+C_{n_2}$ is singular.

PROPOSITION 3.9 *Let C_n be a chain with n vertices and X be a poset with m vertices.* Then the disjoint sum *C_n+X is singular.*

Proof. Let $\mathscr{L}(X)=\{L_1, L_2, \cdots, L_e\}$ be the set of all linear extensions of X. Then note $\mathscr{L}(C_n+X)=\bigcup_{i=1}^{e}\mathscr{L}(C_n+L_i)$ and $\mathscr{L}(C_n+L_i)\cap\mathscr{L}(C_n+L_j)=\phi$ for any distinct i and j. Let $D(C_n+X)=(a_{jk})$ and $D(C_n+L_i)=(i_{jk})$ be $(n+m)\times(n+m)$ corresponding matrices. Then $a_{jk}=\frac{1}{e}\sum_{i=1}^{e} i_{jk}$. Therefore it follows from Proposition 3. 8 that C_n+X is singular.

THEOREM 3. 10. *Let X and Y be posets. Then the disjoint sum $X+Y$ is singular.*

Proof. It can be proved by the same fashion as the proof of Proposition 3. 9.

COROLLARY 3. 11. *Every disconnected poset is singular.*

Proof. It follows immediately from Theorem 3. 10.

COROLLARY 3. 12. *If X is a P-series not a chain, then X is singular.*

Proof. It can be easily obtained from Corollary 3. 7 and Corollary 3. 11.

DEFINITION 3. 13. Let A be a subposet of a poset X and x be a vertex of X. Then x is said to be a *minimal upper* (or *maximal lower*) *bound of* A if x is an upper (or lower) bound of A and there is no upper (or lower) bound which is less (or greater) than x.

THEOREM 3. 14. *Every series-parallel poset not a chain is singular.*

Proof. It follows from Corollary 3. 11 that we may assume that our poset X is connected. If $|X|\leq 3$, then it is clear that X is singular. Suppose that it holds for $|X|<n$. Then we will show this theorem for $|X|=n$. Let $A=\{a_1, \cdots, a_p\}$ be the set of all maximal vertices of X. If $|A|=1$, then $X=(X-A)\oplus A$. By induction $(X-A)$ is singular, and hence X is singular by Theorem 3. 6. Now, we will show this theorem for the case $|A|>1$ with a series of propositions.

PROPOSITION 3. 15. *Let $A=\{a_1, \cdots, a_p\}$ be the set of all maximal vertices of a connected series-parallel poset X. Then there is a maximal lower bound of A.*

Proof. Since X is connected series–parallel, there is a maximal lower bound of A for any maximal vertices a_i and a_j. Assume that it holds for any k maximal vertices of X. Now consider it for maximal vertices $a_1, \cdots, a_{k+1}$ of X, and let x be a maximal lower bound of $\{a_1, \cdots, a_k\}$. Suppose that there is no maximal lower bound of $\{a_1, \cdots, a_{k+1}\}$. Then there are vertices y and z such that $z>x$, $z>y$ and $z<a_{k+1}$. Note that (x, y) and (z, a_{k+1}) are incomparable pairs of vertices. If z is one of a_i' s, then it contradicts the fact that X is series–parallel. Otherwise, we have $z<a_i$ for some a_i in A. This implies that X is not series–parallel, a contradiction. Therefore Proposition 3. 15 holds.

PROPOSITION 3. 16. *Let x be a maximal lower bound of A, where $A=\{a_1, \cdots, a_p\}$ is the set of all maximal vertices of X. Suppose that $B=\{b_1, \cdots b_q\}$ is the set of all upper covers of x, and that $C=\{c_1, \cdots, c_r\}$ is the set of all maximal lower bounds of B. Then we have the following properties*:

(1) *x is in C,*

(2) *C is an antichain,*

(3) *If y is a lower cover of b_j for some b_j, then $y \in C$,*

(4) *If y is incomparable to x, then $y \leq c_j$ for some c_j in C.*

Proof. (1) and (2) are obvious from the hypothesis. The proof of (3) is clear since X is series–parallel. Now, we will prove (4). Suppose that y is not less than b_j for some b_j in B. Since y can not be greater than b_j for all b_j in B, y is incomparable to b_j for all b_j. Also since y is not maximal, it contradicts the fact that X is series–parallel. Hence $y<b_j$ for some b_j in B. Thus $y<b_j$ for all b_j in B, and so y is a lower bound of B. Therefore (4) holds.

PROPOSITION 3. 17. *Theorem* 3. 14 *holds.*

Proof. Let A, B, and C be the sets which are defined in Proposition 3. 16. Let u be an arbitrary vertex of X. If u is incomparable to a vertex c_i in C, then $u \leq c_j$ for some c_j in C by (4) of Proposition 3. 15. Hence $u \leq b_i$ for all $b_i \in B$. Also, if u is comparable to a vertex of C, then $u \leq c_i$ for some c_i in C or $u \geq c_j$ for all c_j in C. Thus $u>c_i$ for all c_i in C or $u<b_j$ for all b_j in B. Now let $Y=\cap \{y : y>c_i\}$. Then $X=(X-Y) \oplus Y$. So by induction X is singular.

COROLLARY 3.18. *Every family X not a chain is singular.*

Proof. It follows from Proposition 2.5 and Theorem 3.13.

EXAMPLE 3.19. If X is a N-free poset which is not a chain, it may not be singular. Let X be a poset with Hasse diagram  . Then X is N-free, but it is neither a series-parallel nor singular.

4. The permanent of a poset

DEFINITION 4.1. Let $A=[a_{ij}]$ be an $n\times n$ matrix over a real number field $\boldsymbol{R}$. Then the *permanent* of A is defined by $\sum a_{1\sigma(1)}\cdot a_{2\sigma(2)}\cdots a_{n\sigma(n)}$, where σ is a permutation on $\{1, 2, \cdots, n\}$ and it is denoted by perm A. If $D(X)$ is a doubly-stochastic matrix of a poset X, then we write perm X for perm $D(X)$.

EXAMPLE 4.2. (1) perm $C_n=1$ and perm $\underline{n}=n!/n^n$.

(2) $\mathrm{perm}(X\oplus Y)=\mathrm{perm}X\cdot\mathrm{perm}Y$, where X and Y are finite posets.

(3) Let $X=\underline{n_1}\oplus\cdots\oplus\underline{n_p}$ be a p-graph. Then $\mathrm{perm}\,X=\prod_{k=1}^{p}(n_k!/n_k^{n_k})$.

From a famous Van der Waerden-Egorycev theorem we get the following theorem:

THEOREM 4.3. *Let X be a poset with n elements Then* $n!/n^n\leq$ *perm* $X\leq 1$, *where the left equality holds only if* $X=\underline{n}$ *and the right equality holds only if* $X=C_n$.

Proof. The first part "$n!/n^n\leq\mathrm{perm}\,X$, where the equality holds only if $X=\underline{n}$" is obtained from [6]. The last part "perm $X\leq 1$, where the equality holds only if $X=C_n$" is easily proved by induction.

5. Entropy

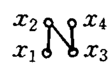
DEFINITION 5.1. Let $X=\{x_1, \cdots, x_n\}$ be a poset. Then the *entropy of a vertex* x_i is defined by $\sum_{k=1}^{n} a_{ik}\log_n a_{ik}$ and it is denoted by $H(x_i)$, where $D(X)=[a_{ij}]$ and $0\log_n 0=0$. In particular, a vertex x_i is called *free* in a linear extension of X if $H(x_i)=-1$ and a vertex x_i is called *fixed* in a linear extension of X if $H(x_i)=0$.

For example, every vertex of an antichain $\underline{n}$ is free. That is, every vertex of $\underline{n}$ can be placed in any position for some linear extension of $\underline{n}$. Also, every vertex of a chain C_n is fixed. That is, every vertex of C_n can be placed in only one fixed position of every linear extension of C_n.

Definition 5. 2. Let $X=\{x_1, \cdots, x_k\}$ be a finite poset. Then the *entropy of* X is defined by $\frac{1}{n}\sum_{k=1}^{n} H(x_i)$ and it is denoted by $H(X)$. In particular, X is called *free* if $H(X)=-1$ and X is called *fixed* if $H(X)=0$.

Proposition 5. 3. *Let x be a vertex of a finite poset X. Then $-1\leq H(x)\leq 0$ and $-1\leq H(X)\leq 0$. In fact, $H(X)=-1$ if and only if $X=\underline{n}$ and $H(X)=0$ if and only if $X=C_n$.*

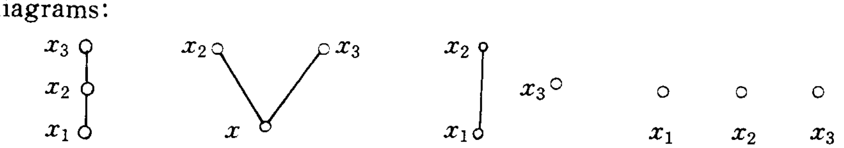
Proof. The proof is easily proved by a simple calculation.

Example 5. 4. Let X be a poset with Hasse diagram 

Then $H(x_2)=H(x_3)\geq H(x_1)=H(x_4)$. That is, the positions of x_2 and x_3 are more restricted than the positions of x_1 and x_4 in a linear extensions of X.

Actually in a finite poset with a small number of vertices, a vertex x is more restricted than a vertex y if and only if $H(x)>H(y)$. Now we have the following conjecture: *Let x and y be vertices of a finite poset X. Then x is more restricted than y if and only if $H(x)>H(y)$.*

Example 5. 5. Let X, Y, Z, and U be posets with following Hasse diagrams:



$X=C_3$ Y Z $U=\underline{3}$

Note that $0=H(X)>H(Y)>H(Z)>H(U)=-1$ and $6=|\mathscr{L}(U)|>|\mathscr{L}(Z)|>|\mathscr{L}(Y)|>|\mathscr{L}(X)|=1$.

Let X and Y be posets with 3 vertices. Then from above example we know that $|\mathscr{L}(X)|>|\mathscr{L}(Y)|$ implies $H(X)<H(Y)$. In fact for any posets X and Y with a small number of vertices, if $|X|=|Y|$ and $|\mathscr{L}(X)|\leq|\mathscr{L}(Y)|$, then $H(X)\geq H(Y)$. The fact that $|X|=|Y|<\infty$ and $|\mathscr{L}(X)|\leq|\mathscr{L}(Y)|$ imply $H(X)\geq H(Y)$ has not been known so far.

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Dankook University
Cheonan 330-180, Korea