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EXACT G-SEQUENCES AND RELATIVE G-PAIRS

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Gottlieb has defined and studied the evaluation subgroups $G_n(X)$ of the homotopy groups $\Pi_n(X)$ of a topological space X [1, 2]. Recently, the group $G_n(X)$ is generalized by the first author and Kim [6] as follows; Let (X, A) be a topological pair. Consider the class of continuous maps $F: A\times S^n \to X$ such that $F(a, s_0)=a$, then the map $h: (S^n, s_0)\to(X, x_0)$ defined by $h(s)=F(x_0, s)$ represents an element $[h]$ in $\Pi_n(X, x_0)$. The set of all elements $[h]\in\Pi_n(X, x_0)$ obtained in the above manner from some F is denoted by $G_n(X, A, x_0)$ and called the generalized evaluation subgroups of the homotopy groups.

The exactness of the homotopy sequence for a topological pair (X, A) plays an important role in algebraic topology. In [8] we have defined and studied the relative evaluation subgroups $G_n^{\mathrm{Rel}}(X, A)$ of a topological pair (X, A). Consider a map

$$H: (X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\to(X, A, x_0)$$

such that $H(x, u)=x$ for $x\in X$ and $u\in J^{n-1}$. Then the map $f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ defined by $f(u) = H(x_0, u)$, where x_0 is a base point of X, represents an element $\alpha=[f]\in\Pi_n(X, A, x_0)$. The set of all elements $\alpha\in\Pi_n(X, A, x_0)$ obtained in the above manner from some H is denoted by $G_n^{\mathrm{Rel}}(X, A, x_0)$.

In particular, we show that if (X, A) is a CW-pair, we obtain a subsequence of the homotopy sequence of (X, A)

$$\begin{array}{ccccccccc}\to\Pi_{n+1}(X, A) & \to & \Pi_n(A) & \to & \Pi_n(X) & \to & \Pi_n(X, A) & \to\cdots \\ \cup & & \cup & & \cup & & \cup & \\ \to G_{n+1}^{\mathrm{Rel}}(X, A) & \to & G_n(A) & \to & G_n(X, A) & \to & G_n^{\mathrm{Rel}}(X, A) & \to\cdots\end{array}$$

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$$\begin{array}{ccccc}\cdots\to \Pi_2(X,A) & \to & \Pi_1(A) & \to & \Pi_1(X)\\ \cup & & \cup & & \cup\\ \cdots\to G_2^{\mathrm{Rel}}(X,A) & \to & G_1(A) & \to & G_1(X,A).\end{array}$$

We call this subsequence the *G-sequence* for a CW-pair (X, A). Furthermore, we show that if the inclusion $i: A \hookrightarrow X$ has a left homotopy inverse, then the G-sequence for (X, A) is exact [8].

In this paper, we study such pairs and show that the fundamental group of A acts trivially on the G-sequence for a CW-pair (X, A). Next, we introduce relative G-pairs and calculate relative evaluation subgroups $G_n^{\mathrm{Rel}}(X, A)$ by the exactness of the G-sequences for those pairs.

Throughout this paper, all spaces are assumed to be pathwise connected with base point and all CW-complexes assumed to be locally finite.

THEOREM 1. *If (X, A) is a CW-pair, the fundamental group $\Pi_1(A)$ operates trivially on the G-sequence of (X, A). In diagrammatic terms, this means that, for each $\xi\in\Pi_1(A)$, all vertical maps in the following diagram are identity maps:*

$$\begin{array}{ccccccccc}\to G_{n+1}^{\mathrm{Rel}}(X,A) & \xrightarrow{\partial} & G_n(A) & \longrightarrow & G_n(X,A) & \longrightarrow & G_n^{\mathrm{Rel}}(X,A) & \longrightarrow\\ \downarrow \tau_{\xi}' & & \downarrow \tau_{\xi} & & \downarrow \tau_{\eta} & & \downarrow \tau_{\xi}' & \\ \to G_{n+1}^{\mathrm{Rel}}(X,A) & \xrightarrow{\partial} & G_n(A) & \longrightarrow & G_n(X,A) & \longrightarrow & G_n^{\mathrm{Rel}}(X,A) & \longrightarrow\end{array}$$

(where $\eta=i_*(\xi)$ and see p. 164 in [6] for τ_ξ', τ_ξ and τ_η).

Proof. Since the fundamental group $\Pi_1(A)$ operates on the homotopy sequence of a pair (X, A), it is sufficient to show that all vertical maps are the identity maps. Let $\xi=[\sigma]$ be any element of $\Pi_1(A)$ and $\eta=i_*(\xi)$.

For $n=1$, $G_1(X, A, x_0)$ is contained in $Z(i_*(\Pi_1(A)), \Pi_1(X))$ (see Theorem 1 [9]), where $Z(H, K)$ denotes the centralizer of a subgroup H of K. Thus we have $\tau_\eta(\alpha)=\eta+\alpha-\eta=i_*(\xi)+\alpha-i_*(\xi)=\alpha$ for any $\alpha\in G_1(X, A, x_0)$. Therefore $\tau_\eta: G_1(X, A, x_0)\to G_1(X, A, x_0)$ is the identity map for any $\xi\in\Pi_1(A)$. Since $G_1(A)=G_1(A, A, x_0)$ is contained in $Z(\Pi_1(A), \Pi_1(A))=Z(\Pi_1(A))$, we also have that τ_ξ is the identity map on $G_1(A)$ for every $\xi\in\Pi_1(A)$.

For $n\geq 2$, we only prove that $\tau_\xi': G_n^{\mathrm{Rel}}(X, A)\to G_n^{\mathrm{Rel}}(X, A)$ is the identity map for any $\S\in\Pi_1(A)$. Let α be any element of $G_n^{\mathrm{Rel}}(X, A)$. Then there is an affiliated homotopy

$$H: (X\times I^n,\ A\times\partial I^n, x_0\times J^{n-1})\to(X, A, x_0)$$

such that $H(x, u)=x$ for every $u\in J^{n-1}$ and $[H(x_0,\ )]=\alpha$. If we difine $F: I^n\times I\to X$ by $F(u, t)=H(\sigma(1-t), u)$, then $F(\partial I^n\times I)\subset A$ and $F(u, t)=\sigma(1-t)$ for every $u\in J^{n-1}$. Since $[F(\ ,0)]=\alpha$ and $F(u, t)=\sigma(1-t)$ for every $u\in J^{n-1}$, we have $\alpha=[F(\ ,1)]=\tau_{\S}'(\alpha)$. Thus $\tau_{\S}'$ is the identity map on $G_n^{\mathrm{Rel}}(X, A)$ for any $\S\in\Pi_1(A)$.

A CW-pair (X, A) is said to be an *H-pair* if and only if X and A are H-spaces and the restriction of the product in X to $B\times B$ is homotopic (and therefore may be assumed equal) to the product in B.

THEOREM 2. *Every H-pair has an exact G-sequence.*

Proof. Let (X, A) be an H-pair. Then $G_n^{\mathrm{Rel}}(X, A, x_0)=\Pi_n(X, A, x_0)$ for $n>1$, by Theorem 5 [8]. Since X and A are H-spaces, $G_n(X)=\Pi_n(X)$ and $G_n(A)=\Pi_n(A)$. Thus the G-sequence for (X, A) is just the homotopy sequence. Therefore the G-sequence for (X, A) is exact.

A space X satisfying $G_n(X)=\Pi_n(X)$ for every n is called a *G-space.* Any H-space is a G-space, but the converse is not true [5]. A pair (X, A) is a *G-pair* if X and A are G-spaces.

THEOREM 3. *Every G-pair (X, A) has an exact G-sequence if ∂-image of $\Pi_n(X, A)$ is equal to ∂-image of $G_n^{\mathrm{Rel}}(X, A)$, where ∂ is the connecting homomorphism in the homotopy sequence of (X, A).*

Proof. Consder the following sequence;

$$\to G_n(A, x_0)\xrightarrow{i_*}G_n(X, A, x_0)\xrightarrow{j_*}G_n^{\mathrm{Rel}}(X, A, x_0)\xrightarrow{\partial'}G_{n-1}(A, x_0)\to$$

for $n>1$. Since X and A are G-spaces, $G_n(A)=\Pi_n(A)$ and $G_n(X, A)=\Pi_n(X)$. Thus it is sufficient to show that the G-sequence is exact at $G_n(A)$ for $n\geq 1$. Since $\partial(G_n^{\mathrm{Rel}}(X, A))=\partial(\Pi_n(X, A))$, $\ker i_*=\mathrm{Im}\,\partial\cap G_n(A)=\mathrm{Im}\ \partial'$, where $\partial': G_n^{\mathrm{Rel}}(X, A)\to G_{n-1}(A)$ is the connecting homomorphism.

A topological pair (X, A) is said to be a *relative G-pair* if $G_n^{\mathrm{Rel}}(X, A)=\Pi_n(X, A)$ for all $n>1$.

Remark. For any topological space X, the pair (X, X) is a relative G–pair. For any CW–complex X, the pair (X, x_0) is a relative G–pair. Because, by Theorem 12 [8], the pair (X, x_0) has an exact G–sequence;

$$\to G_n(x_0, x_0) \to G_n(X, x_0, x_0) \to G_n^{\mathrm{Rel}}(X, x_0, x_0) \to G_{n-1}(x_0, x_0) \to$$

for $n>1$. Since $G_n(x_0, x_0)=0$, $\Pi_n(X, x_0)=G_n(X, x_0, x_0)$ is isomorphic to $G_n^{\mathrm{Rel}}(X, x_0, x_0)$ which is a subgroup of $\Pi_n(X, x_0, x_0)=\Pi_n(X, x_0)$. Thus we have $G_n^{\mathrm{Rel}}(X, x_0, x_0)=\Pi_n(X, x_0, x_0)$.

THEOREM 4. *Every G–pair is a relative G–pair if* $\partial(G_n^{\mathrm{Rel}}(X, A))=\partial(\Pi_n(X, A))$ *for every* $n>1$.

Proof. Let (X, A) be a G–pair. Then $\Pi_n(A)=G_n(A)$ and $\Pi_n(X)=G_n(X, A)$ for $n>1$. Consider the following commutative ladder;

$$\begin{array}{ccccccccccc}
\longrightarrow G_n(A) & \longrightarrow & G_n(X,A) & \longrightarrow & G_n^{\mathrm{Rel}}(X,A) & \longrightarrow & G_{n-1}(A) & \longrightarrow & G_{n-1}(X,A) & \longrightarrow \\
\| & & \| & & \downarrow i & & \| & & \| & \\
\longrightarrow \Pi_n(A) & \longrightarrow & \Pi_n(X) & \longrightarrow & \Pi_n(X,A) & \xrightarrow{\partial} & \Pi_{n-1}(A) & \longrightarrow & \Pi_{n-1}(X) & \longrightarrow
\end{array}$$

for $n>1$, where i is the inclusion.

Let α be any element of $\Pi_n(X, A)$. Then $\partial(\alpha)$ belongs to $\Pi_{n-1}(A)=G_{n-1}(A)$. Since the lower and the upper sequences are exact, there is an element $\beta\in G_n^{\mathrm{Rel}}(X, A)$ such that $\partial(\beta)=\partial(\alpha)$. Thus $\partial(\beta-\alpha)=0$. So for some $\gamma\in\Pi_n(X)=G_n(X, A)$, we have $\beta-\alpha=j_*(\gamma)$. This implies $\beta-\alpha\in G_n^{\mathrm{Rel}}(X, A)$. Therefore α belongs to $G_n^{\mathrm{Rel}}(X, A)$.

For a CW–pair (X, A), if $i : A\to X$ has a left homotopy inverse, then the pair (X, A) has an exact G–sequence. For the CW–pair (B^n, S^{n-1}), $i : S^{n-1}\to B^n$ does not have a left homotopy inverse but the pair (B^n, S^{n-1}) has an exact G–sequence.

THEOREM 5. *The topological pair (B^n, S^{n-1}) has an exact G–sequence.*

Proof. We first show that $(B^n)^{S^{n-1}}$ is contractible to the inclusion $i : S^{n-1}\to B^n$. If we define a homotopy $H : (B^n)^{S^{n-1}}\times I\to(B^n)^{S^{n-1}}$ by $H(f, t)(u)=f(u)t+u(1-t)$, then H is well-defined and continuous. Since

$$H(f, 0)(u)=f(u)0+u(1-0)=u=i(u)$$
$$H(f, 1)(u)=f(u)$$
$$H(i, t)(u)=ut+u(1-t)=u=i(u).$$

Thus $(B^n)^{S^{n-1}}$ is contractible to i.

Since (B^n, S^{n-1}) is a CW–pair, we have the following commutative diagram

$$\begin{array}{ccc} ((B^n)^{B^n}, R((B^n)^{B^n}), 1_{B^n}) & \xrightarrow{p} & ((B^n)^{S^{n-1}}, (S^{n-1})^{S^{n-1}}, 1_{S^{n-1}}) \\ & w\searrow \quad \swarrow w & \\ & (B^n, S^{n-1}, x_0) & \end{array}$$

where, w is the evaluation map from $(B^n)^{B^n}$ (or $(B^n)^{S^{n-1}}$) to B^n and $R((B^n)^{B^n})$ is the subspace of $(B^n)^{B^n}$ such that $f\in R((B^n)^{B^n})$ iff $f(S^{n-1})\subset S^{n-1}$.

Since $p^{-1}((S^{n-1})^{S^{n-1}})=R((B^n)^{B^n})$, $p_* : \Pi_n((B^n)^{B^n}, R((B^n)^{B^n}), 1_{B^n})\to\Pi_n((B^n)^{S^{n-1}}, (S^{n-1})^{S^{n-1}}, 1_{S^n})$ is onto. So

$$\begin{aligned} G_n^{\mathrm{Rel}}(B^n, S^{n-1}, x_0) &= w_*(\Pi_n((B^n)^{B^n}), R((B^n)^{B^n}), 1_{B^n}) \\ &= w_*p_*(\Pi_n((B^n)^{B^n}), R((B^n)^{B^n}), 1_{B^n}) \\ &= w_*(\Pi_n((B^n)^{S^{n-1}}, (S^{n-1})^{S^{n-1}}, 1_{S^{n-1}}). \end{aligned}$$

Consider the following commutative ladder;

$$\begin{array}{ccccccc} \cdots\to\Pi_m((B^n)^{S^{n-1}}, i) & \to & \Pi_m((B^n)^{S^{n-1}}, (S^{n-1})^{S^{n-1}}, i) & \xrightarrow{\partial} & \Pi_{m-1}((S^{n-1})^{S^{n-1}}, i) & \xrightarrow{i_*} & \cdots \\ \downarrow w_* & & \downarrow w_* & & \downarrow w_* & & \\ \cdots\to G_m(B^n, S^{n-1}) & \xrightarrow{j_*} & G_m^{\mathrm{Rel}}(B^n, S^{n-1}) & \xrightarrow{\partial'} & G_{m-1}(S^{n-1}) & \xrightarrow{i_*} & \cdots \\ \cap & & \cap & & \cap & & \\ \cdots\to\Pi_m(B^n, x_0) & \xrightarrow{j_*} & \Pi_m(B^n, S^{n-1}, x_0) & \xrightarrow{\partial'} & \Pi_{m-1}(S^{n-1}, x_0) & \to & \cdots \end{array}$$

By the property of evaluation subgroups, the vertical homomorphisms w_* are epimorphisms. Since B^n is contractible, $\Pi_m(B^n, x_0)$ is the trivial group for $m\geq 1$. Thus ∂' is a monomorphism. Thus it is sufficient to show that ∂' is an epimorphism. Since $\Pi_m((B^n)^{S^{n-1}}, i)=0$, ∂ is an isomorphism. Thus ∂' is an epimorphism.

Corollary 6. $G_n^{\mathrm{Rel}}(B^n, S^{n-1})=\begin{cases} 0 & \textit{if } n : \textit{odd number greater than } 1 \\ 2Z\subset Z & \textit{if } n : \textit{even and } n\neq 2, 4, 8 \\ Z=\Pi_n(B^n, S^{n-1}) & \textit{if } n=2, 4, 8. \end{cases}$

Proof. By the above theorem, $G_n^{\mathrm{Rel}}(B^n, S^{n-1})=G_{n-1}(S^{n-1})$.

Theorem 7. *If a pair (X, A) has an exact G-sequence and X is deformable into A relative to a point $x_0\in A$, then*

$$G_n(A, x_0)=G_n(X, A, x_0)\oplus G_{n+1}^{\mathrm{Rel}}(X, A, x_0)$$

for $n\geq 2$ and $i_* : G_n(A, x_0)\to G_n(X, A, x_0)$ is an epimorphism for $n\geq 1$.

Proof. According to the hypothesis, there exists a homotopy $h_t : X\to X$, $(0\leq t\leq 1)$, such that $h_0(x)=x$, $h_1(x)\in A$, $h_t(x_0)=x_0$, for each $x\in X$ and $t\in I$. Since $ih_1=h_0$ is homotopic to $h_0(\text{rel } x_0)$, we have $i_*h_{1*}=1$. Hence h_{1*} is a monomorphism and i_* is an epimorphism for every $n\geq 1$. Thus $G_n(X, A, x_0)$ and $G_{n+1}^{\text{Rel}}(X, A, x_0)$ are abelian for $n\geq 1$. Thus we have the following split short exact sequence;

$$0\longrightarrow G_{n+1}^{\text{Rel}}(X, A, x_0)\longrightarrow G_n(A, x_0)\underset{h_{1*}}{\overset{i_*}{\rightleftarrows}} G_n(X, A, x_0)\rightarrow 0.$$

Therefore, $G_n(A, x_0)=G_n(X, A, x_0)\oplus G_{n+1}^{\text{Rel}}(X, A, x_0)$.

THEOREM 8. *Let Y be a Lie group, B be a subgroup, and G be any closed subgroup contained in B. If $p:(Y, B, e)\rightarrow(Y/G, B/G, \bar{e})$ is a quotient pair map, then $p_*(\Pi_n(Y, B, e))$ is contained in $G_n^{\text{Rel}}(Y/G, B/G, \bar{e})$, where e is the unit element of Y and $\bar{e}=eG$.*

Proof. Note that the natural map $\pi: Y/G\times Y\rightarrow Y/G$ given by $\Pi(aG, b)=baG$ is continuous. If $\alpha\in p_*(\Pi_n(Y, B, e))$, then there is a map $f:(I^n, \partial I^n, J^{n-1})\rightarrow(Y, B, e)$ such that $\alpha=[pf]$. Then the map $H=\Pi(1\times f): Y/G\times I^n\rightarrow Y/G$ is an affiliated map for α. In fact,

$$H(bG, u)=\Pi(1\times f)(bG, u)=\Pi(bG, f(u))=f(u)bG\in B/G$$

for $b\in B$ and $u\in\partial I^n$,

$$H(G, u)=\Pi(G, f(u))=f(u)G=pf(u) \text{ for } u\in I^n,$$
$$H(yG, v)=\Pi(yG, e)=eyG=yG \text{ for } v\in J^{n-1}.$$

Thus we have that α belongs to $G_n^{\text{Rel}}(Y/G, B/G, \bar{e})$.

In particular, if G is a finite subgroup of B, we have the following;

THEOREM 9. *Let Y be a Lie group and B be a subgroup. If G is a fnite subgroup contained in B, then we have*

$$G_n^{\text{Rel}}(Y/G, B/G, \bar{e})=\Pi_n(Y/G, B/G, \bar{e}) \textit{ for } n>2.$$

Proof. Since the quotient map $p: Y\rightarrow Y/G$ is a fiber map with fiber G, $p_*:\Pi_n(Y, G, e)\rightarrow\Pi_n(Y/G, \bar{e})$ is an isomorphism. Similarly, $\Pi_n(B, G, e)$ is isomorphic to $\Pi_n(Y/G, \bar{e})$ by p_*. Since (B, G) and (Y, G) are H-pairs, $\Pi_n(B, G, e)=G_n^{\text{Rel}}(B, G, e)$ and $\Pi_n(Y, G, e)=G_n^{\text{Rel}}(Y, G, e)$ for $n>1$. Consider the following commutative ladder;

$$\begin{array}{ccccccccccc}
\rightarrow\Pi_n(B, G) & \rightarrow & \Pi_n(Y, G) & \rightarrow & \Pi_n(Y, B) & \rightarrow & \Pi_{n-1}(B, G) & \rightarrow & \Pi_{n-1}(Y, G) & \rightarrow \\
\downarrow p_* & & \downarrow p_* & & \downarrow p_* & & \downarrow p_* & & \downarrow p_* & \\
\rightarrow\Pi_n(B/G) & \rightarrow & \Pi_n(Y/G) & \rightarrow & \Pi_n(Y/G, B/G) & \rightarrow & \Pi_{n-1}(B/G) & \rightarrow & \Pi_{n-1}(Y/G) & \rightarrow
\end{array}$$

By five lemma, $p_*:\Pi_n(Y, B, e)\longrightarrow\Pi_n(Y/G, B/G, \bar{e})$ is an isomorphism for $n>2$. If we use Theorem 8, we have

$$\Pi_n(Y/G, B/G, e) = G_n^{\mathrm{Rel}}(Y/G, B/G, \bar{e}) \text{ for } n>2.$$

THEOREM 10. *If (X, A) and (Y, B) are relative G-pairs, then $(X\times Y, A\times B)$ is also a relative G-pair.*

Proof.

$$\begin{aligned}\Pi_n(X\times Y, A\times B, x_0\times y_0) &= \Pi_n(X, A, x_0)\oplus \Pi_n(Y, B, y_0)\\ &= G_n^{\mathrm{Rel}}(X, A, x_0)\oplus G_n^{\mathrm{Rel}}(Y, B, y_0)\\ &= G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0).\end{aligned}$$

for $n>1$.

THEOREM 11. *If (X, A) and (Y, B) have exact G-sequences, then $(X\times Y, A\times B)$ has an exact G-sequence.*

Proof. Consider the following ladder of sequences;

$$\begin{array}{ccccc}\to G_n(A\times B) & \xrightarrow{i_*} & G_n(X\times Y, A\times B) & \xrightarrow{j_*} & G_n^{\mathrm{Rel}}(X\times Y, A\times B)\to \\ \downarrow (p_{1*}', p_{2*}') & & \downarrow (p_{1*}'', p_{2*}'') & & \downarrow (p_{1*}, p_{2*}) \\ \to G_n(A)\oplus G_n(B) & \xrightarrow[i_{1*}\oplus i_{2*}]{} & G_n(X, A)\oplus G_n(Y, B) & \xrightarrow[j_{1*}\oplus j_{2*}]{} & G_n^{\mathrm{Rel}}(X, A)\oplus G_n^{\mathrm{Rel}}(Y, B)\to\end{array}$$

Since $(p_{1*}'', p_{2*}'')i_*[f]=(p_{1*}'', p_{2*}'')[f]=([p_1f], [p_2f])$ and $(i_{1*}\oplus i_{2*})(p_1', p_2')[f]=(i_{1*}\oplus i_{2*})([p_1'f], [p_2'f])=([p_1f], [p_2f])$, the retangle on the left is commutative.

Similarly, the middle retctangle and the right rectangle are commutative. Since (X, A) and (Y, B) have exact G-sequence, the lower sequence is exact. By Theorem 10[8], all vertical homomorphisms are isomorphisms. Thus the upper G-sequence for $(X\times Y, A\times B)$ is exact.

COROLLARY 12. *For any CW-complexes X, Y, $(X\times Y, X)$ and $(X\times Y, Y)$ have exact G-sequences.*

References

1. Gottlieb, D. H., *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840-856.
2. ______, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729-756.
3. Hu, S. T., Homotopy theory, Academic press, New York, 1959.
4. Lang, G. E., *Evaluation subgroups of factor spaces*, Pac. J. of Math. **42**

(1972), 701-709.

5. Siegel, J., *G-spaces, H-spaces and W-spaces,* Pac. J. Math. **31**(1969), 209-214.
6. Whitehead, G. W., Elements of homotopy theory, Springer-Verlag, New York, 1978.
7. Woo, Moo Ha and Kim, J. R., *Certain subgroups of homotopy groups,* J. Korean Math. Soc. **21**(1984), 109-120.
8. _______ and Lee, K. Y., *On the relative evaluation subgroups of a CW-pair,* J. Korean Math. Soc. **25**-1 (1988), 149-160.
9. Woo, Moo Ha, *Properties of the generalized evaluation subgroup of a topological pair,* Bull. Korean Math. Soc. **27**(1990), 121-126.

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