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ON A CERTAIN CLASS OF UNIVALENT FUNCTIONS

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1. Introduction

Let f be analytic in a convex domain D. If f satisfies the condition

$$\mathrm{Re}(f'(z))>0 \tag{1.1}$$

for all $z\in D$, then it is well known (see [14], [18] and others) that f is univalent in D. MacGregor [9] investigated the properties, e.g., coefficient estimates, radius of convexity etc. for functions f analytic in the unit disc $U=\{z : |z|<1\}$ having power series representation

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.2}$$

and satisfying (1.1) for $z\in U$. We denote the class of such functions by R. Analogous properties have also been obtained in [9] for analytic functions with initial zero coefficients in (1.2) and satisfying (1.1) for $z\in U$. Ezrohi [4] and Martynov [11] obtained the radius of convexity along with the other properties for the class R of functions $f(z)$ that are analytic and satisfy

$$\mathrm{Re}(f'(z))>\alpha \tag{1.3}$$

for $0\leqslant\alpha<1$, $z\in U$. Several other subclasses of R have also been obtained by Caplinger and Causey [2], Goel [5, 6], MacGregor [10], Padmanabhan [15], Shaffer [16] and others.

Let N be the class of functions

$$f(z)=z+\sum_{n=k+1}^{\infty}a_n z^n \tag{1.4}$$

that are analytic in U. In this paper, we propose a unified approach to the study of various subclasses of univalent functions whose derivatives have a positive real part in U. Thus, we introduce the class

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$R_k(\alpha, \beta, A, B)$ which, for different values of the parameters α, β, A, B ($0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$), not only gives rise to the classes studied by the above mentioned workers but also gives rise to many new subclasses of univalent functions. Thus we have the following:

Definition. Let $f(z) \in N$. Then $f \in R_k(\alpha, \beta, A, B)$ if the condition

$$\left| \frac{f'(z)-1}{(B-A)\beta(f'(z)-\alpha)+A(f'(z)-1)} \right| < 1 \qquad (1.5)$$

is satisfied for some α, β ($0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$), $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$ and for all $z \in U$. We note that $R_k(\alpha, \beta, -1, 1) = R_k(\alpha, \beta)$, is the class studied by Mogra [12].

It is easy to check that $R_1(\alpha, 1, -1, 1)$ is the class R_α studied by Ezrohi [4], Martynov [11] etc.; $R_1(0, 1, -1, 1) = R$, $R_1(0, \frac{1}{2}, -1, 1)$, $R_k(0, 1, -1, 1)$ and $R_k(0, 1-\delta, -1, 1)$, where $0 \leqslant \delta < 1$, give rise to the classes introduced by MacGregor [9, 10], Shaffer [16], while the cases $(\alpha, \beta) = \left(0, \frac{2\delta-1}{2\delta}\right)$, $\delta > \frac{1}{2}$ and $(\alpha, \beta) = \left(\frac{1-\gamma}{1+\gamma}, \frac{1+\gamma}{2}\right)$, $0 \leqslant \gamma < 1$ with $k=1$, $A=-1$ and $B=1$ lead respectively to the classes studied earlier by Goel [5], Padmanabhan [15], Caplinger and Causey [2] etc.; also $k=1$, $A=-1$, $B=1$ and a replacement of α by $1-\alpha$ and β by $\frac{1}{2}$ in (1.5) gives the class introduced by Goel [6].

We further, observe that by special choices of α, β, A and B our class $R_k(\alpha, \beta, A, B)$ give rise to the following new subclasses of R:

$$1-R^*_{k,\delta,\alpha} = R_k\left(\alpha, \frac{2\delta-1}{2\delta}, -1, 1\right)$$
$$= \left\{ f \in N : \left| \frac{f'(z)-\alpha}{1-\alpha} - \delta \right| < \delta,\ \delta > \frac{1}{2},\ 0 \leqslant \alpha < 1,\ z \in U \right\},$$

$$2-R_k(\gamma, A, B) = R_k\left(\frac{-A+A\gamma}{B\gamma-A}, \frac{B\gamma-A}{B-A}, A, B\right)$$
$$= \left\{ f \in N : \left| \frac{f'(z)-1}{Bf'(z)-A} \right| < \gamma,\ 0 < \gamma \leqslant 1,\ -1 \leqslant A < B \leqslant 1,\ 0 < B \leqslant 1,\ z \in U \right\},$$

$$3-R_{k,\alpha}(A, B) = R_k(\alpha, 1, A, B)$$
$$= \left\{ f \in N : \left| \frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]} \right| < 1,\ z \in U \right\},$$

$$4-R_{k,\alpha,\beta}(A,B)=R_k\left(\frac{-A+A\beta-(A-B)\alpha\beta}{B\beta-A},\ \frac{B\beta-A}{B-A},\ A,B\right)$$
$$=\left\{f\in N:\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta,\right.$$
$$\left.0\leqslant\alpha<1,\ 0<\beta\leqslant 1,\ -1\leqslant A<B\leqslant 1,\ 0<B\leqslant 1,\ z\in U.\right\}$$

REMARK. From the class $R_{k,\alpha,\beta}(A,B)$ we note that:

(i) $R_{1,\alpha,\beta}(A,B)=R(\alpha,\beta,A,B)$, is the class of functions $f(z)$ given by (1.2) and satisfying

$$\left|\frac{f'(z)-1}{Bf'(z)-[B+(A-B)(1-\alpha)]}\right|<\beta \tag{1.6}$$

for some $\alpha,\beta(0\leqslant\alpha<1,\ 0<\beta\leqslant 1)$, $-1\leqslant A<B\leqslant 1$, $0<B\leqslant 1$ and $z\in U$. The class of functions $f(z)$ satisfying (1.6) was introduced and studied by Aouf and Owa [1].

(ii) $R_{1,\alpha,\beta}(-1,1)=R(\alpha,\beta)$, is the class of functions $f(z)$ given by (1.2) and satisfying

$$\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right|<\beta \tag{1.7}$$

for some $\alpha,\beta(0\leqslant\alpha<1,\ 0<\beta\leqslant 1)$ and $z\in U$. The class of functions $f(z)$ satisfying (1.7) was introduced and studied by Juneja and Mogra [7].

From the definition given above it is clear that $R_k(\alpha,\beta,A,B)$ is a subclass of the class of functions whose derivatives have a positive real part in U. Also $R_k(\alpha,\beta,A,B)\subset R_k(\alpha,\beta',A,B)$ for $\beta\leqslant\beta'$. It is easily seen that for $f\in R_k(\alpha,\beta,A,B)$, the values $f(z)$ lie inside the circle in the right half plane with center at

$$\frac{1-[(B-A)\alpha\beta+A][(B-A)\beta+A]}{1-[(B-A)\beta+A]^2}\ \text{and radius}\ \frac{(B-A)\beta(1-\alpha)}{1-[(B-A)\beta+A]^2}.$$

Further, we assume that $f(z)\in R_k(\alpha,\beta,A,B)$. Setting

$$z^{k-1}h(z)=\frac{1-f'(z)}{(B-A)\beta(f'(z)-\alpha)+A(f'(z)-1)},$$

we see that the function $h(z)$ is analytic in the unit disc U, satisfies $|h(z)|<1$ for $z\in U$ and $h(0)=0$. Consequently, by using Schwarz's Lemma [13], we have $h(z)=z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disc U and satisfies $|\phi(z)|<1$ for $z\in U$. Thus we get

$$f'(z)=\frac{1+[(B-A)\alpha\beta+A]z^k\phi(z)}{1+[(B-A)\beta+A]z^k\phi(z)}.$$

In the present paper, we determine sharp coefficient estimates for functions in $R_k(\alpha, \beta, A, B)$, radius of convexity etc. for functions in $R_1(\alpha, \beta, A, B)$. A sufficient condition for a function to be in $R_k(\alpha, \beta, A, B)$ has also been obtained. For different values of the parameters α, β, A, B $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ -1 \leqslant A < B \leqslant 1,\ 0 < B \leqslant 1)$ our results generalize the corresponding results obtained by Caplinger and Causey [2], Ezrohi [4], Goel [5, 6], Kaczmarski [8], MacGregor [9, 10], Martynov [11], Padmanabhan [15], Shaffer [16] and Mogra [12].

2. Coefficient estimates

THEOREM 1. *If* $f(z) \in N$ *is in* $R_k(\alpha, \beta, A, B)$, *then*

$$|a_n| \leqslant \frac{(B-A)\beta(1-\alpha)}{n}$$

for $n \geqslant k+1$, $k = 1, 2, \ldots$. *The bounds are sharp for the functions*

$$f_n(z) = \int_0^z \frac{1 + [-A-(B-A)\alpha\beta]t^{n-1}}{1-[(B-A)\beta+A]t^{n-1}} dt$$

for $n \geqslant k+1$ *and* $z \in U$.

The proof of the above theorem is similar to that of Clunie [3], and hence is omitted.

REMARK. Different values of the parameters α, β, A, B and $k=1$ in Theorem 1 lead to the coefficient estimates obtained earlier by Caplinger and Causey [2], Goel [5], MacGregor [9, 10], Padmanabhan [15] and Mogra [12].

3. A sufficient condition for a function to be in $R_k(\alpha, \beta, A, B)$

THEOREM 2. *Let* $f(z) \in N$. *If for some* α, β, A, B $\left(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant \frac{-A}{(B-A)},\ -1 \leqslant A < B \leqslant 1,\ 0 < B \leqslant 1\right)$,

$$\sum_{n=k+1}^{\infty} (1-A-(B-A)\beta) n |a_n| \leqslant (B-A)\beta(1-\alpha), \tag{3.1}$$

then $f(z)$ *belongs to* $R_k(\alpha, \beta, A, B)$.

Proof. Suppose (3.1) holds for some α, β, A, B $\left(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant \frac{-A}{(B-A)},\ -1 \leqslant A < B \leqslant 1,\ 0 < B \leqslant 1\right)$ and that

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n,$$

then for $z \in U$,

$$\begin{aligned}
&|f'(z)-1|-|(B-A)\beta(f'(z)-\alpha)+A(f'(z)-1)| \\
&\leqslant \sum_{n=k+1}^{\infty} n|a_n|r^{n-1}-\Big\{(B-A)\beta(1-\alpha)+ \\
&\qquad \sum_{n=k+1}^{\infty}(-A-(B-A)\beta)n|a_n|r^{n-1}\Big\} \\
&< \sum_{n=k+1}^{\infty} n|a_n|-(B-A)\beta(1-\alpha)+\sum_{n=k+1}^{\infty}(-A-(B-A)\beta)n|a_n| \\
&= \sum_{n=k+1}^{\infty}(1-A-(B-A)\beta)n|a_n|-(B-A)\beta(1-\alpha)\leqslant 0,
\end{aligned}$$

by (3.1). Hence it follows that

$$\left|\frac{f'(z)-1}{(B-A)\beta(f'(z)-\alpha)+A(f'(z)-1)}\right|<1,$$

so that $f \in R_k(\alpha, \beta, A, B)$. Hence the theorem.

REMARK. Since $f(z) \in R_k(\alpha, \frac{-A}{(B-A)}, A, B)$ implies $f \in R_k(\alpha, \beta, A, B)$ for $\frac{-A}{(B-A)} \leqslant \beta \leqslant 1$, the condition (3.1) for $\beta = \frac{-A}{(B-A)}$, that is, the condition

$$\sum_{n=k+1}^{\infty} n|a_n| \leqslant (-A)(1-\alpha) \tag{3.2}$$

can also be used as a sufficient condition for a function to be in $R_k(\alpha, \beta, A, B)$ for $0 \leqslant \alpha < 1$, $\frac{-A}{(B-A)} \leqslant \beta \leqslant 1$, $-1 \leqslant A < B \leqslant 1$, $0 < B \leqslant 1$. The condition (3.2) with $k=1$, $\alpha=0$, $A=-1$, $B=1$ may be found in [19] as a sufficient condition for a function to be in R.

4. Radius of convexity for function in $R_1(\alpha, \beta, A, B)$

Let Ω denote the class of analytic functions $w(z)$ in U which satisfy the conditions (i) $w(0)=0$ and (ii) $|w(z)|<1$ for $z \in U$. We require the following lemmas.

LEMMA 1[17]. *If $w \in \Omega$, then for $z \in U$*

$$\left|zw'(z)-w(z)\right| \leqslant \frac{|z|^2-|w(z)|^2}{1-|z|^2}. \tag{4.1}$$

LEMMA 2. *Let $w(z)\in\Omega$. Then we have*

$$Re\left\{\frac{zw'(z)}{(1+[(B-A)\beta+A]w(z))(1+[(B-A)\alpha\beta+A]w(z))}\right\}$$
$$\leqslant\frac{-1}{(B-A)^2(1-\alpha)^2\beta^2}Re\left\{[(B-A)\beta+A]p(z)+\frac{[(B-A)\alpha\beta+A]}{p(z)}\right.$$
$$\left.-[(B-A)\beta(1+\alpha)+2A]\right\}$$
$$+\frac{r^2|[(B-A)\beta+A]p(z)-[(B-A)\alpha\beta+A]|^2-|1-p(z)|^2}{(B-A)^2(1-\alpha)^2\beta^2(1-r^2)|p(z)|},$$

where $p(z)=\dfrac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}$, $r=|z|$ *and* $0\leqslant\alpha<1$, $0<\beta\leqslant1$, $-1\leqslant A<B\leqslant1$, $0<B\leqslant1$.

The proof of the above lemma follows from (4.1) immediately. So we omit it.

REMARK. The transformation

$$p(z)=\frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}$$

maps the circle $|w(z)|\leqslant r$ onto the circle

$$\left|p(z)-\frac{1-[(B-A)\alpha\beta+A][(B-A)\beta+A]r^2}{1-[(B-A)\beta+A]^2r^2}\right|$$
$$\leqslant\frac{(B-A)\beta(1-\alpha)r}{1-[(B-A)\beta+A]^2r^2}.$$

THEOREM 3. *Let $f(z)\in R_1(\alpha,\beta,A,B)$, $0\leqslant\alpha<1$, $0<\beta\leqslant1$, $-1\leqslant A<B\leqslant1$, $0<B\leqslant1$, then f is convex in $|z|<r_0$, where r_0 is the smallest positive root of*

$$\sqrt{(1+[(B-A)\beta+A])(1+[(B-A)\alpha\beta+A])(1-[(B-A)\beta+A]r^2)\cdot(1-[(B-A)\alpha\beta+A]r^2)}$$
$$-(1-[(B-A)\alpha\beta+A][(B-A)\beta+A]r^2)+[-A-(B-A)\alpha\beta](1-r^2)$$
$$=0$$

if $R_0\geqslant R_1$ and r_0 is the smallest positive root of

$$1-2[-A-(B-A)\alpha\beta]r+[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2=0$$

if $R_0\leqslant R_1$, where

$$R_0=\left\{\frac{(1+[(B-A)\alpha\beta+A])(1-[(B-A)\alpha\beta+A]r^2)}{(1+[(B-A)\beta+A])(1-[(B-A)\beta+A]r^2)}\right\}^{\frac{1}{2}}$$

and

$$R_1=\frac{1+[(B-A)\alpha\beta+A]r}{1+[(B-A)\beta+A]r},\quad |z|=r<1.$$

All the above estimates are sharp.

Proof. Since $f\in R_1(\alpha,\beta,A,B)$, we have by Schwarz's Lemma [13]

$$f'(z)=\frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)},\tag{4.2}$$

where $w\in\Omega$. Differentiating (4.2) logarithmically, we get

$$1+\frac{zf''(z)}{f'(z)}=1-(B-A)\beta(1-\alpha)\cdot$$
$$\cdot\left\{\frac{zw'(z)}{1+[(B-A)\beta+A]w(z))(1+[(B-A)\alpha\beta+A]w(z))}\right\}.\tag{4.3}$$

An application of Lemma 2 to the above equation gives

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqslant\frac{1}{(B-A)\beta(1-\alpha)}\cdot$$
$$\cdot[\mathrm{Re}\left\{[(B-A)\beta+A]\,p(z)+\frac{[(B-A)\alpha\beta+A]}{p(z)}\right\}$$
$$-\frac{r^2|[(B-A)\beta+A]p(z)-A-(B-A)\alpha\beta|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|}$$
$$+2\frac{-A-(B-A)\alpha\beta}{(B-A)\beta(1-\alpha)},\tag{4.4}$$

where $p(z)=\frac{1+[(B-A)\alpha\beta+A]w(z)}{1+[(B-A)\beta+A]w(z)}$. Setting $p(z)=a+\xi+i\eta$, $R^2=(a+\xi)^2+\eta^2$, where $a=\frac{1-[(B-A)\alpha\beta+A][(B-A)\beta+A]r^2}{1-[(B-A)\beta+A]^2r^2}$ and denoting the expression on the right hand side of (4.4) by $S(\xi,\eta)$, we get

$$S(\xi,\eta)=2\ \frac{-A-(B-A)\alpha\beta}{(B-A)\beta(1-\alpha)}$$
$$+\frac{1}{(B-A)\beta(1-\alpha)}[(B-A)\beta+A](a+\xi)+[(B-A)\alpha\beta+A]\cdot$$
$$\cdot(a+\xi)R^{-2}-\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}(d^2-\xi^2-\eta^2)R^{-1}],\tag{4.5}$$

where $d=\frac{(B-A)\beta(1-\alpha)r}{1-[(B-A)\beta+A]^2r^2}$. Differentiating (4.5) partially w.r.

t. η, we get

$$\frac{\partial S}{\partial \eta}=\frac{1}{(B-A)\beta(1-\alpha)}\eta R^{-4}T(\xi,\eta) \tag{4.6}$$

where

$$T(\xi,\eta)=-2[A+(B-A)\alpha\beta](a+\xi)+\frac{1-[(B-A)\beta+A]^2r^2}{(1-r^2)}\cdot \cdot R(d^2-\xi^2-\eta^2)+2\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}R^3.$$

It is easy to check that $T(\xi,\eta)>0$ and so (4.6) gives that the minimum of $S(\xi,\eta)$ inside the disc $\xi^2+\eta^2\leqslant d^2$ is attained on the diameter $\eta=0$. On putting $\eta=0$ in (4.5), we obtain

$$U(R)=S(\xi,0)=2\frac{-A-(B-A)\alpha\beta}{(B-A)\beta(1-\alpha)}+ \frac{1}{(B-A)\beta(1-\alpha)}[([(B-A)\beta+A]+\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2})R + \frac{(1+[(B-A)\alpha\beta+A])(1-[(B-A)\alpha\beta+A]r^2)}{1-r^2}R^{-1} -2a\frac{1-[(B-A)\beta+A]^2r^2}{1-r^2}],$$

where $R=a+\xi$ and $a-d\leqslant R\leqslant a+d$. Thus the absolute minimum of $U(R)$ in $(0,\infty)$ is attained at

$$R_0=\sqrt{\frac{(1+[(B-A)\alpha\beta+A])(1-[(B-A)\alpha\beta+A]r^2)}{(1+[(B-A)\beta+A])(1-[(B-A)\beta+A]r^2)}} \tag{4.7}$$

and the value of this minimum is

$$U(R_0)=\frac{2}{(B-A)\beta(1-\alpha)(1-r^2)} \{\sqrt{(1+[(B-A)\beta+A])(1+[(B-A)\alpha\beta+A])(1-[(B-A)\beta+A]r^2(1-[(B-A)\alpha\beta+A]r^2)}-(1-[(B-A)\alpha\beta+A]\cdot \cdot[(B-A)\beta+A]r^2)+[-A-(B-A)\alpha\beta](1-r^2)\}. \tag{4.8}$$

It is easily seen that $R_0<a+d$, but R_0 is not always greater than $a-d$. In such a case when $R_0\notin[a-d,\ a+d]$, the minimum of $U(R)$ on the segment $[a-d,\ a+d]$ is attained at $R_1=a-d$ since $U(R)$ increases with R on this segment. The value of this minimum equals

$$U(R_1)=U(a-d)=\frac{1-2[-A-(B-A)\alpha\beta]r+[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2}{(1+[(B-A)\beta+A]r)(1+[(B-A)\alpha\beta+A]r)}. \tag{4.9}$$

It follows from what has been said that the bound r_0 of convexity for the class $R_1(\alpha, \beta, A, B)$ is determined either from the equation $U(R_0)=0$ or from the equation $U(R_1)=0$. Also, $U(R_0)=U(R_1)$ for such values of $\alpha(0\leqslant\alpha<1)$, $\beta(0<\beta\leqslant1)$ and $-1\leqslant A<B\leqslant1$, $0<B\leqslant1$ for which $R_0=R_1$.

From (4.4), (4.8) and (4.9), we have

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$

$$\geqslant\begin{cases}\dfrac{2}{(B-A)\beta(1-\alpha)(1-r^2)}\Big\{\sqrt{(1+[(B-A)\beta+A])(1+[(B-A)\alpha\beta+A])(1-[(B-A)\beta+A]r^2)(1-[(B-A)\alpha\beta+A]r^2)}-(1-[(B-A)\alpha\beta+A][(B-A)\beta+A]r^2)+[-A-(B-A)\alpha\beta](1-r^2)\Big\} & \text{if } R_0\geqslant R_1 \\ \dfrac{1-2[-A-(B-A)\alpha\beta]r+[(B-A)\beta+A][(B-A)\alpha\beta+A]r^2}{(1+[(B-A)\beta+A]r)(1+[(B-A)\alpha\beta+A]r)} & \text{if } R_0\leqslant R_1.\end{cases} \tag{4.10}$$

Now the theorem follows easily from (4.10). The functions given by

$$f'(z)=\frac{1-[(B-A)\alpha\beta+A]z}{1-[B-A)\beta+A]z},$$

$$f'(z)=\frac{1-[1+(B-A)\alpha\beta+A]bz+[(B-A)\alpha\beta+A]z^2}{1-[1+(B-A)\beta+A]bz+[(B-A)\beta+A]z^2},$$

where b is determined by the relation

$$\frac{1-[1+(B-A)\alpha\beta+A]br+[(B-A)\alpha\beta+A]r^2}{1-[1+(B-A)\beta+A]br+[(B-A)\beta+A]r^2}=R_0$$

show that the results obtained in the theorem are sharp.

REMARK. Taking different values of the parameters α, β, A, B $(0\leqslant\alpha<1$, $0<\beta\leqslant1$, $-1\leqslant A<B\leqslant1$, $0<B\leqslant1)$ in Theorem 3, we get the radii of convexity for functions in different classes obtained earlier by Caplinger and Causey [2], Ezrohi [4], Goel [5, 6], Kaczmarski [8], MacGregor [9, 10], Martynov [11], Padmanabhan [15], Shaffer [16],

Aouf and Owa [1], Juneja and Mogra [7] and Mogra [12].

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