J. Korean Math. Soc. 27(1990), No. 2, pp. 137-144

EVALUATION FORMULAS FOR CONDITIONAL ABSTRACT WIENER INTEGRALS II

DONG MYUNG CHUNG* AND SOON JA KANG**

1. Introduction and preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let B denote the completion of H with respect to a measurable norm $\|\cdot\|$ on H. As H is identified as a dense subspace of B, we identified the topological dual B^* of B as a dense subspace of $H^*\approx H$ in the sense that, for all y in B^* and x in H $\langle y,x\rangle=(y,x)$, where $(\cdot,\cdot)$ is the natural dual pairing between B and B^*. Thus we have a triple $B^*\subset H^*\approx H\subset B$. Gross [4] proved that B carries a mean zero Gaussian measure, called as the abstract Wiener measure, which is characterized by the probability measure on the Borel σ-algebra $\mathscr{B}(B)$ of B such that

$$\int_B e^{i(y,x)}d\nu(x)=\exp\left\{-\frac{1}{2}|y|^2\right\} \text{ for every } y\in B^*.$$

The triple (H,B,ν) is called an *abstract Wiener space.* For more details, see [4, 6]. Let $\boldsymbol{R}^n$ and $\boldsymbol{C}$ denote an n-dimensional Euclidean space and the complex numbers, respectively.

Let $(C[0,T],\ \mathscr{B}(C[0,T]),\ m_w)$ denote Wiener space, i.e. $C[0,T]$ denotes the Banach space $\{x(\cdot) : x$ is a real valued continuous function with $x(0)=0\}$ with the supremum norm and m_w denotes the Wiener measure on the Borel σ-algebra $\mathscr{B}(C[0,T])$ of $C[0,T]$ (see [10]). Let $C'[0,T]=\{x\in C[0,T] : x(s)=\int_0^s f(u)du,\quad f\in L^2[0,T]\}$. Then it is a real separable infinite dimensional Hilbert space with inner product $\langle x_1,x_2\rangle=\int_0^T Dx_1(\tau)\cdot Dx_2(\tau)d\tau$, where $Dx=\frac{dx}{d\tau}$. As is known,

Received July 5, 1989.
*Research partially supported by KOSEF Grant, 1989.
**Research partially supported by the BSRIP, MOE, 1989.

$(C'[0, T],\ C[0, T],\ m_w)$ is one of the most important examples of abstract Wiener space [see [6]).

Let $\{e_j\ ;\ j \geq 1\}$ be a complete orthonormal set in H such that e_j's are in B^*. For each $h \in H$ and $x \in B$, let

$$(h, x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j \rangle\ (e_j, x), & \text{if the limit exists} \\ 0 & , \text{otherwise.} \end{cases}$$

Then it is shown that for each $h(\neq 0)$ in H, $(h, \cdot)^{\sim}$ is a Gaussian random variable on B with mean zero, variance $|h|^2$, and that $(h, x)^{\sim}$ is essentially independent of the choice of the complete orthonormal set used in its definition, and further that $(h, \lambda x)^{\sim} = \lambda (h, x)^{\sim}$ for all $\lambda \in \boldsymbol{R}^1$. It is known [2, 4, 9] that if $\{h_1, h_2, \cdots, h_n\}$ is an orthogonal set in H, then the random variables $(h_i, x)^{\sim}$'s are independent, and that if $B=C[0, T]$, $H=C'[0, T]$, then

$$(h, x)^{\sim} = \int_0^T Dh(s)\tilde{d}x(s)$$

where $\int_0^T Dh(s)\tilde{d}x(s)$ is the Paley-Wiener-Zygmund integral of Dh.

Let A be a self-adjoint, trace class operator with eigenvalues $\{\alpha_k\}$ and corresponding eigenfunctions $\{e_k\}$. Let

$$(x, Ax)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j [(e_j, x)^{\sim}]^2, & \text{if the limit exists} \\ 0 & , \text{otherwise.} \end{cases}$$

For more details, see [4, 5, 8].

Let X be a $\boldsymbol{R}^n$-valued measurable function and Y a $\boldsymbol{C}$-valued integrable function on $(B, \mathcal{B}(B), \nu)$. Let $\mathcal{F}(X)$ denote the σ-algebra generated by X. Then by the definition of conditional expectation, the conditional expectation of Y given $\mathcal{F}(X)$, written $E[Y|X]$, is any real valued $\mathcal{F}(X)$-measurable function on B such that

$$\int_E Y\, d\nu = \int_E E[Y|X] d\nu \ \text{ for } E \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and P_X-integrable function ψ on $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n),\ P_X)$ such that $E[Y|X] = \psi \circ X$, where $\mathcal{B}(\boldsymbol{R}^n)$ denotes the Borel σ-algebra of $\boldsymbol{R}^n$ and P_X is the probability distribution of X defined by $P_X(A) = \nu(X^{-1}(A))$ for $A \in \mathcal{B}(\boldsymbol{R}^n)$. The function $\psi(\xi)$, $\xi \in \boldsymbol{R}^n$ is unique up to Borel null sets in $\boldsymbol{R}^n$. Following Yeh [10] the function $\psi(\xi)$, written $E[Y|X=\xi]$, will be called the *conditional abstract Wiener integral of* Y given X.

This paper is a continuation of the paper [1, 3]. In this paper, we first establish a general formula for evaluating conditional abstract Wiener integrals which has all the results given in [1] as corollaries. We next generalize translation theorem for conditional Wiener integrals to abstract Wiener spaces.

2. A evaluation formula for conditional abstract Wiener integrals

In this section we will give a general formula for evaluating conditional abstract Wiener integral which has all the results given in [1] as corollaries.

THEOREM 2.1. *Let* $\{g_1, g_2, \cdots, g_n\}$ *be an orthonormal set in* H. *Let* X *and* Z *be measurable functions on* $(B, \mathcal{B}(B))$ *defined, respectively, by*

(2.1) $$X(x)=((g_1, x)^\sim, \ (g_2, x)^\sim, \cdots, (g_n, x)^\sim)$$

and

(2.2) $$Z(x)=F((Sh, x)^\sim)$$

where $h\in H$, S *is a bounded linear operator on* H *and* F *is a Borel measurable function on* $\boldsymbol{R}^1$ *such that* $E[|Z|]<\infty$. *Then*

(2.3) $$E[Z|X=\vec{\xi}]=\frac{1}{\sqrt{2\pi|p|^2}}\int_{-\infty}^{\infty} F\left[\sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j+u\right]exp\left\{-\frac{u^2}{2|p|^2}\right\}du$$

where $\vec{\xi}=(\xi_1, \xi_2, \cdots, \xi_n)\in\boldsymbol{R}^n$ *and* $|p|^2=|Sh|^2-\sum_{j=1}^{n}\langle Sh, g_j\rangle^2$.

Proof. We first note that $E[Z|X]$ exists since $E[|Z|]<\infty$. Let $Sh=k$. Then k can be written as

$$k=\sum_{j=1}^{n}\langle k, g_j\rangle g_j+p, \quad p\in[g_1, g_2, \cdots, g_n]^\perp$$

where $[g_1, g_2, \cdots, g_n]^\perp$ stands for the orthogonal comlpement of the subspace of H spanned by $\{g_1, g_2, \cdots, g_n\}$. So we have

$$F((k, x)^\sim)=F\left[\sum_{j=1}^{n}\langle k, g_j\rangle(g_j, x)^\sim+(p, x)^\sim\right].$$

Since $(p, x)^\sim$ and $(g_j, x)^\sim$'s are independent,

$$E[Z|X=\vec{\xi}]=E\left[F\left[\sum_{j=1}^{n}\langle k, g_j\rangle\xi_j+(p, x)^\sim\right]\right]$$
$$=\int_{-\infty}^{\infty} F\left[\sum_{j=1}^{n}\langle k, g_j\rangle\xi_j+u\right]\frac{1}{\sqrt{2\pi|p|^2}}\exp\left\{-\frac{u^2}{2|p|^2}\right\}du.$$

But $|p|^2=|k-\sum_{j=1}^{n}\langle k, g_j\rangle g_j|^2=|k|^2-\sum_{j=1}^{n}\langle k, g_j\rangle^2$. Hence we establish the equation (2.3) as desired.

COROLLARY 2.2. [1] *Let X be as in Theorem 2.1. Then we have*

$$(2.4)\qquad E[(Sh, x)^\sim|X=\vec{\xi}]=\sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j,$$

where S is as in Theorem 2.1.

The function in the following corollary is not a form of function in (2, 2) ; however its conditional abstract Wiener integral can be evaluated by using Theorem 2.1.

COROLLARY 2.3. [1] *Let X be as in Theorem 2.1. Then we have*

$$(2.5)\qquad E[(x, Ax)^\sim|X=\vec{\xi}] = TrA+<\sum_{j=1}^{n}\xi_j g_j,\ A(\sum_{j=1}^{n}\xi_j g_j)>-\sum_{j=1}^{n}\langle g_j, Ag_j\rangle$$

where A is a self-adjoint, trace class operator on H and TrA stands for the trace of A.

Proof. Let $\{e_m\}$ be the orthonormal eigenvectors and $\{\alpha_m\}$ be the corresponding eigenvalues of A. Let $\langle g_j, e_m\rangle=a_{mj}$. Since $(x, Ax)^\sim=\sum_{m=1}^{\infty}\alpha_m((e_m, x)^\sim)^2$, a.e. $x\in B$, we have, by using $S=I$, $h=e_m$ in Theorem 2.1

$$\begin{aligned}E[(x, Ax)^\sim|X=\vec{\xi}] &= \sum_{m=1}^{\infty}\alpha_m\ E\left[((e_m, x)^\sim)^2|X=\vec{\xi}\right]\\ &= \sum_{m=1}^{\infty}\alpha_m\left[(\sum_{j=1}^{n}a_{mj}\xi_j)^2+(1-\sum_{j=1}^{n}a_{mj}^2)\right].\end{aligned}$$

Hence we estalish equation (2.5) as desired.

COROLLARY 2.4. [1] *Let X be as in Theorem 2.1. Then*

$$(2.5)\qquad E[exp\{\lambda(Sh, x)^\sim\}|X=\vec{\xi}] = exp\left\{\lambda\sum_{j=1}^{n}\langle Sh, g_j\rangle\xi_j+\frac{\lambda^2}{2}\left[|Sh|^2-\sum_{j=1}^{n}\langle Sh, g_j\rangle^2\right]\right\}$$

where $\lambda\in \mathbf{C}$ and S is as in Theorem 2.1.

REMARK 2.1. If we specialize our results in Corollaries 2.2~2.4 to classical abstract Wiener space $C[0, T]$, then we can evaluate various conditional Wiener integrals studied in [7] (see [1]).

3. Translation of conditional abstract Wiener integrals

The abstract Wiener space version of the Cameron-Martin translation theorem (see [6], p. 113) states that if $h\in H$ and if $T: B\rightarrow B$ is given by $T(x)=x+h$, then for any integrable function F on B and any Γ in $\mathcal{B}(B)$

$$(3.1)\qquad \int_{\Gamma}F(y)d\nu(y)=\int_{T^{-1}\Gamma}F(x+h)J(h,x)d\nu(x)$$

where

$$(3.2)\qquad J(h,x)=\exp\left\{-\frac{1}{2}|h|^2-(h,x)^{\sim}\right\}.$$

The following is abstract Wiener space version of translation theorem for conditional Wiener integrals studied in [7].

THEOREM 3.1. *Let X be as in Theorem 2.1 and let $h\in H$. If F is integrable, then*

$$(3.3)\qquad E[F(y)\,|\,X(y)=\vec{\xi}\,]$$
$$=E[F(x+h)J(h,x)\,|\,X(x+h)=\vec{\xi}\,]exp\left\{-\frac{1}{2}|G(h)|^2+\langle G(h),\vec{\xi}\rangle\right\}$$

where $G(h)=(\langle g_1,h\rangle,\cdots,\langle g_n,h\rangle)$ *and* $\vec{\xi}=(\xi_1,\cdots,\xi_n)\in\boldsymbol{R}^n$.

Proof. Since $\mathcal{F}(X)\subset\mathcal{B}(B)$, (3.1) shows that for any $A\in\mathcal{B}(\boldsymbol{R}^n)$,

$$\int_{X^{-1}(A)}F(y)d\nu(y)=\int_{T^{-1}(X^{-1}(A))}F(x+h)J(h,x)d\nu(x).$$

So by the definition of conditional abstract Wiener integral, we have

$$\int_{X^{-1}(A)}F(y)d\nu(y)=\int_{A}E[F(y)\,|\,X(y)=\vec{\xi}\,]d\nu\circ X^{-1}(\vec{\xi})$$

and

$$\int_{T^{-1}(X^{-1}(A))}F(x+h)J(h,x)d\nu(x)$$
$$=\int_{A}E[F(x+h)J(h,x)\,|\,X\circ T(x)=\vec{\xi}\,]d\nu\circ(X\circ T)^{-1}(\vec{\xi}).$$

Hence for any $A\in\mathcal{B}(\boldsymbol{R}^n)$,

$$\int_{A}E[F(y)\,|\,X(y)=\vec{\xi}\,]d\nu\circ X^{-1}(\vec{\xi})$$

$$=\int_A E[F(x+h)J(h,x)\,|\,X\circ T(x)=\vec{\xi}]d\nu\circ(X\circ T)^{-1}(\vec{\xi})$$

$$=\int_A E[F(x+h)J(h,x)\,|\,X\circ T(x)=\vec{\xi}]\cdot\frac{d\nu\circ(X\circ T)^{-1}}{d\nu\circ X^{-1}}(\vec{\xi})d\nu\circ X^{-1}(\vec{\xi})$$

from which we have

(3.4) $$E[F(y)\,|\,X(y)=\vec{\xi}]$$
$$=E[F(x+h)J(h,x)\,|\,X(x+h)=\vec{\xi}]\cdot\frac{d\nu\circ(X\circ T)^{-1}}{d\nu\circ X^{-1}}(\vec{\xi}).$$

But

$$\frac{d\nu\circ(X\circ T)^{-1}}{d\mu}(\vec{\xi})=\left[\frac{1}{\sqrt{2\pi}}\right]^n\exp\left\{-\frac{1}{2}|\vec{\xi}-G(h)|^2\right\}$$

and

$$\frac{d\nu\circ X^{-1}}{d\mu}(\vec{\xi})=\left[\frac{1}{\sqrt{2\pi}}\right]^n\exp\left\{-\frac{1}{2}|\vec{\xi}|^2\right\}$$

where μ denotes the Lebesgue measure on $\boldsymbol{R}^n$. So we have

$$\frac{d\nu\circ(X\circ T)^{-1}}{d\nu\circ X^{-1}}(\vec{\xi})=\exp\left\{-\frac{1}{2}(|G(h)|^2+\langle G(h),\vec{\xi}\rangle)\right\}$$

This together with (3.4) gives the equality (3.3).

The following example shows that Corollary 2.4 can also be obtained by using Theorem 3.1.

Example 3.1. Let X be as in Theorem 2.1 and let $Z(x)=\exp\{\lambda(Sh,x)^{\sim}\}$ where $\lambda\in\boldsymbol{R}^1$, $h\in H$ and S is a bounded linear operator on H. By choosing $F\equiv1$ and $\vec{\xi}=\vec{\eta}+G(h)$ in Theorem 3.1,

$$1=E[F(x)\,|\,X(x)=\vec{\eta}+G(h)]$$
$$=E[J(h,x)\,|\,X(x)=\vec{\eta}]\cdot\exp\left\{-\frac{1}{2}|G(h)|^2+\langle G(h),\ \vec{\eta}+G(h)\rangle\right\}.$$

Hence we have, by using (3.2)

(3.5) $$E[\exp\{-(h,x)^{\sim}\}\,|\,X(x)=\vec{\eta}]$$
$$=\exp\left\{\frac{1}{2}|h|^2\right\}\cdot\exp\left\{-\frac{1}{2}|G(h)|^2-\langle G(h),\ \vec{\eta}\rangle\right\}$$

By replacing h by $-\lambda(Sh)$, we obtain

$$E[Z\,|\,X=\vec{\xi}]=\exp\left\{\frac{\lambda^2}{2}|Sh|^2-\frac{\lambda^2}{2}|G(Sh)|^2+\lambda\langle G(Sh),\ \vec{\xi}\rangle\right\}.$$

It can be shown that using analytic continuation, this result coincides with Corollary 2.4.

In the next example, using Theorem 3.1 we obtain the corresponding result of Park and Skoug's for Wiener space [7].

EXAMPLE 3.2. Let B be the Wiener space $C[0, T]$ with Wiener measure m_w. Let us fix a partition $\{0=t_0<t_1<\cdots<t_n=T\}$ of $[0, T]$ and let $g_j\in C'[0, T]$ be defined by

$$g_j(\tau)=\frac{1}{\sqrt{t_j-t_{j-1}}}\int_0^\tau 1_{[t_{j-1}, t_j]}(u)\,du.$$

Then $\{g_1, \cdots, g_n\}$ is an orthonormal set in $C'[0, T]$. Let $h\in C'[0, T]$ be defined by $h(\tau)=\int_0^\tau f(u)\,du$ for some $f\in L^2[0, T]$. Then for any Wiener integrable function F on $C[0, T]$, we have

$$\begin{aligned}
&E[F(y)\mid y(t_1)=\xi_1, \cdots, y(t_n)=\xi_n]\\
&=E\left[F(y)\mid (g_1, y)^\sim=\frac{\xi_1-\xi_0}{\sqrt{t_1-t_0}}, \cdots, (g_n, y)^\sim=\frac{\xi_n-\xi_{n-1}}{\sqrt{t_n-t_{n-1}}}\right], \quad \xi_0=0\\
&=E\left[F(x+h)J(h, x)\mid (g_j, x)^\sim+\langle g_j, h\rangle=\frac{\xi_j-\xi_{j-1}}{\sqrt{t_j-t_{j-1}}}, \ j=1, \cdots, n\right]\\
&=E\left[F(x+h)J(h,x)\mid (g_j,x)^\sim=\frac{\xi_j-\xi_{j-1}}{\sqrt{t_j-t_{j-1}}}-\frac{h(t_j)-h(t_{j-1})}{\sqrt{t_j-t_{j-1}}}, j=1,\cdots,n\right]\\
&\qquad\cdot\exp\left\{-\frac{1}{2}\sum_{j=1}^n\langle g_j, h\rangle^2+\sum_{j=1}^n\langle g_j, h\rangle\frac{\xi_j-\xi_{j-1}}{\sqrt{t_j-t_{j-1}}}\right\}\\
&=E[F(x+h)J(h, x)\mid x(t_j)=\xi_j-h(t_j), \ j=1, 2, \cdots, n]\\
&\qquad\cdot\prod_{j=1}^n\exp\left\{-\frac{(h(t_j)-h(t_{j-1}))^2}{2(t_j-t_{j-1})}+\frac{(h(t_j)-h(t_{j-1}))(\xi_j-\xi_{j-1})}{t_j-t_{j-1}}\right\}
\end{aligned}$$

which agrees with the result in Theorem 4 [7].

References

1. Dong M. Chung and Soon J. Kang, *Evaluation of some conditional abstract Wiener integrals*, Bull. Korean Soc., **26**, (1989), No. 2, 151-158.
2. Dong M. Chung, *Scale-invariant measurability in abstract Wiener spaces*, Pacific J. Math., **130**, (1987), 27-40.
3. Dong M. Chung and Soon J. Kang, *Evaluation formulas for conditional abstract Wiener integrals*, Stochastic Analysis and Applications, **7**, (1989),

125–144.
4. L. Gross, *Abstract Wiener space*, Proc. 5th, Berkeley Sym. Math. Stat. Prob. **2**, (1965), 31–42.
5. G. Kallianpur, D. Kannan and R. L. Karnadikar, *Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces and a Cameron-Martin formula*, Ann. Institut H. Poincaré, **21**, (1985), 323–361.
6. H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics, No. 463, Springer, Berlin, (1975).
7. C. Park and D. Skoug, *A simple formula for conditional Wiener integrals with applications*, Pacific J. of Math., **135**, (1988). No. 2, 381–394.
8. M. A. Piech, *A fundamental solution of the parabolic equation on Hilbert space*, J. Func. Anal. **3**, (1969), 85–114.
9. I. E. Segal, *Tensor algebra over Hilbert spaces*, Trans. Amer. Math. Soc. **81**, (1956), 106–134.
10. J. Yeh, *Inversion of conditional Wiener integrals*, Pacific J. Math. **59**, (1975), 623–638.
11. J. Yeh, *Transformation of conditional Wiener integrals under translation and the Cameron-Martin translation theorem*, Tohoku Math. J., **30**, (1978), 505–515.

Sogang University
Seoul 121–742, Korea
and
Cheonnam National University
Kwangju 500–757, Korea