J. Korean Math. Soc. 27(1990), No. 1, pp. 111–117

THE ROLE OF THE BASE POINT OF THE FUNDAMENTAL GROUP OF A TRANSFORMATION GROUP

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In [2], Rhodes introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that if G is abelian, then $\sigma(X, x_0, G)$ and $\sigma(X, x_1, G)$ are isomorphic for every point x_1 of GX_0, where X_0 is the path-connected component of x_0. He also showed that in order to satisfy the above result it is necessary to impose the condition that G be abelian.

In this paper, we show that the Rhodes' result contains an unnecessary condition that G be abelian.

Let (X, G) be a transformation group. Given any element g of G, a *path f of order g* with base point x_0 is a continuous map $f: I \to X$ such that $f(0)=x_0$ and $f(1)=gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1+g_1f_2$ of order g_1g_2 defined by the equations

$$(f_1+g_1f_2)(s)=\begin{cases} f_1(2s), & 0\leqslant s\leqslant 1/2 \\ g_1f_2(2s-1), & 1/2\leqslant s\leqslant 1. \end{cases}$$

Two paths f and f' of the same order g are said to be ***homotopic*** if there is a continuous map $F: I^2 \to X$ such that

$$\begin{aligned} F(s,0)&=f(s) && 0\leqq s\leqq 1,\\ F(s,1)&=f'(s) && 0\leqq s\leqq 1,\\ F(0,t)&=x_0 && 0\leqq t\leqq 1,\\ F(1,t)&=gx_0 && 0\leqq t\leqq 1. \end{aligned}$$

The homotopy class of a path f of order g is denoted by $[f : g]$. Two homotopy classes of paths of different order g_1 and g_2 are distinct, even if $g_1x_0=g_2x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group,

Received September 1, 1989.
This research is supported by the Ministry of Education, 1988~90.

where $*$ is defined by $[f_1 ; g_1]*[f_2 ; g_2]=[f_1+g_1f_2 ; g_1g_2]$ [2]. This group is denoted by $\sigma(X, x_0, G)$ and is called the *fundamental group* of a transformation group (X, G) with base point x_0.

Rhodes [2] showed the following two theorems;

THEOREM 1. *If x_1 belongs to the path-connected component X_0 of x_0, and λ is a path from x_0 to x_1, then λ induces an isomorphism from $\sigma(X, x_0, G)$ to $\sigma(X, x_1, G)$.*

THEOREM 2. *If (X, G) is a transformation group whose group G is abelian, and if x_1 belongs to GX_0, where X_0 is the path-connected component of x_0, then $\sigma(X, x_0, G)$ and $\sigma(X, x_1, G)$ are isomorphic.*

In the following theorem, we show that Theorem 2 obtained by Rhodes contains an unnecessary condition that G is abelian.

THEOREM 3. *If (X, G) is a transformation group and x_1 belongs to GX_0, where X_0 is the path-connected component of x_0, then $\sigma(X, x_0, G)$ and $\sigma(X, x_1, G)$ are isomorphic.*

Proof. In view of Theorem 1, it is sufficient to prove that if $x_1=gx_0$ then $\sigma(X, x_0, G)$ and $\sigma(X, x_1, G)$ are isomorphic. For an element g of G such that $gx_0=x_1$, we define the map

$$g_* : \sigma(X, x_0, G)\to\sigma(X, x_1, G)$$

by $g_*([f ; h])=[gf ; ghg^{-1}]$.

Clearly g_* is well defined. Since

$$\begin{aligned} g_*([f_1 ; g_1]*[f_2 ; g_2]) &= g_*([f_1+g_1f_2 ; g_1g_2]) \\ &= [gf_1+gg_1f_2 ; gg_1g_2g^{-1}] \\ &= [gf_1 ; gg_1g^{-1}]*[gf_2 ; gg_2g^{-1}] \\ &= g_*([f_1 ; g_1])*g_*([f_2 ; g_2]), \end{aligned}$$

g_* is a homomorphism. Assume that $g_*([f_1 ; g_1])=g_*([f_2 ; g_2])$. Then $gg_1g^{-1}=gg_2g^{-1}$ and gf_1 is homotopic to gf_2. So $g_1=g_2$ and f_1 is homotopic to f_2. Thus $[f_1 ; g_1]=[f_2 ; g_2]$. For any element $[f ; h]$ in $\sigma(X, x_1, G)$, there exists an element $[g^{-1}f ; g^{-1}hg]$ in $\sigma(X, x_0, G)$ such that $g_*([g^{-1}f ; g^{-1}hg])=[f ; h]$. Therefore g_* is an isomorphism.

Let $\sigma'(X, x_0, G)$ be the subgroup of $\sigma(X, x_0, G)$ generated by elements of the form $[1+g1\rho ; g]$, where 1 is a path from x_0 to a point x which is a fixed point of g and $e(t)=1-t$. Then $\sigma'(X, x_0, G)$ is a normal subgroup of $\sigma(X, x_0, G)$. The quotient group $\bar{\sigma}(X, x_0, G)=\sigma(X, x_0, G)/\sigma'(X, x_0, G)$ is called the *reduced fundamental group* [2] of (X, G). Rhodes showed that if λ is a path from x_0 to a point x_1 then the isomorphism $\lambda_* : \sigma(X, x_0, G) \to \sigma(X, x_1, G)$ induces an isomorphism $\lambda_* : \bar{\sigma}(X, x_0, G) \to \bar{\sigma}(X, x_1, G)$. He also stated that if G is abelian, $\bar{\sigma}(X, x_0, G)$ is isomorphic to $\bar{\sigma}(X, x_1, G)$ for $x_1 \in GX_0$.

The following corollary show that this also contains an unnecessary condition that G is abelian.

Corollary 4. *If (X, G) is a transformation group and x_1 belongs to GX_0, where X_0 is the path-connected component of x_0, then $\bar{\sigma}(X, x_0, G)$ and $\bar{\sigma}(X, x_1, G)$ are isomorphic.*

Proof. Let X_0 be the path-connected component of x_0. Since $\bar{\sigma}(X, x_0, G)$ is isomorphic to $\bar{\sigma}(X, x_1, G)$ for $x_1 \in X_0$, it is sufficient to prove that if $x_1 = gx_0$ then $\bar{\sigma}(X, x_0, G)$ and $\bar{\sigma}(X, x_1, G)$ are isomorphic. Let $[1+h1\rho\,;h]$ be a generator of the normal subgroup $\sigma'(X, x_0, G)$ and 1 be a path from x_0 to x which is a fixed point of h. Then gx is a fixed point of ghg^{-1} and $[g1+gh1\rho\,;ghg^{-1}]=[g1+ghg^{-1}g1\rho\,;ghg^{-1}]$. Thus $g_*([1+h1\rho\,;h])=[g1+gh1\rho\,;ghg^{-1}]$ is a generator of the normal subgroup $\sigma'(X, x_1, G)$. By Theorem 3, it was proved.

In [1], a homotopy $H: X \times I \to X$ is called a *cyclic homotopy* if $H(x, 0)=H(x, 1)=x$. The first author and Yoon generalized this concept on transformation groups. A continuous map $H: X \times I \to X$ is called a *homotopy of order* g if $H(x, 0)=x$, $H(x, 1)=gx$, where g is an element of G. If H is a homotopy of order g, then the path $f: I \to X$ such that $f(t)=H(x_0, t)$ will be called the *trace* of H. In [3], the subgroup $E(X, x_0, G)$ was defined by the set of all elements $[f\,;g] \in \sigma(X, x_0, G)$ such that f is the trace of a homotopy of order g, where $g \in G$.

The first author and Yoon showed the following;

Theorem 5. *If X is a CW-complex and λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : E(X, x_0, G) \to E(X, x_1, G)$.*

If we use this result, we obtain the following theorem.

Theorem 6. *If X is a CW-complex and x_1 belongs to GX_0, where*

X_0 is the path-connected component of x_0, then $E(X, x_0, G)$ is isomorphic to $E(X, x_1, G)$.

Proof. In view of Theorem 5, it is sufficient to prove that if $x_1 = gx_0$, then $E(X, x_0, G)$ is isomorpshic to $E(X, x_1, G)$. Since g_* is an isomorphism from $\sigma(X, x_0, G)$ to $\sigma(X, x_1, G)$ by Theorem 3, g_* is an monomorphism. Thus it is sufficient to show that $g_*(E(X, x_0, G)) \subset E(X, x_1, G)$. Let $[f\,;h]$ be an element of $E(X, x_0, G)$. Then there exists a homotopy $H : X \times I \to X$ of order h with trace f. If we construct a homotopy $K : X \times I \to X$ by $K = gH(g^{-1} \times 1)$, then $K(x, 0) = gH(g^{-1} \times 1)(x, 0) = gH(g^{-1}x, 0) = x$ and $K(x, 1) = gH(g^{-1} \times 1)(x, 1) = gH(g^{-1}x, 1) = ghg^{1}x$. Thus K is a homotopy of order ghg^{-1} with trace gf, because $K(x_1, t) = gH(g^{-1}x_1, t) = gH(x_0, t) = gf(t)$. Thus $g_*([f\,;h]) = [gf\,; ghg^{-1}]$ belongs to $E(X, x_1, G)$.

In [2], a transformation group (X, G) is said to admit a *family K of preferred paths* at x_0 if it is possible to associate with every element g of G a paths at x_0 such that the path k_e associated with the identity element e of G is x_0' which is the constant map such that $x_0'(t) = x_0$ for each $t \in I$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$. In [4], a family K of preferred paths at x_0 is called a *family of preferred traces* at x_0 if for every preferred path k_g in K, k_g is the trace of a homotopy of order g. A family K of preferred paths at x_0 is called a *family of preferred strong paths* at x_0 if for each loop f at x_0 and each k_g in K, f is homotopic to $k_g\rho + gf + k_g$.

The first author showed that if λ is a path from x_0 to x_1, then a family of preferred strong paths at x_0 gives rise to a family of preferred strong paths at x_1 and a family of preferred traces at x_0 induces a family of preferred traces at x_1 (Theorem 7 in [4]).

We improve the above result as the following theorem.

THEOREM 7. *If (X, G) is a transformation group and x_1 belongs to GX_0, where X_0 is the path-connected component of x_0, then a family of preferred strong paths at x_0 gives rise to a family of preferred strong paths at x_1 and a family of preferred traces at x_0 induces a family of preferred traces at x_1.*

Proof. In view of Theorem 7 in [4], it is sufficient to prove that if $x_1=hx_0$, then a family of preferred strong paths (traces) at x_0 gives rise to a family of strong paths (traces) at x_1. Let $K^0=\{k_g^0 : g\in G\}$ be a family of preferred strong paths at x_0. For each g in G, g can be expressed by $hg'h^{-1}$ since $h_* : G\to G$ given by $h_*(g)=hgh^{-1}$ is an automorphism. Let $k^1_{(hgh^{-1})}=hk_g^0$ be a path from $hgh^{-1}x_1$ to x_1, where $k_g^0\in K^0$. Then $k^1=\{k^1_{(hgh^{-1})} : hgh^{-1}\in G\}$ is a family of preferred paths at x_1, because for any two element hg_1h^{-1}, hg_2h^{-1} in G we have that

$$\begin{aligned}(hg_1h^{-1})k^1_{(hg_2h^{-1})}+k^1_{(hg_1h^{-1})}\\ =hg_1h^{-1}(hk^0_{g_2})+hk^0_{g_1}\\ =h(g_1k^0_{g_2}+k^0_{g_1})\end{aligned}$$

is homotopic to $hk^0_{(g_1g_2)}=k^1_{(hg_1g_2h^{-1})}=k^1_{(hg_1h^{-1})(hg_2h^{-1})}$.

Let f be any loop at x_1 and $k^1_{hgh^{-1}}$ be any element of K^1. Then we have that

$$\begin{aligned}k^1_{(hgh^{-1})}\rho+hgh^{-1}f+k^1_{(hgh^{-1})}=hk^0_g\rho+hgh^{-1}f+hk^0_g\\ =h(k^0_g\rho+gh^{-1}f+k^0_g)\end{aligned}$$

is homotopic to $h(h^{-1}f)=f$ since K^0 is a family of strong preferred paths at x_0 and $h^{-1}f$ is a loop at x_0. Thus f is homotopic to $k^1_{(hgh^{-1})}\rho+hgh^{-1}f+k^1_{(hgh^{-1})}$. Therefore K^1 is a family of preferred strong paths at x_1.

Next, let $K^0=\{k_g^0 : g\in G\}$ be a family of preferred traces at x_0 and $k^1_{(hgh^{-1})}=hk_g^0$. By Theorem 6, the induced isomorphism h_* carries $E(X,x_0,G)$ isomorphically onto $E(X,x_1,G)$. Hence

$$\begin{aligned}h_*([k_g^0\rho\ ;\ g])=[hk_g^0\rho\ ;\ hgh^{-1}]\\ =[k^1_{(hgh^{-1})}\rho\ ;\ hgh^{-1}]\end{aligned}$$

belongs to $E(X,x_1,G)$. Thus $K^1=\{k^1_{(hgh^{-1})} : hgh^{-1}\in G\}$ is a family of preferred traces at x_1.

The first author obtained the following theorems in [4].

THEOREM 8. *A transformation group (X,G) admits a family K of preferred strong paths at x_0 if and only if there exists an isomorphism $\emptyset_K : \sigma(X,x_0,G)\to\pi_1(X,x_0)\times G$ such that the diagram commutes:*

$$\begin{tikzcd}[column sep=small] & & \sigma(X,x_0,G) \arrow[d, "\emptyset_K"] \arrow[dr, "j_G"] & & \\ 0 \arrow[r] & \pi_1(X,x_0) \arrow[ur, "i"] \arrow[r, "i_1"'] & \pi_1(X,x_0)\times G \arrow[r, "p_2"'] & G \arrow[r] & 0\end{tikzcd}$$

THEOREM 9. *A transformation group (X, G) admits a family K of preferred traces at x_0 if and only if there is an isomorphism $\emptyset_K$: $E(X, x_0, G) \to G(X, x_0) \times G$ such that the diagram commutes:*

$$
\begin{tikzcd}[row sep=small]
 & E(X, x_0, G) \arrow[dd, "\emptyset_K"] \arrow[dr] & \\
0 \to G(X, x_0) \arrow[ur] \arrow[dr] & & G \to 0. \\
 & G(X, x_0) \times G \arrow[ur] &
\end{tikzcd}
$$

If we use the above two theorems and Theorem 7, we have the following corollary:

COROLLARY 10. *If a transformation group (X, G) admits a family of preferred strong paths (traces) at x_0, then the representation of $\sigma(X, x_0, G)$ $(E(X, x_0, G))$ by $\pi_1(X, x_0)$ $(G(X, x_0))$ and G is natural with respect to change of base point in the sense that for any $x \in GX_0$, the following diagram is commutative:*

$$
\begin{array}{ccc}
\sigma(X, x_0, G) & \to & \sigma(X, x, G) \\
\downarrow & & \downarrow \\
\pi_1(X, x_0) \times G & \to & \pi_1(X, x) \times G
\end{array}
\quad
\left(
\begin{array}{ccc}
E(X, x_0, G) & \to & E(X, x, G) \\
\downarrow & & \downarrow \\
G(X, x_0) \times G & \to & G(X, x) \times G
\end{array}
\right)
$$

Proof. Let $x = hx_1$ for some $x_1 \in X_0$ and λ be a path from x_0 to x_1. The first author showed that the following diagram commutes:

$$
\begin{array}{ccc}
\sigma(X, x_0, G) & \xrightarrow{\lambda_*} & \sigma(X, x_1, G) \\
\Big\downarrow \emptyset_K & & \Big\downarrow \emptyset_H \\
\pi_1(X, x_0) \times G & \xrightarrow{\lambda_*} & \pi_1(X, x_1) \times G
\end{array}
\quad
\left(
\begin{array}{ccc}
E(X, x_0, G) & \xrightarrow{\lambda_*} & E(X, x_1, G) \\
\Big\downarrow \emptyset_K & & \Big\downarrow \emptyset_H \\
G(X, x_0) \times G & \xrightarrow{\lambda_*} & G(X, x_1) \times G
\end{array}
\right),
$$

where K is a family of preferred strong paths (traces) at x_0 and H is the family of preferred strong paths (traces) at x_1 induced by K and λ (see Theorem 7 in [4]).

By Theorem 3, h_* is an isomorphism from $\sigma(X, x_1, G)$ to $\sigma(X, x, G)$ given by $h_*([f\,;\,g]) = [hf\,;\,hgh^{-1}]$. Let H be the family of preferred strong paths (traces) at x induced by H and h (see Theoerm 7). By Theorem 8 and 9, we have isomorphisms $\emptyset_H : E(X, x, G) \to G(X, x) \times G$. If we define $\bar{h}_*$ from $\pi_1(X, x_1) \times G$ to $\pi_1(X, x) \times G$ $(G(X, x_1) \times G$ to $G(X, x) \times G)$ by $\bar{h}_*([f], g) = ([hf], hgh^{-1})$, then we have the following commutative diagram:

$$\begin{array}{ccc} \sigma(X, x_1, G) & \xrightarrow{h_*} & \sigma(X, x, G) \\ \downarrow \phi_H & \bar{h}_* & \downarrow \phi_H \\ \pi_1(X, x_1)\times G & \longrightarrow & \pi_1(X, x)\times G \end{array} \qquad \left(\begin{array}{ccc} E(X, x_1, G) & \xrightarrow{h_*} & E(X, x, G) \\ \downarrow \phi_H & \bar{h}_* & \downarrow \phi_H \\ G(X, x_1)\times G & \longrightarrow & G(X, x)\times G \end{array}\right)$$

If we use the above two diagrams, we obtain Corollary 10.

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