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# A CHARACTERIZATION OF REAL HYPERSURFACES OF TYPE $C, D$ AND $E$ OF A COMPLEX PROJECTIVE SPACE

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## 0. Introduction

A complex $n(\geqq 2)$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$ is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a comlpex projective space $P_n\boldsymbol{C}$, a complex Euclidean space $\boldsymbol{C}_n$ or a complex hyperbolic space $H_n\boldsymbol{C}$, according as $c>0$, $c=0$ or $c<0$. The induced almost contact metric structure of a real hypersurface of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$.

Now, there exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space $P_n\boldsymbol{C}$ by Takagi [14], who showed that these hypersurfaces of $P_n\boldsymbol{C}$ could be divided into six types which are said to be type $A_1, A_2, B, C, D$ and $E$, and in [5] Kimura proved that they were realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or rank 2. Namely, he proved the following

THEOREM A. *Let $M$ be a homogeneous real hypersurface of $P_n\boldsymbol{C}$, on which the structure vector $\xi$ is principal. Then $M$ is locally congruent to one of the following:*

($A_1$) *a geodesic hypersphere (that is, a tube over a hyperplane $P_{n-1}\boldsymbol{C}$),*
($A_2$) *a tube over a totally geodesic $P_k\boldsymbol{C}$ $(1\leqq k\leqq n-2)$,*
($B$) *a tube over a complex quadric $Q_{n-1}$,*
($C$) *a tube over $P_1\boldsymbol{C}\times P_{(n-1)/2}\boldsymbol{C}$ and $n(\geqq 5)$ is odd,*
($D$) *a tube over a complex Grassmann $G_{2,5}\boldsymbol{C}$ and $n=9$,*

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(*E*) *a tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n=15$.

In particular, real hypersurfaces of type $A_1, A_2$ or $B$ of $P_n\boldsymbol{C}$ have been studied by many authors (for example, [2], [3], [6], [8], [9], [12], [13] and so on).

On the other hand, Berndt [1] showed recently that all real hypersurfaces with constant principal curvatures a complex hyperbolic space $H_n\boldsymbol{C}$ are realized as the tubes over constant radius over certain submanifolds. Namely, he proved the following.

THEOREM B. *Let M be a real hypersurface with constant principal curvatures of* $H_n\boldsymbol{C}$, *on which the structure vector* $\xi$ *is principal. Then M is locally congruent to one of the following*:

($A_0$) *a self-tube, that is, a horosphere,*

($A_1$) *a geodesic hypersphere or a tube over a hyperplane* $H_{n-1}\boldsymbol{C}$,

($A_2$) *a tube over a totally geodesic* $H_n\boldsymbol{C}$ $(1\leqq k\leqq n-2)$,

(*B*) *a tube over a totally real hyperbolic space* $H_n\boldsymbol{R}$.

Real hypersurfaces of type $A_0, A_1, A_2$ and $B$ of $H_n\boldsymbol{C}$ have also been investigated by some authors (for example, [3], [4], [10], [11], [13] and so on).

In particular, the shape operator $A$ of the real hypersurface $M$ of $M_n(c)$, $c\neq 0$, is said to be *η-parallel*, if it satisfies

$$g(\nabla_X A(Y), Z)=0 \text{ for any } X, Y \text{ and } Z \text{ in } \xi^{\perp},$$

where $\xi^{\perp}$ denotes the orthogonal complement of the tangent bundle $TM$ with respect to $\xi$. As the characterization of homogeneous real hypersurfaces of type $A_1, A_2$ or $B$ in $P_n\boldsymbol{C}$ and real hypersurfaces of $H_n\boldsymbol{C}$, Kimura and Maeda [8] and Suh [13] proved recently the following

THEOREM C. *Let M be a real hypersurface of* $M_n(c)$, $c\neq 0$. *Then the shape operator is η-parallel and the structure vector* $\xi$ *is principal if and only if M is locally congruent to one of homogeneous hypersurfaces of type* $A_1, A_2$ *or B of* $P_n\boldsymbol{C}$ *or real hypersurfaces of type* $A_0, A_1, A_2$ *or B of* $H_n\boldsymbol{C}$.

The purpose of this paper is to give a characterization of real hypersurfaces of type $C, D$ or $E$ of $P_nC$. In §1 the theory of real hypersurfaces of a complex space form is recalled and in §2 the property of the operator $P_f$ defined by $A^2-fA$ is analyzed, where $f$ is a smooth function. As an application of properties obtained in §2 a generalization of Theorem $C$ is in §3 proved. In §4 another characterization of real homogeneous hypersurfaces of type $A_1, A_2$ or $B$ in $P_nC$ or real hypersurfaces of type $A_0, A_1, A_2$ or $B$ of $H_nC$ is given. In the last section, we treat with the characterization of real hypersurfaces of type $C, D$ or $E$.

## 1. Preliminaries

We begin with recalling basic properties of real hypersurfaces of a complex space form. Let $M$ be a real hypersurface of an $n(\geqq 2)$ dimensional complex space form $M_n(c)$ of constant holomorphic curvature $c(\neq 0)$ and let $C$ be a unit normal field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the transformations of $X$ and $C$ under $J$ can be represented as

$$JX=\phi X+\eta(X)\xi, \quad JC=-\xi,$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X)=\eta(X)$, where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies then

$$(1.1) \qquad \phi^2=-I+\eta\otimes\xi, \quad \phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1,$$

where $I$ denotes the identity transformation. Accordingly, the set is an almost contact metric structure. Furthermore the covariant derivatives of the structure tensors are given by

$$(1.2) \qquad \nabla_X\phi(Y)=\eta(Y)AX-g(AX, Y)\xi, \quad \nabla_X\xi=\phi AX,$$

where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to $C$ on $M$.

Since the ambient space is of constant holomorphic curvature $c$, the equations of Gauss and Codazzi are respectively given as follows:

$$(1.3) \quad R(X, Y)Z=c\{g(Y, Z)X-g(X, Z)Y$$

$$+g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}/4$$
$$+g(AY, Z)AX - g(AX, Z)AY,$$

(1.4) $\nabla_X A(Y) - \nabla_Y A(X) = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_X A$ is the covariant derivative of the shape operetor $A$ with respect to $X$.

The Ricci tensor $S'$ of $M$ is a tensor of type $(0, 2)$ given by $S'(X, Y) = \mathrm{tr}\{Z \to R(Z, X)Y\}$. But it may be also regarded as the tensor of type $(1, 1)$ and denoted by $S: TM \to TM$; it satisfies $S'(X, Y) = g(SX, Y)$. By the Gauss equation, (1.1) and (1.2) the Ricci tensor $S$ is given by

(1.5) $$S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where $h$ is the trace of the shape operator $A$. The covariant derivative of $S$ is also given by

(1.6) $$\nabla_X S(Y) = -3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}/4 + dh(X)AY + (hI - A)\nabla_X A(Y) - \nabla_X A(AY).$$

Now, some fundamental properties about the structure vector $\xi$ are stated here for later use. First of all, we have the following fact, which is proved by Maeda [8] and Ki and Suh [4], according as $c > 0$ and $c < 0$.

Proposition D. *Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$. If the structure vector $\xi$ is principal, then the corresponding principal curvature $\alpha$ is locally constant.*

In the sequel, assume that the structure vector $\xi$ is principal and denote by $\alpha$ the corresponding principal curvature. Namely, $A\xi = \alpha\xi$ is assumed. It follows from (1.4) that we have

(1.7) $$2A\phi A = c\phi/2 + \alpha(A\phi + \phi A)$$

and therefore, if $AX = \lambda X$ for any vector field $X$, then we have

(1.8) $$(2\lambda - \alpha)A\phi X = (\alpha\lambda + c/2)\phi X.$$

Accordingly, it turns out that in the case where $\alpha^2 + c \neq 0$, $\phi X$ is also a principal vector with principal curvature $\mu = (\alpha\lambda + c/2)/(2\lambda - \alpha)$, namely, we have

(1.9) $$2\lambda - \alpha \neq 0,$$
$$A\phi X = \mu\phi X, \quad \mu = (\alpha\mu + c/2)/(2\lambda - \alpha).$$

On the other hand, for any principal curvature $\lambda$ we find

(1.10) $$d\lambda(\xi) = 0$$

by the Codazzi equation (1.4) and Proposition $D$. In fact, the Codazzi equation gives $\nabla_X A(\xi) - \nabla_\xi A(X) = -c\phi X/4$ for any $X$ orthogonal to $\xi$. Accordingly, for any principal vector $X$ in $\xi^\perp$ with principal curvature $\lambda$, we have $g(\nabla_X A(\xi) - \nabla_\xi A(X), X) = (\alpha - \lambda) g(\nabla_X \xi, X) + d\lambda(X) g(X, X)$, which implies that $d\lambda(X) = 0$, because of (1.2). This is due to Kimura and Maeda [8].

From the Codazzi equation (1.4) it follows that the restriction $A_0$ of the shape operator to the orthogonal complement $\xi^\perp$ satisfies $g(\nabla_X A_0(Y), Z) = g(\nabla_Y A_0(X), Z)$. Then $A_0$ is called a *Codazzi tensor of type* (1, 1). For this Codazzi tensor, we define a subset $M_0$ of $M$ consisting of points $x$ so that there exists a neighborhood $U_x$ of $x$ such that the multiplicity of each principal curvature is constant on $U_x$. Then it is seen that $M_0$ is the open and dense subset of $M$. Given a point $x$ in $M$ and an eigenvalue $\lambda$ of $A_0$, let $A_x(\lambda)$ in $\xi_x^\perp \subset T_x M$ be a corresponding eigenspace of $A_0$. In every connected component of the open and dense subset $M_0$ the principal curvatures of $A_0$ form mutually distinct smooth eigenvalue functions and for such a function $\lambda$, the assignment $x \to A_x(\lambda(x))$ defines a smooth eigenspace distribution. Thus the orthogonal complement $\xi^\perp$ of the tangent bundle $TM$ can be decomposed as

$$\xi^\perp = A(\lambda_1) \oplus A(\lambda_2) \oplus \cdots \oplus A(\lambda_p) \tag{1.11}$$

on a connected component of $M_0$, which means that all principal curvatures may be regarded as smooth functions, unless otherwise provided.

It is easily seen that in the case where $\alpha^2 + c \neq 0$, we get $\phi A(\lambda) = A(\mu)$ and in the case where $\alpha^2 + c = 0$, we have

$$\mu = \alpha/2, \tag{1.12}$$

provided that $2\lambda - \alpha \neq 0$.

## 2. Operators

Let $M$ be a real hypersurface of an $n$-dimensional complex space form $M_n(c)$, $c \neq 0$, and assume that the structure vector $\xi$ is principal. An operator $P_f$ is defined by $P_f = A^2 - fA$ for some smooth function $f$ which is called *the operator determined by the smooth function*. In this section we are concerned with fundamental properties of this operator.

At any point $x$ of $M$, $P_f : T_xM \to T_xM$ is the self-adjoint linear transformation and $\xi$ is the eigenvector of $P_f$ associated with the eigenvalue $\beta_0=\alpha^2-f\alpha$, because $\xi$ is principal. Since the restriction of $P_f$ to the orthogonal complement $\xi^\perp$ is also self-adjoint, $\xi^\perp$ can be orthogonally decomposed as

(2.1) $$\xi^\perp=P_f(\beta_1)\oplus P_f(\beta_2)\oplus\cdots\oplus P_f(\beta_q),$$

on every connected component of the open and dense subset $M_0$ of $M$, where $P_f(\beta_r)$ denotes an eigenspace distribution of $P_f$ corresponding to the eigenvalue $\beta_r$ $(1\leqq r\leqq q)$. Taking account of the definition of $P_f$, we see that for any index $r$ there exist indices $a$ and $b$ $(1\leqq a, b\leqq p)$ such that

(2.2) $$P_f(\beta_r)=A(\lambda_a) \text{ or } P_f(\beta_r)=A(\lambda_a)\oplus A(\lambda_b).$$

Because of $\phi A(\lambda)=A(\mu)$, the restriction of $P_f$ to the eigenspace $P_f(\beta_r)$, namely, the transformation $\phi|P_f(\beta_r) : P_f(\beta_r)\to P_f(\beta_s)$ is bijective. In every connected component of $M_0$ the eigenvalue $\beta$ of $P_f$ is smooth and we suppose that $\phi P_f(\beta)=P_f(\beta')$. Then $P_f(\beta)$ and $P_f(\beta')$ are said to be *$\phi$-related.*

LEMMA 2.1. *If the eigenspace $A(\lambda)$ is contained in $P_f(\beta)$ and if $P_f(\beta)$ is $\phi$-invariant, then $\lambda$ depends on $c, \alpha$ and $f$.*

*Proof.* For any $X$ in $A(\lambda)$, the vector $\phi X$ belongs to $P_f(\beta)$, because it is $\phi$-invariant. Furthermore $\lambda$ and $\beta$ satisfy the following relationship:

(2.3) $$\lambda^2-f\lambda=\beta.$$

Without loss of generality, we may suppose that $\alpha^2+c\neq 0$ or $\alpha^2+c=0$, $\lambda\neq\alpha/2$. The equation (1.9) gives $A\phi X=\mu X$ and $\mu=(\alpha\lambda+c/2)/(2\lambda-\alpha)$, and therefore we have

(2.4) $$\mu^2-f\mu=\beta.$$

From (2.3) and (2.4) it follows that we have $(\lambda-\mu)(\lambda+\mu-f)=0$. Accordingly, the following equation is derived from the simple calculation:

(2.5) $$(\lambda^2-\alpha\lambda-c/4)\{\lambda^2-f\lambda+(c+2\alpha f)/4\}=0.$$

This completes the proof.

The operator $P_f$ is said to be *$\eta$-parallel,* if it satisfies $g(\nabla_X P_f(Y), Z)=0$ for any $X$, $Y$ and $Z$ orthogonal to $\xi$.

LEMMA 2.2. *Let $P_f$ be the operator determined by a smooth function $f$. If $P_f$ is $\eta$-parallel and if $f$ depends only on principal curvatures, then all eigenvvalues of $P_f$ are constant.*

*Proof.* For a unit vector $Y$ in $P_f(\beta)$ we have $PY=\beta Y$, which yields that $\nabla_X P_f(Y)+P_f\nabla_X Y=d\beta(X)Y+\beta\nabla_X Y$. From which together with the fact that $Y$ is unit it follows that we have

(2.6) $$g(\nabla_X P_f(Y), Y)=d\beta(X).$$

Since the operator $P_f$ is $\eta$-parallel, it reduces to

(2.7) $$d\beta(X)=0 \text{ for any } X \text{ orthogonal to } \xi.$$

On the other hand, the equation (2.3) means that $\beta$ depends only upon all principal curvatures and hence it follows from (1.10) that we have $d\beta(X)=0$, which together with (2.6) implies that $\beta$ is constant.

LEMMA 2.3. *Let $P_f$ be the operator determined by the smooth function $f$ depending only on principal curvatures. If an operator $P_f(\beta)+\varepsilon P_f(\beta')$ is $\eta$-parallel and if $P_f(\beta)$ and $P_f(\beta')$ are $\phi$-related, then $\beta+\varepsilon\beta'$ is constant,* $\varepsilon=\pm 1$.

*Proof.* The formula (1.2) means that the structure tensor $\phi$ is $\eta$-parallel, namely we have

(2.8) $$g(\nabla_X\phi(Y), Z)=0$$

for any vectors $X, Y$ and $Z$ in $\xi^\perp$. Since the operator $P_f\phi-\phi P_f$ is $\eta$-parallel, we have

$$g(\nabla_X P_f(\phi Y), Z)-g(\phi\nabla_X P_f(Y), Z)=0$$

by (2.8). Accordingly we have

(2.9) $$g(\nabla_X P_f(Y), Z)=g(\nabla_X P_f(\phi Y), \phi Z),$$

because $\phi$ is skew-symmetric. By the similar discussion to that of (2.6), we have $g(\nabla_X P_f(\phi Y), \phi Y)=d\beta(X)$ for a unit vector $Y$, which completes the proof.

## 3. Real hypersurfaces of type $A$ and $B$

This section is concerned with a generalization of Theorem C due to Kimura and Maeda [8] and Suh [13]. Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$. It is easily seen that if the structure vector $\xi$ is principal, the orthogonal complement $\xi^\perp$ is $\phi$-invariant and $A$-invariant. Since the structure tensor $\phi$ is $\eta$-parallel by (1.2), it is seen that if $\xi$ is principal and if the shape operator $A$ is $\eta$-parallel, then so is the operator $A\phi+\phi A$. We prove here the following

THEOREM 3.1. *Let $M$ be a real hypersurface of $P_nC$. Then the operator $A\phi+\phi A$ is $\eta$-parallel and the structure vector $\xi$ is principal if and only if $M$ is locally congruent to one of homogeneous real hypersurfaces of type $A_1$, $A_2$ and $B$.*

REMARK 3.1. (1) Let $M$ be a real hypersurface of type $B$ of $P_nC$. Then the shape operator $A$ satisfies $A\phi+\phi A=k\phi$, where $k=-c/\alpha$ is constant, which means that $A\phi+\phi A$ is $\eta$-parallel, because $\phi$ is $\eta$-parallel.

(2) Suppose that $A\phi-\phi A$ is $\eta$-parallel and $\xi$ is principal. Then the fact that $A$ is also $\eta$-parallel can be derived from the simple algebraic calculation. In fact, by the supposition that $A\phi-\phi A$ is $\eta$-parallel we have

$$g(\nabla_X A(\phi Y), Z)-g(\phi\nabla_X A(Y), Z)=0$$

for any $X, Y$ and $Z$ in $\xi^\perp$, and hence

$$(g(\nabla_X A(Y), \phi Z)=-g(\nabla_X A(Z), \phi Y),$$

which shows that $g(\nabla_X A(Y), \phi Z)$ is symmetric with respect to $X$ and $Y$, because of the Codazzi equation (1.4) and it is also skewsymmetric with respect to $Y$ and $Z$. This implies

$$g(\nabla_X A(Y), \phi Z)=0 \text{ for any } X, Y \text{ and } Z \text{ in } \xi^\perp,$$

which yields that $A$ is $\eta$-parallel, because the orthogonal complement $\xi^\perp$ is $\phi$-invariant.

In order to prove Theorem 3.1 we prepare for two lemmas. First of all, we shall prove

LEMMA 3.2. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$. If $A\phi+\phi A$ is $\eta$-parallel and if $\xi$ is principal, then all principal curvatures are constant.*

*Proof*. Since $A\phi+\phi A$ is $\eta$-parallel, we have

$$g(\nabla_X A(\phi Y), Z+g(\phi\nabla_X A(Y), Z)=0$$

for any vector fields $X, Y$ and $Z$ in $\xi^\perp$, because $\phi$ is also $\eta$-parallel. Hence it reduces to

(3.1) $$g(\nabla_X A(Y), Z)+g(\nabla_X A(\phi Y), \phi Z)=0.$$

For any unit vector field $Y$ in $A(\lambda)$ such that $\phi Y$ in $A(\mu)$ we get $g(\nabla_X A(Y), Y)=g(\nabla_X(AY)-A\nabla_X Y, Y)=d\lambda(X)$ and similarly $g(\nabla_X A(\phi Y), \phi Y)=d\mu(X)$, from which together with above equatioa it follows that we have $d(\lambda+\mu)(X)=0$ for any vector field $X$ in $\xi^{\perp}$. By this property and the fact (1.10) that each principal curvature is constant along the $\xi$-direction, it is seen that $\lambda+\mu$ is constant, say $a$.

Suppose that $\alpha^2+c\neq 0$. By (1.9) the principal curvature $\mu$ is given by $(\alpha\lambda+c/2)/(2\lambda-\alpha)$ and therefore the equation $\lambda+\mu=a$ is reduced to

$$2\lambda^2-2a\lambda+(c/2+a\alpha)=0.$$

Since the principal curvature $\alpha$ is constant by Proposition D, it is a quadric equation with constant coefficients, which yields that $\lambda$ is constant.

Next, suppose that $\alpha^2+c=0$. In this case it suffices to show the property that a principal curvature $\lambda$ different from the value $\alpha/2$ is constant. Then (1.12) implies that $\mu=\alpha/2$ and moreover the argument in the above case can be applied in this situation. It means that $\lambda$ is constant.

Let $D_x$ be a subspace of the tangent space $T_xM$ at any point $x$ consisting of vectors $Y$ in $\xi_x^{\perp}$ which satisfy $(A^2+aA+bI)Y=0$ for some constants $a$ and $b$, where $I$ denotes the identity transformation. On a connected component of $M_0$ a distribution $D$ can be defined by $x\to D_x$.

LEMMA 3.3. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$. If the constants $a$ and $b$ satisfy the condition $a^2\neq 4b$, then we have*

$$g(\nabla_X A(Y), Z)=0 \text{ for any vector fields } X, Y \text{ and } Z \text{ in } D.$$

*Proof*. Differentiating covariantly the equation $(A^2+aA+bI)Y=0$, we get

(3.2) $$\nabla_X A(AY)+A\nabla_X A(Y)+a\nabla_X A(Y)+(A^2+aA+bI)\nabla_X Y=0.$$

Taking account of the fact that $A$ and $A^2$ are both self-adjoint, we have

(3.3) $$g(\nabla_X A(AY)+A\nabla_X A(Y)+a\nabla_X A(Y), Z)=0.$$

Since the distribution $D$ is $A$-invariant, we can substitute $AX$ into $X$ in (3.3) and we have

$$g(\nabla_{AX}A(AY)+A\nabla_{AX}A(Y)+a\nabla_{AX}A(Y), Z)=0.$$

Since $g(\nabla_X A(Y), Z)$ is symmetric with respect to $X, Y$ and $Z$ in $\xi^\perp$ by means of the Codazzi equation (1.4), the first term of the above equation can be deformed as follows:

$$\begin{aligned} g(\nabla_Z A(AX), AY) &= -g(A\nabla_Z A(X)+a\nabla_Z A(X), AY) \\ &= g(\nabla_Z A(X), aAY+bY)-ag(\nabla_Z A(X), AY) \\ &= bg(\nabla_Z A(X), Y), \end{aligned}$$

where the definition of the distribution and (3.3) are used. It turns out that we have $2bg(\nabla_X A(Y), Z)+ag(\nabla_Y A(Z), AX)=0$ for any $X$, $Y$ and $Z$ in $D$. Accordingly it is seen that $g(\nabla_Y A(Z), AX)$ is symmetric with respect to $X$, $Y$ and $Z$, from which together with (3.3) it follows that $2g(\nabla_X A(Y), AZ)+ag(\nabla_X A(Y), Z)=0$. By the last two equations we have

$$(a^2-4b)g(\nabla_X A(Y), Z)=0$$

for any vector fields in $D$.

REMARK 3.2. (1) In [11] Montiel and Romero proved that in a Lorentz hypersurface $M_1{}^m$ of an anti-De Sitter space $H_1{}^{m+1}$, if the shape operator $A$ satisfies a polynomial $p(x)=x^2-ax+1$ for some constant $a$ such taht $a^2\neq 4$, then $A$ is parallel. The proof of Lemma 3.3 is essentially the similar method to that of thier result.

(2) In [13] Suh proves the following property: Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$. If the shape opertor $A$ satisfies a polynomial $p(x)=x^2+ax+b$ for some constants $a$ and $b$ such that $a^2\neq 4b$ and if $\xi$ is principal, then $A$ is $\eta$-parallel. Lemma 3.3 is a generalization of Suh's result without the assumption that $\xi$ is principal.

Now, we will here prove Theorem 3.1. We may consider, without loss of generality, that the constant holomorphic curvature of the ambient space $P_n\boldsymbol{C}$ is equal to 4. Assume that $A\phi+\phi A$ is $\eta$-parallel and $\xi$ is principal. By Lemma 3.2 all principal curvatures are constant. According to Kimura's theorem [6], the hypersurface is locally congruent to one of homogeneous real hypersurfaces of $P_n\boldsymbol{C}$. On the other hand, due to Takagi's clasification theorem [14] of homogeneous real hypersurfaces of $P_n\boldsymbol{C}$, they can be divided into six types $A_1, A_2, B$, $C, D$ and $E$.

If $M$ is of type $A_1, A_2$ or $B$, then it is seen by Kimura and Maeda's theorem [8] that the shape operator is $\eta$-parallel. As is remarked in the first of this section, the operator $A\phi+\phi A$ is then $\eta$-parallel. So, in order to prove Theorem 3.1 it suffices to show that the case of type $C, D$ or $E$ can not occur.

Suppoe that $M$ is a homogeneous real hypersurface of type $C, D$ or $E$. Due to Takagi's table [14], the hypersurface has distinct five constant principal curvatures: say $\alpha=\cot 2\theta$, $\lambda_1=\cot\theta$, $\mu_1=-\tan\theta$, $\lambda_2=\cot(\theta-\pi/4)$, $\mu_2=-\tan(\theta-\pi/4)$, $0<\theta<\pi/4$. Let $D_1$ and $D_2$ be distributions defined by $D_1=A(\lambda_1)+A(\mu_1)$ and $D_2=A(\lambda_2)+A(\mu_2)$, respectively. Then the vector field $Y$ (resp. $Z$) belonging to $D_1$ (resp. $D_2$) satisfies $(A^2-A\alpha-I)Y=0$ (resp. $A^2+4A/\alpha-I)Z=0$), which implies that $D_1$ and $D_2$ are both $A$-invariant and $\phi$-invariant. Since $\xi^\perp$ can be orthogonally decomposed by $\xi^\perp=D_1+D_2$, we have

$$(3.4)\qquad (A\phi-\phi A)(A\phi+\phi A-k\phi)=0,\ k=-4/\alpha,$$

of which the covariant derivationt gives rise to

$$(\nabla A\phi+A\nabla\phi-\nabla\phi A-\phi\nabla A)(A\phi+\phi A-k\phi)$$
$$+(A\phi-\phi A)(\nabla A\phi+A\nabla\phi+\nabla\phi A+\phi\nabla A-k\nabla\phi)=0.$$

Acting the above equation to any vector field $Y$ in $A(\lambda_1)$ and taking account of the inner product of it and any vector $Z$ in $A(\lambda_2)$, we have

$$(2\lambda_1-k)g((\nabla_X A\phi-\phi\nabla_X A)(\phi Y), Z)$$
$$+(\mu_2-\lambda_2)g((\nabla_X A\phi+\phi\nabla_X A)(Y), \phi Z)=0,$$

because $\phi$ is $\eta$-parallel and, $A\phi Y=\lambda_1 Y$ and $A\phi Z=\mu_2 Z$.

Accordingly it is equivalent to

$$(\mu_2-\lambda_1)g(\Delta_X A(Y), Z)+(\lambda_1-\lambda_2)g(\nabla_X A(\phi Y), \phi Z)=0,$$

because of $k=\lambda_2+\mu_2$. On the other hand, since $A(\lambda_1)$ is $\phi$-invariant, we can substitute $\phi Y$ into $Y$ in the last equation. Thus we have

$$(3.5)\qquad (\lambda_1-\lambda_2)g(\nabla_X A(Y), \phi Z)=(\mu_2-\lambda_1)g(\nabla_X A(Z), \phi Y)$$

for $Y\in A(\lambda_1)$ and $Z\in A(\lambda_2)$. Similarly we have the following equations:

$$(3.6)\qquad (\mu_1-\lambda_2)g(\nabla_X A(Y), \phi Z)=(\mu_2-\mu_1)g(\nabla_X A(Z), \phi Y),\quad Y\in A(\mu_1),\ Z\in A(\lambda_2),$$

$$(3.7)\qquad (\lambda_1-\mu_2)g(\nabla_X A(Y), \phi Z)=(\lambda_2-\lambda_1)g(\nabla_X A(Z), \phi Y)\quad Y\in A(\lambda_1),\ Z\in A(\mu_2),$$

$$(3.8)\qquad (\mu_1-\mu_2)g(\nabla_X A(Y), \phi Z)=(\lambda_2-\mu_1)g(\nabla_X A(Z), \phi Y)\quad Y\in A(\mu_1),\ Z\in A(\mu_2).$$

Suppose that $g(\nabla_X A(D_1), D_2)=0$ for any vector field $X$ in $\xi^\perp$. Any vector fields $X, Y$ and $Z$ are decomposed as $X=X_1+X_2$, $Y=Y_1+Y_2$ and $Z=Z_1+Z_2$ such that $X_a$, $Y_a$ and $Z_a \in D_a$ $(a=1, 2)$. By the above supposition we get $g(\nabla_X A(Y_b), \phi Z_c)=0$ provided that $b \neq c$, from which together with Lemma 3. 3 it follows that we have

$$\begin{aligned} g(\nabla_X A(Y), \phi Z) &= g(\nabla_{X_1} A(Y_2), \phi Z_2)+g(\nabla_{X_2} A(Y_1), \phi Z_1) \\ &= g(\nabla_{Y_2} A(X_1), \phi Z_2)+g(\nabla_{Y_1} A(X_2), \phi Z_1) \\ &= 0. \end{aligned}$$

Thus the shape operator $A$ must be $\eta$-parallel. By a theorem of Kimura and Maeda [8], $A$ is not $\eta$-parallel in the hypersurface of type $C, D$ and $E$, which yields that there exist vector fields $X$ in $\xi^\perp$, $Y$ in $D_1$ and $Z$ in $D_2$ such that $g(\nabla_X A(Y), \phi Z) \neq 0$. This means that without loss of generality we may suppose that there are vector fields $Y$ in $A(\lambda_1)$ and $Z$ in $A(\lambda_2)$ such that $g(\nabla_X A(Y), \phi Z) \neq 0$. By (3. 5) we have

$$(\mu_2-\lambda_1)g(\nabla_X(A\phi)YZ,)+(\lambda_1-\lambda_2)g(\nabla_X(\phi A)Y, Z)=0,$$
$$g(\nabla_X(\phi A)Y, Z)\neq 0, \quad g(\nabla_X(A\phi)Y, Z)\neq 0.$$

Because of $\mu_2-\lambda_1 \neq \lambda_1-\lambda_2$, it enables us to show

$$g(\nabla_X(A\phi+\phi A)Y, Z)\neq 0,$$

which means that $A\phi+\phi A$ is not $\eta$-parallel. It completes the proof.

Remark 3. 3. In Theorem 3. 1 the assumption that $\xi$ is principal can not be omitted. In fact, in ruled hypersurfaces of $P_n\boldsymbol{C}$ constructed by Kimura [5], the shape operator $A$ is $\eta$-parallel and hence so is $A\phi+\phi A$, but $\xi$ is not principal.

On the other hand, for a real hypersurface of $H_n\boldsymbol{C}$, Lemmas 3. 2 and 3. 3 mean that Berndt's classification theorem can be applied. Thus one finds the following

Theorem 3. 4. *Let $M$ be a real hypersurface of a complex hyperbolic space $H_n\boldsymbol{C}$ on which the structure vector field $\xi$ is principal. Then the operator $A\phi+\phi A$ is $\eta$-parallel if and only if $M$ is locally congruent to one of real hypersurfaces with constant principal curvatures of $H_n\boldsymbol{C}$.*

## 4. The Ricci tensor

Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$. In contract with a theorem of Kimura and Maeda [8] for the $\eta$-parallel shae operator, another characterization of real hypersurfaces of type $A_1$, $A_2$ or $B$ in $P_nC$ or real hypersurfaces of type $A_0 \sim B$ of $H_nC$ is recently given by Suh [13], who proved that the shape operator is $\eta$-parallel if and only if the Ricci tensor is $\eta$-parallel. On the other hand, Ki and Suh [4] treated with real hypersurfaces satisfying the condition $S\phi + \phi S = k_1\phi$, where $k_1$ is constant. This section is concerned with the generalization of these results.

THEOREM 4.1. *Let $M$ be a real hypersurface of $P_nC$, $n \geqq 3$. Then the operator $S\phi + \phi S$ is $\eta$-parallel and $\xi$ is principal if and only if $M$ is locally congruent to one of homogeneous hypersurfaces of type $A_1, A_2$ and $B$.*

REMARK 4.1. (1) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies $S\phi + \phi S = k_1\phi$, where $k_1$ is constant, then $S\phi + \phi S$ is $\eta$-parallel.

(2) Let $M$ be a real hypersurface of type $B$. Then it is easily seen it satisfies $S\phi + \phi S = k_1\phi$, where $k_1$ is constant, because of $A\phi + \phi A = k\phi$, where $k = -c/\alpha$.

(3) If the shape operator $A$ is $\eta$-parallel, then so is $S\phi + \phi S$. Accodingly, real hypersurfaces of type $A_1$ and $A_2$ in $P_nC$ admit this property, for example.

For any $X$, $Y$ and $Z$ in $\xi^{\perp}$, the fact that the operator $S\phi + \phi S$ is $\eta$-parallel implies $g(\nabla_X(S\phi)Y, Z) + g(\phi\nabla_X S(Y), Z) = 0$ and hence we have

(4.1) $$g(\nabla_X S(Y), \phi Z) = g(\nabla_X S(Z), \phi Y).$$

Substituting $\phi Y$ for $Y$ in the above equation, one gets

(4.2) $$g(\nabla_X S(Y), Z) + g(\nabla_X S(\phi Y), \phi Z) = 0.$$

Since the Riccic tensor $S$ is expressed as $S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 - P$, where we put $P = A^2 - hA$ and $h = \mathrm{Tr}\, A$, the covaraint derivation is given by $\nabla_X S(Y) = -3c\nabla_X \eta(Y)\xi/4 - \nabla_X P(Y)$. Accordingly, (4.2) is equivalent to

(4.3) $$g(\nabla_X P(Y), Z) + g(\nabla_X P(\phi Y), \phi Z) = 0.$$

An eigenspace of $P$ corresponding to an eigenvalue $\beta$ is denoted by $P(\beta)$. Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, whose Ricci tensor satisfies the condition $S\phi+\phi S$ is $\eta$-parallel. Then, by Lemma 2.3 it is seen that if $\xi$ is principal and if $P(\beta)$ and $P(\beta')$ are $\phi$-related, then $\beta+\beta'$ is constant. Moreover, concerning with principal curvatures, one finds

Lemma 4.2. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$. If $S\phi+\phi S$ is $\eta$-parallel and if $\xi$ is principal, then all curvatures are constant on $M$.*

*Proof.* First we consider the case where $\alpha^2+c\neq 0$. For a principal curvature $\lambda$ and eigenvalues $\beta$ and $\beta'$ such that $A(\lambda)\subset P(\beta)$ and $\phi P(\beta)=P(\beta')$, we have

$$\lambda^2-h\lambda=\beta,\quad \mu^2-h\mu=\beta', \tag{4.4}$$

where $A\phi X=\mu\phi X$ for any $X$ in $A(\lambda)$. Eliminating the function $h$ in the second equation of (4.4) together with (1.9), we have

$$\begin{aligned}2\alpha\lambda^4+(4\beta'+c-2\alpha^2)\lambda^3-\{2(\beta+2\beta')+3c/2\}\alpha\lambda^2\\ +\{\alpha^2(\beta+\beta')-c\beta-c^2/4\}\lambda+c\alpha\beta/2=0.\end{aligned} \tag{4.5}$$

If we suppose that $\alpha\neq 0$, then it is a quartic equation of $\lambda$. In the case where $\alpha=0$, $\lambda$ satisfies the root of the cubic equation at a point $x$ such that $4\beta'(x)+c\neq 0$, otherwise it is seen that there are at most two distinct principal curvatures, say $\lambda$ and $\mu$, different from $\alpha$ and they satisfy $\lambda+\mu=h$. It enables us to give $(n-2)h=0$ and hence principal curvatures are both constant by (4.4).

Let $\beta_1,\cdots,\beta_q$ be eigenvalues of $P$. Since any principal curvature is smooth on every connected component of the open and dense subset $M_0$ of $M$, the eigenvalues of $P$ may be supposed to be smooth. Then, for any $\beta$, the eigenspace $P(\beta)$ is $\phi$-invariant or there is another $\beta'$ such that $P(\beta')=\phi P(\beta)$. Therefore we see

$$\xi^{\perp}=\sum_{r=1}^{q}(P(\beta_r)\oplus P(\beta_r'))\oplus\sum_{r=2q+1}^{q}P(\beta_r),$$

where $P(\beta_r)$ and $P(\beta_r')$ is $\phi$-related for $r\leqq q_0$ and $P(\beta_r)$ is $\phi$-invariant for $r>q_0$. Suppose that $P(\beta)$ is $\phi$-invariant. Since $\beta$ is equal to $\beta'$, the remark before Lemma 4.2 implies that $\beta$ is constant, which means that (4.5) is the quartic equation with constant coefficients and the principal curvature $\lambda$ is constant, from which it turns out that $h$ is constant by (4.4). On the other hand, suppose that $P(\beta)$ and $P(\beta')$ re $\phi$-related and let $\lambda$ and $\mu$ are principal curvatures satisfying (4.4).

Then they satisfy the relationship:

$$\lambda^2+\mu^2-h(\lambda+\mu)=\beta+\beta', \tag{4.6}$$

from which together with the second equation of (1.9) it follows that we have

$$4\lambda^4-4(\alpha+h)\lambda^3+2(\alpha^2+h\alpha-2c)\lambda^2 \\ +(c\alpha-ch+4b\alpha)\lambda+(c^2+2ch\alpha-4b\alpha^2)/4=0, \tag{4.7}$$

where $b=\beta+\beta'$. Consequently, if $2q_0<q$, then $h$ is constant on $M$ and therefore all principal curvatures are also constant by (4.8). We consider the case of $2q_0=q$. Then it suffices to show that $h$ is constant. Suppose that $P(\beta)=A(\lambda)\oplus A(\lambda')$ $(\lambda\neq\lambda')$. Then $\lambda^2-h\lambda=\beta$ and $\lambda'^2-h\lambda'=\beta$ and hence we have $\lambda+\lambda'=h$. While it is seen that $P(\beta')=A(\mu)\oplus A(\mu')$, because $P(\beta)$ and $P(\beta')$ are $\phi$-related, and hence $\mu+\mu'=h$. On the other hand, since they are the roots of (4.7), the elementary relation of the equation (4.7) gives rise to $\lambda+\lambda'+\mu+\mu'=\alpha+h$ and hence $h=\alpha$, which yields that $\lambda$ is constant on $M$ by (4.7). Next, suppose that $P(\beta)=A(\lambda)$ and there are not less than two sets of the pair $(P(\beta), P(\beta'))$, namely, $q\geqq 4$, where $P(\beta')=A(\mu)$. Then the number of distinct principal curvatures is at least four and the fact shows that the equations (4.5) and (4.7) must be equivalent, from which it follows that each coefficients can be compared. Thus we have $4\beta'=-c-2\alpha h$ and $2\beta'=-\alpha^2-\alpha h-3c/2$, which implies $\alpha^2+c=0$, a contradiction. It means that there is only a pair $(P(\beta), P(\beta'))$ such that $P(\beta)=A(\lambda)$ and $P(\beta')=A(\mu)$. As the multiplicites of $\lambda$ and $\mu$ are equal, say $m=n-1$, we have $h=\alpha+m(\lambda+\mu)$, which implies

$$2m\lambda^2-2(h-\alpha)\lambda+mc/2+\alpha(h-\alpha)=0,$$
$$2m\mu^2-2(h-\alpha)\mu+mc/2+\alpha(h-\alpha)=0.$$

Adding above two equations and taking account of (4.6), we get

$$2(n-2)(h-\alpha)^2+4(n-1)\alpha(h-\alpha)+(n-1)^2(2b+c)=0,$$

which yields that $h$ is constant and so is $\lambda$. Therefore all principal curvatures are constant on the whole $M$.

In the case where $\alpha^2+c=0$, we may suppose that there is a principal curvature $\lambda$ different from $\alpha/2$. Then $\mu=\alpha/2$ and (4.6) is reduced to

$$\lambda^2-h\lambda+\alpha^2/4-\alpha h/2-b=0. \tag{4.8}$$

This means that the number of principal curvatures different from $\alpha/2$ and $\alpha$ is at most two, say $\lambda_1$ and $\lambda_2$ with multiplicities $n_1$ and $n_2$. Then they satisfy $\lambda_1+\lambda_2=h=n_1\lambda_1+n_2\lambda_2+(2n-n_1-n_2)\alpha/2$, which together with (4.8) it follows that $h$ is constant.

This concludes the proof.

We shall here prove Theorem 4.1. Since Lemma 4.2 shows that all principal curvatures are constant and the structure vector field $\xi$ is principal, $M$ is locally congruent to one of homogeneous real hypersurfaces of $P_nC$ according to Kimura's theorem [6]. On the other hand, due to Takagi's classification theorem [14] of homogeneous real hypersurfaces of $P_nC$, $M$ is of type $A_1, A_2, B, C, D$ and $E$.

In the case of type $A_1$, $A_2$ or $B$, it is seen that shape operator is $\eta$-parallel and hence so is $S\phi+\phi S$. In order to prove this theorem we shall show that a hypersurface of type $C, D$ or $E$ can not occur. Let $M$ be a real hypersurface of type $C, D$ or $E$ of $P_nC$. Suppose that the operator $S\phi+\phi S$ is $\eta$-parallel. Then all principal curvatures different from $\alpha$ are roots of the equation $(x^2-\alpha x-c/4)(x^2+cx/\alpha-c/4)=0$, and hence the shape operator $A$ satisfies the equation $(A\phi-\phi A)(A\phi+\phi A-k\phi)=0$, where $k=-c/\alpha$, which is deformed as $Q+\phi Q\phi-(c+\alpha^2)=0$ by (1.7), where $Q=P_k$ denotes the operator defined by $A^2-kA$. It is equivalent to

$$Q\phi-\phi Q=0. \tag{4.9}$$

Accordingly the operator $S\phi+\phi S$ is expressed as $(2n+1)c\phi/2+(h-k)A\phi+(h+k)\phi A-2\phi A^2$. Since it is $\eta$-parallel, we have

$$\{2(\mu+\sigma')-h-k\}\,g(\nabla_X A(Y),\phi Z)+(h-k)g(\nabla_X A(Z),\phi Y)=0$$

for $Y\in A(\mu)$, $A\in A(\sigma)$, $\phi Y\in A(\mu')$ and $\phi Z\in A(\sigma')$, in which we can exchange $Y$ and $Z$ and we get

$$\{2(\mu+\sigma')-h-k\}\,g(\nabla_X A(Z),\phi Y)+(h-k)g(\nabla_X A(Y),\phi Z)=0.$$

Thus there exists a function $F_1$ depending only on principal curvatures, which satisfies $F_1 g(\nabla_X A(Y), \phi Z)=0$. Similarly, there is a function $F_2$ depending only on principal curvatures, which satisfies $F_2 g(\nabla_Y A(X), \phi Z)=0$, where $F_1\neq F_2$, provided that $\lambda\neq\mu$. Thus we have $g(\nabla_X A(Y), \phi Z)=0$ for any $X\in A(\lambda)$, $Y\in A(\mu)$, $(\lambda\neq\mu)$ and any $Z$ in $\xi^\perp$. Since it is easily seen that $g(\nabla_X A(X), Y)=0$ for any $X, Y\in A(\lambda)$ and any $Z$ in $\xi^\perp$, it turns out that $A$ is $\eta$-parallel, a contradiction. Consequently, in the real hypersurface of type $C$, $D$ or $E$ the operator $S\phi+\phi S$ is not $\eta$-parallel.

This concludes the proof.

In the complex hyperbolic space, Berndt's classification theorem [1] can be applied and the following theorem is verified by Lemmas 2.3 and 4.2.

THEOREM 4.3. *Let $M$ be a real hypersurface of $H_nC$, $n\geqq 3$. Then $S\phi+\phi S$ is $\eta$-parallel and $\xi$ is principal if and only if $M$ is locally congruent to one of real hypersurfaces of type $A_0, A_1, A_2$ or $B$.*

## 5. Hypersurfaces of type $C, D$ or $E$

This section is devoted to the investigation of a characterization of real hypersurfaces of type $C, D$ or $E$ in $P_n\boldsymbol{C}$. Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, and assume that the structure vector $\xi$ is principal. Let $P_f$ be an operator introduced in §2, that is, $P_f=A^2-fA$, where $f$ is a smooth function. Then the subspace $\xi^{\perp}$ can be orthogonally decomposed into $\xi^{\perp}=P_f(\beta_1)+\cdots+P_f(\beta_q)$, where $P_f(\beta_r)$ denotes the eigenspace distribution of $P_f$ corresponding to the eigenvalue $\beta_r$.

Now, it is proved by Suh [13] that the Ricci tensor is $\eta$-parallel if and only if $M$ is locally congruent to one of real hypersurfaces of type $A_1\sim B$ or $A_0\sim B$. On the other hand, Kimura [7] proved that real hypersurfaces of $P_n\boldsymbol{C}$ satisfiying the condition $S\phi-\phi S=0$ are completely classified. First of all, we shall here prove the following

THEOREM 5.1. *Let $M$ be a real hypersurface with constant mean curvature of $M_n(c)$, $c\neq 0$, on which $\xi$ is principal. The Ricci tensor $S$ is not $\eta$-parallel and $S\phi-\phi S$ is $\eta$-parallel if and only if $c$ is positive and $M$ is locally congruent to one of a tube of radius $r$ over the following Kaehler submanifolds:*

(1) $P_1\boldsymbol{C}\times P_{(n-1)/2}\boldsymbol{C}$, *where* $0<r<\pi/4$, $\cot^2 2r=1/(n-2)$ *and* $n(\geqq 5)$ *is odd,*

(2) *a complex Grassmann* $G_{2,5}\boldsymbol{C}$, *where* $0<r<\pi/4$, $\cot^2 2r=3/5$ *and* $n=9$,

(3) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0<r<\pi/4$, $\cot^2 2r=5/9$ and $n=15$.

In order to verify Thererem 5.1, the following lemma is prepared.

LEMMA 5.2. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, on which $\xi$ is principal. For an operator $P_k=A^2-kA$, where $k$ is constant, if $P_k\phi-\phi P_k$ is $\eta$-parallel, then all principal curvatures are constant.*

*Proof*. Suppose that $\alpha^2+c\neq 0$. For a principal curvature $\lambda$ and eigenvalues $\beta$ and $\beta'$ such that $A(\lambda)\subset P_k(\beta)$ and $\phi P_k(\beta)=P_k(\beta')$, we have (4.5). By Lemma 2.3, $\beta-\beta'$ is constant, say $b'$, and hence we have $\lambda^2-\mu^2-k(\lambda-\mu)=b'$, which is equivalent to

$$(5.1)\quad 4\lambda^4-4(\alpha+k)\lambda^3+(6\alpha k-4b')\lambda^2 + \{(4b'-c)\alpha+k(c-2\alpha^2)\}\lambda-(c^2+2ck\alpha+4b'\alpha^2)/4=0.$$

Since $k$ is constant, (5.1) is the quartic equation with constant coefficients and $\lambda$ is constant. It turns out that all principal curvatures are constant on the whole $M$.

It is easily seen that it holds in the case where $\alpha^2+c=0$.

Consequently, using the classification theorems due to Takagi [14], Kimura [5] and Berndt [1], $M$ is locally congruent to one of real hypersurfaces of type $A_1\sim E$ or $A_0\sim B$, according as $c>0$ or $c<0$. The characterization theorems of the $\eta$-parallel shape operator by Kimura and Maeda [8] and Suh [13] yield that $A$ is $\eta$-parallel if and only if $M$ is of type $A_1\sim B$ or $A_0\sim B$ according as $c>0$ or $c<0$. This shows that if $P_k$ is not $\eta$-parallel, then these hypersurfaces can not occur, because if $A$ is $\eta$-parallel, then so is $P_k$. Thus one finds the following

PROPOSITION 5.3. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, on which $\xi$ is principal. If the operator $P_k=A^2-kA$ is not $\eta$-parallel, where $k$ is constant, and if $P_k\phi-\phi P_k$ is $\eta$-parallel, then $c$ is positive and $M$ is locally congruent to one of real hypersurfaces $C, D$ and $E$.*

From now on, we shall investigate the operator $P_k=Q$ in the real hypersurface of type $C, D$ or $E$ in $P_nC$, where $k=-c/\alpha$ is constant. According to the classification theorem due to Takagi [14], the hypersurface has five distinct principal curvatures and furthermore $A$ satisfies the equation $(A\phi-\phi A)(A\phi+\phi A-k\phi)=0$, $k=-c/\alpha$, which is deformed as $Q+\phi Q\phi=0$ on $\xi^\perp$ by (1.7). It is equivalent to

$$(5.2)\qquad Q\phi-\phi Q=0 \text{ on } \xi^\perp.$$

Taking account of the above property, the following characterization of real hypersurfaces of type $C, D$ and $E$ can be asserted.

THEOREM 5.4. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, on which the structure vector $\xi$ is principal. Then, for the operator $Q=A^2-kA$, where $k=-c/\alpha$ is constant, $Q$ is not $\eta$-parallel and $Q\phi-\phi Q$ is $\eta$-parallel if and only if $c$ is positive and $M$ is locally congruent to one of homogeneous real hypersurfaces of type $C, D$ or $E$.*

*Proof.* In order to prove Theorem 5.4, it suffices to verify the "if„ part, that is, to show that the operator $Q$ in the real hypersurface of type $C, D$ and $E$ is not $\eta$-parallel. Suppose that $M$ be a real hypersurface of type $C, D$ or $E$ and $P_k=A^2-kA$ is $\eta$-parallel, where $k$ is constant. Since we have $\nabla_X P_k(Y)=\nabla_X A(AY)+A\nabla_X A(Y)-k\nabla_X A(Y)$, the following equation

$$g(\nabla_X P_k(Y), Z)=(\mu+\sigma-k)g(\nabla_X A(Y), \phi Z)=0$$

is derived for any $X\in A(\lambda)$, $Y\in A(\mu)$ and $Z\in A(\sigma)$. Exchanging $X$ and $Y$ in the above equation, we get

$$g(\nabla_Y P_k(X), \phi Z)=(\lambda+\sigma-k)g(\nabla_Y A(X), Z)=0.$$

Combining together with above two equations, we have $(\lambda-\mu)g(\nabla_X A(Y), Z)=0$, from which it follows that $g(\nabla_X A(Y), Z)=0$ for any $X\in A(\lambda)$, $Y\in A(\mu)$, $\lambda\neq\mu$, and any $Z$. While it is easily seen that it holds for any $X, Y\in A(\lambda)$ and any $Z$. Thus we have $g(\nabla_X A(Y), Z)=0$ for any $X, Y$ and $Z$ and hence $A$ is $\eta$-parallel. This is a contradiction to the result of Kimura and Maeda [8]. Thus $P_k$ is not $\eta$-parallel.

It completes the proof.

As a direct consequence of Thoerem 5.4, we can prove Theorem 5.1.

The careful discussion of the proof of Theorem 5.4 can derive the slight generalization of Suh's theorem.

THEOREM 5.5. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, on which $\xi$ is principal. For an operator $P_f=A^2-fP$, where $f$ is a smooth function depending only on principal curvatures, $P_f$ is $\eta$-parallel if and only if $M$ is locally congruent to one of real hypersurfaces of type $A_1\sim B$ or of type $A_0-B$, according as $c>0$ or $c<0$.*

The sketch of the proof. By Lemma 2. 2 all eigenvalues of the operator $P_f$ are constant. Suppose $\alpha^2+c\neq 0$. For any principal curvatures $\lambda$ and $\mu$ such that $A(\lambda)\in P_f(\beta)$ and $A(\mu)\in\phi P_f(\beta)=P_f(\beta')$, we have (4. 6), from which it follows that any principal curvature $\lambda$ is constant. It is easy that the fact holds in the case where $\alpha^2+c=0$.

The conclusion is complete by means of the proof of Theorem 5. 4.

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