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# INVARIANT SUBMEANS AND SEMIGROUPS OF NONEXPANSIVE MAPPINGS ON UNIFORMLY CONVEX BANACH SPACES

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## 1. Introduction

Let $S$ be a semitopological semigroup i.e., $S$ is a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \to ts$ and $t \to st$ from $S$ into $S$ are continuous. Let $E$ be a uniformly convex Banach space and $\mathscr{T} = \{T_t ; t \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of $E$ into $C$, i.e., $T_{ts}x = T_t T_s x$, $t, s \in S$, $x \in C$, and the mapping $(t, x) \to T_t x$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology. Let AP$(S)$ be all continuous almost periodic functions on $S$. i.e., $f \in C(S)$ such that $\{r_s f ; s \in S\}$ is relatively compact in the norm topology.

Lau[1], in 1985, proved that if the space of almost periodic functions on $S$ has a left invariant mean, $C$ is a closed convex subset of a Hilbert space $H$, and there exist $x \in C$ with relatively compact orbit, then $C$ contains a common fixed point for $\mathscr{T} = \{T_s : s \in S\}$.

In this paper we prove that if AP$(S)$ has an invariant submean, $\mathscr{T} = \{T_s ; s \in S\}$ is a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of an uniformly convex, uniformly smooth Banach space and $C$ contains an element of relatively compact orbit, then $C$ contains a common fixed point for $S$.

## 2. Preliminaries

Let $S$ be a semitopological semigroup and $B(S)$ be the Banach space of all bounded real valued functions on $S$ with supremum norm. Let $D$ be a subspace of $B(S)$ containing constants. A real valued function

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$\mu$ on $D$ is called ***submean*** on $D$ if the following conditions are satisfied:

1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in D$.
2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in D$ and $\alpha \geq 0$.
3) For $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$.
4) $\mu(c) = c$ for every constant $c$.

Let $\mu$ be a submean on $D$ and $f \in D$. Then, according to times and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$.

For $s \in S$ and $f \in B(S)$, we define $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in S$. Let $D$ be a subspace of $B(S)$ containing constants which is $l_s$-invariant, i.e., $l_s(D) \subset D$ for each $s \in S$. Then a submean $\mu$ on $D$ is said to be ***left invariant*** if $\mu(f) = \mu(l_s f)$ for all $s \in S$ and $f \in D$. Similarly, we can define a ***right invariant submean*** on a $r_s$-invariant subspace of $B(S)$ containing constants. A left and right invariant submean is called an ***invariant submean.***

Let $E$ be a Banach space, and let $E^*$ be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. With each $x \in E$, we associate the set

$$J(x) = \{f \in E^*,\ \ \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for each $x \in E$. The multi-valued operator $J : E \to E^*$ is called the ***duality mapping***. As well known ([2, p. 130]), if $E^*$ is uniformly convex (or equivalently, $E$ is uniformly smooth), $J$ is single-valued, and $J$ is uniformly continuous on each bounded subset of $E$ when $E$ has the strong topology while $E^*$ has the weak* topology.

Let $E$ be a uniformly smooth Banach space with duality mapping $J : E \to E^*$. A map $T$ with domain $D(T)$ is said to be ***accretive*** if, for any $x, y \in E$ and all $\lambda > 0$,

$$\|\lambda x + Tx - (\lambda y + Ty)\| \geq \lambda\|x - y\|.$$

Equivalently, $T$ is accretive if and only if

$$(Tx - Ty,\ J(x-y)) \geq 0$$

for all $x, y \in D(T)$ (see [5, p. 245]). The range of $\lambda I + T$, $R(\lambda I + T)$, is known to be all of either for all $\lambda > 0$, or no $\lambda > 0$ (see [4]) ; in the first case, $T$ is called $m$-acceretive. In this case, the resolvent $J_\lambda{}^T = (I + \lambda T)^{-1}$ is a nonexpansive mapping defined on $E$ for each positive $\lambda$.

Let $\mathfrak{F}$ be a family of $m$-accretive operators with common domain

$D \subset E$. Let $S(\mathfrak{F})$ be the semigroup of nonexpansive mappings on $E$ generated by $\{J_\lambda^T ; T \in \mathfrak{F}\}$. Equip $S(\mathfrak{F})$ with the strong operator topology. Then $S(\mathfrak{F})$ is a topological semigroup i.e., the multiplicative on $S(\mathfrak{F})$ is jointly coninuous.

## 3. Lemmas

LEMMA 3.1. *Let $S$ be a semitopological semigroup, let $D$ be a subspace of $B(S)$ containing constants and let $\mu$ be a submean on $D$. Let $\{x_t ; t \in S\}$ be a bounded subset of a Banach space $E$ and let $C$ be a closed convex subset of $E$. Suppose that for each $x \in C$, the real-valued function $G$ on $C$ by*

$$G(x) = \mu_t \|x_t - x\|^2.$$

*Then the real-valued function $G$ on $C$ is continuous and convex.*

*Proof*. Let $x_n \to x$, and $M = \sup\{\|x_t - x_n\| + \|x_t - x\| ; n = 1, 2, \cdots,$ and $t \in S\}$. Then, since

$$\begin{aligned} \|x_t - x_n\|^2 - \|x_t - x\|^2 &= (\|x_t - x_n\| + \|x_t - x\|) \\ &\quad (\|x_t - x_n\| - \|x_t - x\|) \\ &\leq M \left| \|x_t - x_n\| - \|x_t - x\| \right| \\ &\leq M \|x_n - x\| \end{aligned}$$

for every $n = 1, 2, \cdots$ and $t \in S$, we have

$$\mu_t \|x_t - x_n\|^2 \leq \mu_t \|x_t - x\|^2 + M \|x_n - x\|.$$

Similarly, we have

$$\mu_t \|x_t - x\|^2 \leq \mu_t \|x_t - x_n\|^2 + M \|x_n - x\|.$$

So, we have $|G(x_n) - G(x)| \leq M \|x_n - x\|$. This implies that $G$ is continuous on $C$.

Let $\alpha$ and $\beta$ be nonnegative numbers with $\alpha + \beta = 1$ and $x, y \in C$. Then, since

$$\|x_t - (\alpha x + \beta y)\|^2 \leq \alpha \|x_t - x\|^2 + \beta \|x_t - y\|^2$$

we have $G(\alpha x + \beta y) \leq \alpha G(x) + \beta G(y)$. This implies that $G$ is convex on C.

LEMMA 3.2. *Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $E$, let $S$ be a semitopological semigroup, and let $\{x_t ; t \in S\}$ be a bounded subset of $E$. Let $D$ be a subspace of $B(S)$ such that $D$ contains constants and for any $z \in C$ and $u \in E$, functions $h$*

*and $g$ defined by $h(t)=\|x_t-z\|^2$, $g(t)=\langle u, J(x_t-z)\rangle$ for all $t\in S$ are in $D$ such that $\lim_t\|x_t-z\|^2$ exists for all $z\in C$. Let $\mu$ be a submean on $D$ satisfying the following condition: if $\lim_t x_t=\alpha$ and $\lim_t y_t=\beta$ then $\mu_t(x_t\pm y_t)=\mu_t(x_t)\pm\mu_t(y_t)$. Let $z_0\in C$ and $\mu_t\|x_t-z_0\|^2=\min_{y\in C}\mu_t\|x_t-y\|^2$. Then $\mu_t\langle z-z_0, J(x_t-z_0)\rangle\leq 0$ for all $z\in C$.*

*Proof*. For $z$ in $C$ and $\lambda: 0\leq\lambda\leq 1$, we have

$$\begin{aligned}\|x_t-z_0\|^2&=\|x_t-\lambda z_0-(1-\lambda)z+(1-\lambda)(z-z_0)\|^2\\&\geq\|x_t-\lambda z-(1-\lambda)z\|^2\\&\quad+2(1-\lambda)\langle z-z_0, J\langle x_t-\lambda z_0-(1-\lambda)z\rangle\rangle\end{aligned}$$

since $J(x)$ is the subdifferential of the convex function $\frac{1}{2}\|x\|^2$ ([2, p. 97]). Since $E$ is uniformly smooth, the duality map is uniformly continuous on bounded subset of $E$ from the strong topology of $E$ to the weak* topology of $E^*$. Therefore

$$|\langle z-z_0, J(x_t-\lambda z_0-(1-\lambda)z)-J(x_t-z_0)\rangle|<\varepsilon$$

if $\lambda$ is closed enough to 1. Consequently, we have

$$\begin{aligned}&\langle z-z_0, J(x_t-z_0)\rangle\\&\quad<\varepsilon+\langle z-z_0, J(x_t-\lambda z_0-(1-\lambda)z)\rangle\\&\quad\leq\varepsilon+\frac{1}{2(1-\lambda)}\{\|x_t-z_0\|^2-\|x_t-\lambda z_0-(1-\lambda)z\|^2\}.\end{aligned}$$

Hence, by hypothesis,

$$\begin{aligned}\mu_t\langle z-z_0, J(x_t-z_0)\rangle&<\varepsilon+\frac{1}{2(1-\lambda)}\{\mu_t\|x_t-z_0\|^2-\mu_t\|x_t\\&\qquad-\lambda z_0-(1-\lambda)z\|^2\}\\&<\varepsilon\end{aligned}$$

since $\lim_t\|x_t-z_0\|^2$ and $\lim_t\|x_t-\lambda z_0-(1-\lambda)z\|^2$ exists.

LEMMA 3.3. *Let $C$ be a closed convex subset of a uniformly convex and uniformly smooth Banach space $E$, let $S$ be a semitopological semigroup, and let $\{x_t;t\in S\}$ be a bounded set of $E$. Let $D$ be a subspace of $B(S)$ such that $D$ contains constants and for any $z\in C$ and $u\in E$, functions $h$ and $g$ defined by $h(t)=\|x_t-z\|^2$ and $f(z)=\langle u, J(x_t-z)\rangle$ for all $t\in S$ are in $D$ such that $\lim_t\|x_t-z\|^2$ exists for all $z\in C$. Let $\mu$ be a submean on $D$ satisfying the following condition: if $\lim_t x_t=\alpha$*

*and* $\lim_t y_t=\beta$ *then* $\mu_t(x_t\pm y_t)=\mu_t(x_t)\pm\mu_t(y_t)$. *Then, the set*

$$M=\{u\in C\ ;\ \mu_t\|x_t-u\|^2=\min_{z\in C}\ \mu_t\|x_t-z\|^2\}$$

*consists of one point.*

*Proof*. Let $g(z)=\mu_t\|x_t-z\|^2$ for every $z\in C$ and $\gamma=\inf\{g(z)\ ;\ z\in C\}$. Then, since the function $g$ on $C$ is convex, continuous and $g(z)\to\infty$ as $\|z\|\to\infty$ from [6, p. 79], there exists $u\in C$ with $g(u)=\gamma$. Therefore $M$ is nonempty. From lemma 3. 2 and $u\in M$,

$$\mu_t\langle z-u,\ J(x_t-u)\rangle\leq 0$$

for all $z\in C$. We show that $M$ consists of one point. Let $u, v\in M$ and suppose $u\neq v$. Then by [3, Theorem 1], there exists a positive number $k$ such that

$$\langle x_t-u-(x_t-v),\ J(x_t-u)-J(x_t-v)\rangle\geq k$$

for all $t\in S$. Therefore

$$\mu_t\langle v-u,\ J(x_t-u)-J(x_t-v)\rangle\geq k>0.$$

On the other hand, since $u, v\in M$, we have $\mu_t\langle v-u,\ J(x_t-u)\rangle<0$ and $\mu_t\langle u-v,\ J(x_t-v)\rangle<0$. Since

$$\begin{aligned}&\langle v-u,\ J(x_t-u)-J(x_t-v)\rangle\\ &\quad=\langle v-u,\ J(x_t-u)\rangle+\langle u-v,\ J(x_t-v)\rangle,\\ &\mu_t\langle v-u,\ J(x_t-u)-J(x_t-v)\rangle\\ &\quad\leq\mu_t\langle v-u,\ J(x_t-u)\rangle+\mu_t\langle u-v,\ J(x_t-v)\rangle\\ &\quad<0.\end{aligned}$$

This is a contradiction. Therefore $u=v$.

## 4. Semigroup of nonexpansive mappings with bounded orbit

Let $S$ be a semitopological semigroup. Let $C(S)$ be the Banach space of bounded continuous real-valued functions on $S$. Let AP($S$) denote the space of all continuous almost periodic functions on $S$. i. e., $f\in C(S)$ such that $\{r_sf\ ;\ s\in S\}$ is relatively compact in the norm topology where $(r_sf)(t)=f(ts)$.

THEOREM 4. 1. *Let $S$ be a semitopological semigroup. Let $\mathscr{T}=\{T_s\ ;\ s\in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of a uniformly convex uniformly smooth Banach space $E$ into $C$. If $AP(S)$ has an invariant submean, and $x\in C$ with* *relatively compact orbit, then there exists* $u \in C$ *such that* $T_s u = u$ *for all* $s \in S$.

*Proof*. We first prove that for any $z \in C$ and $y \in E$ the function $h$ and $g$ defined by $h(t) = \|T_t x - z\|^2$ and $g(t) = \langle y, \ J(T_t x - z)\rangle$ for all $t \in S$ are in AP$(S)$. It is clear that $h \in C(S)$. Let $h_x(t) = \|T_t x - z\|^2$. Then $r_s h_x(t) = h_w(t)$ where $w = T_s x$. Let $\tau : x \to h_x(t)$. If we can show that $\tau$ is continuous when $C(S)$ has the supnorm topology, the $\tau(O(x))$ is a compact subset of $C(S)$ containing $\{r_s h \ ; s \in S\}$ where $O(x) = \{T_s x \ ; s \in S\}$. In particular, $h \in$ AP$(S)$. To see that $\tau$ is continuous, let $\{x_n\}$ be a sequence in $C, x_n \to x$ and $M = \sup_{t \in S} \|T_t x - z\|$ for all $x \in C$ with relatively compact orbit, then

$$\begin{aligned}\|\tau(x_n) - \tau(x)\| &= \sup_{t \in S} | \, \|T_t x_n - z\|^2 - \|T_t x - z\|^2 | \\ &= \sup_{t \in S} | (\|T_t x_n - z\| - \|T_t x - z\|) \\ &\qquad (\|T_t x_n - z\| + \|T_t x - z\|) | \\ &\leq 2M \ \sup_{t \in S} \|T_t x_n - T_t x\| \\ &\leq 2M \|x_n - x\|\end{aligned}$$

by nonexpansive of $T_t$, $t \in S$. Hence $\|\tau(x_n) - \tau(x)\| \to 0$ as $x_n \to x$. Thus we have $h \in$ AP$(S)$.

Similarly, let $g_x(t) = \langle y, \ J(T_t x - z)\rangle$. Then $r_s g_x(t) = g_w(t)$ where $w = T_s x$. Let $\eta : x \to g_x(t)$ and $x_n \to x$. Then we have

$$\begin{aligned}\|\eta(x_n) - \eta(x)\| &= \sup_{t \in S} \ | \langle y, \ J(T_t x_n - z)\rangle \\ &\qquad - \langle y, \ J(T_t x - z)\rangle | \\ &= \sup_{t \in S} \ | \langle y, \ J(T_t x_n - z) - J(T_t x - z)\rangle |.\end{aligned}$$

Since $J$ is uniformly continuous on bounded sets when $E$ has its strong topology while $E^*$ has its weak* topology and

$$\begin{aligned}\|(T_t x_n - z) - (T_t x - z)\| &= \|T_t x_n - T_t x\| \\ &\leq \|x_n - x\|.\end{aligned}$$

Hence $\|\eta(x_n) - \eta(x)\| \to 0$ as $x_n \to x$. Thus we have $g \in$ AP$(S)$.

Let $\mu$ be an invariant submean on AP$(S)$. Then, the set

$$M = \{u \in C \ ; \ \mu_t \|T_t x - u\|^2 = \min_{z \in C} \ \mu_t \|T_t x - z\|^2\}$$

is invariant under every $T_s$, $s \in S$. In fact, if $u \in M$ then for each $s \in S$ we have

$$\mu_t \|T_t x - T_s u\|^2 = \mu_t \|T_{st} x - T_s u\|^2$$

$$
\begin{aligned}
&= \mu_t \|T_s T_t x - T_s u\|^2 \\
&\leq \mu_t \|T_t x - u\|^2
\end{aligned}
$$

and hence $T_s u \in M$. On the other hand, by Lemma 3. 3, we know that $M$ consists of one point. Therefore this point is a common fixed point of $T_s$, $s \in S$.

THEOREM 4. 2. *Let $E$ be a uniformly convex uniformly smooth Banach space. Let $\mathfrak{I}$ be a family of m–accretive mappings with common domain $D$ in $E$. Suppose that $AP(S)$ has an invariant submean, and there exists a sequence $\{x_n\}$ in $D$ such that $T(x_n) \to 0$ for each $T \in \mathfrak{I}$, then there exists $v \in E$ such that $T(v)=0$ for all $T \in \mathfrak{I}$.*

*Proof*. Define a function $g : E \to R$ by $g(z) = \mu_t \|x_n - z\|$ for each $z \in E$ and $\gamma = \inf\{g(z) \,;\, z \in E\}$, where $\mu_t \|x_n - z\|$ denotes the value of $\mu$ at the bounded sequence $\{\|x_n - z\|\}$. Then, since the function g on $E$ is continuous, convex and $g(z) \to \infty$ as $\|z\| \to \infty$, it follows from [6, p. 79], there exists $v \in E$ with $g(v) = \gamma$. So, putting $M = \{v \in E \,;\, g(v) = \gamma\}$, $M$ is nonempty, bounded, closed, and convex. Let $v \in M$, $T \in \mathfrak{I}$, and $J = J_1{}^T = (I+T)^{-1}$. Then

$$
\begin{aligned}
\mu_t \|x_n - Jv\| &= \mu_t \|x_n - Jx_n + Jx_n - Jv\| \\
&\leq \mu_t \|x_n - Jx_n\| + \mu_t \|x_n - v\| \\
&\leq \mu_t \|Tx_n\| + \mu_t \|x_n - v\| \\
&\leq \mu_t \|x_n - v\|
\end{aligned}
$$

since $Tx_n \to 0$ by assumption. Therefore, $M$ is invariant under $J_1{}^T$ for each $T \in \mathfrak{I}$. In particular, $M$ is invariant under the semigroup $S$ generated by $\{J_1{}^T \,;\, T \in \mathfrak{I}\}$. If AP($S$) has an invariant submean, then by Theorem 4. 1, there exists $v \in M$ with $J_1{}^T(v) = v$ for all $T \in \mathfrak{I}$. i. e., $T(v)=0$ for all $T \in \mathfrak{I}$.

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