韓國數學敎育學會誌<數學敎育>
1990. 6, 제 29 권 제1호 63-68

Journal of the Korea Society of Mathematical Education
June 1990: Vol. 29, No.1, 63-68

Vertical Lift of Vector Fields to the Frame Bundle

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ABSTRACT. Let M be a differentiable manifold, TM its tangent bundle and FM its frame bundle. The theory of complete lifts and Horizontal lifts to FM of vector fields on M ahs been studied by many authors. In this paper, vertical lifts of functions vector fields and 1-forms on M to FM are studied.

0. Introduction

Let M be an n-dimensional C^∞ manifold and FM its frame bundle. The differential geometry of FM was studied by T. Okubo [9, 10], Terrier [3], Mok [1, 2], Cordero [4, 5]. In [2], Mok introduced a Riemannian metric on the frame bundle of a Riemannian manifold. This metric is similar to that defined by Sasaki [8] for the tangent bundle TM of a smooth manifold M. In 1977, Mok [1] introduced the complete lifts of vector fields, different types of tensor fields and linear connections on M to FM. Moreover, in 1984, Cordero and Manual De Leon [4, 5] introduced the Horizontal ifrts of vectors fields, tensor fields and connections on M to FM. One of the present authors [6, 7] has studied vector fields and lifts of different types of tensor fields on Complex tangent bundle TM_{2n} of a Complex manifold M_{2n}. The main purpose of the present paper is to study the vertical lifts of function, vector field and 1-form of M to FM.

1. Preliminaries

In this section, we summarize all the basic definitions and results that are used later. Indices $a, b, c, \ldots, i, j, k, \ldots, \alpha, \beta, \gamma, \ldots$ have range in $\{1, \ldots, n\}$. Summation over repeated indices is always implied. Entries of matrices are written as A^i_j, A_{ij}, or A^{ij}, and in all cases, i is the row index while j is the column index. R^n is the euclidean n-space $Gl(n, R)$ the general linear group and $Gl(n, R)$ the Lie algebra of all $n \times n$ square matrices. Coordinate systems in M are denoted by (U, x^i), where U is the coordinate neighbourhood and x^i are the coordinate functions. Components in (U, x^i) of geometric objects on M will be referred to simply as components in U, or just components. We denote the partial differentiation $\partial/\partial x^i$ by ∂_i, and Lie derivative by $\mathfrak{L}_x$.

Let T_xM be the tangent space at a point $x \in M$, $(X_\alpha) = (X_1, \ldots, X_n)$ a linear frame at x and FM the frame bundle over M, that is the set of all frames at all points of M. Let $\pi : FM \to M$ be the Canonical parjection of FM on to M; for the coordinate system (U, x^i) in M'_i, we put $FU = \pi^{-1}U$. A frame (X_α) at x can be expressed uniquely in the form $X_\alpha = X^i_\alpha(\partial/\partial x^i)_x$. The induced coordinate system in FM is $\{FU, (x^i, X^i_\alpha)\}$. The matrix $[X^i_\alpha]$ is non-singular and its inverse will be written as $[X^\alpha_i]$.

Suppose (U, x^i) and $(U', x^{i'})$ are two coordinate systems in M and let $\{FU, (x^i; x^i_\alpha)\}$ and $\{FU', (x^{i'}; X^{i'}_\alpha)\}$ be the induced coordinate system in FM; by a routine calculation, one easily gets

$$\frac{\partial}{\partial x^i} = P^{i'}_i \frac{\partial}{\partial x^{i'}} + P^{i'}_{ij} X^i_\alpha \frac{\partial}{\partial X^{i'}_\alpha}, \quad \frac{\partial}{\partial X^i_\alpha} = P^{i'}_i \delta^\beta_\alpha \frac{\partial}{\partial X^{i'}_\beta}$$

on $FU \cap FU'$, where δ^β_α is the Kronecker delta and

$$P^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}; \quad P^{i'}_{ij} = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}$$

with a given linear connection Γ on M, we can define two sets of global 1-forms on FM, namely θ^γ and ω^ρ_σ. Their expression on FU or

$$\theta^\gamma = X^\gamma_i dx^i$$
$$\omega^\rho_\sigma = X^\rho_h (\Gamma^h_{ji} X^i_\sigma dx^j + dX^h_\sigma)$$

and these $n + n^2$ global 1-forms are linearly independent every where. Actually, $\theta = (\theta^\gamma)$ is the Canonical 1-forms of FM and $\omega = (\omega^\rho_\sigma)$ is the connection form of Γ. Let E_α, E^μ_λ be the $n + n^2$ global vector fields on FM dual to θ^γ, ω^ρ_σ; they span respectively the horizontal and vertical distribution on FM and their expression on FU are

$$E_\alpha = X^i_\alpha \left(\frac{\partial}{\partial x^i} - \Gamma^j_{ik} X^k_\beta \frac{\partial}{\partial x^i_\beta} \right)$$
$$E^\mu_\lambda = X^i_\lambda \frac{\partial}{\partial X^i_\mu}.$$

2. Vertical Lifts

2.1 Vertical Lifts of Functions

If ϕ is a function in M, we write ϕ^V for the function in FM obtained by forming the composition of $\pi : FM \to M$ and $\phi : M \to R$, so that

(2.1) $$\phi^V = \phi \circ \pi$$

Thus, if a point $x \in FU$ has induced coordinates (x^i, x^i_α), the

(2.2) $$\phi^V(x) = \phi^V(x, X_\alpha) = \phi \cdot \pi(x) = \phi(x)$$

Thus the value of $\phi^V(x)$ is constant along each frame $T_x M$ and equal to the value $\phi^V(x)$ is constant along each frame $T_x M$ and equal to the value $\phi(x)$ of ϕ at the point $x = \pi(x) \in M$. We call ϕ^V the vertical lift of the function ϕ. Thus we have, from (2.2),

(2.3) $$(\psi\phi)^V = (\psi)^V (\phi)^V$$

for any ψ, ϕ on M. We now see from (2.2) that the mapping $\phi \to \phi^V$ deternimes a linear isomorphism of M into FM with respect to constant coefficients.

If τ is a 1-form in M, it is regarded, in a natural way, as a function in FM, which we denote by $\gamma\tau$. If τ has the local expression $\tau = \tau_i dx^i$ in a coordinate neighbourhood U of M, then $\gamma\tau$ has the local expression

$$\gamma\tau = \tau_i X^i_\alpha \tag{2.4}$$

with respect to the induced coordinates in FU.

Thus if ϕ is a function in M, then $\gamma(d\phi)$ hjas the local expression

$$\gamma(d\phi) = \partial i X^i_\alpha \tag{2.5}$$

with respect to the induced coordinates in FU.

Proposition 2.1. *Let X and Y be vector fields in FM such that*

$$X(\gamma(d\phi) = Y(\gamma(d\phi)),$$

for an arbitrary function ϕ in M. Then $X = Y$.

Proof: If $X(\gamma(d\phi)) = 0$ for any function ϕ in M; then $X = 0$. If $\begin{matrix} X^i \\ X^{i'} \end{matrix}$ are components of X with respect to the induced coordinates (x^i, X^i_α) in FU. Then we have from $X(\gamma(d\phi)) = 0$

$$X^j(\partial_j\partial_i\phi)X^i_\alpha + X^j\partial_j\phi = 0$$

If this holds for any $\phi \in M$, $\partial_j\phi$ and $\partial_j\partial_i\phi$ taking any preassigned values at a fixed point, we have

$$X^j X^i_\alpha + X^i X^j_\alpha = 0, \qquad X_{j'} = 0 \tag{2.6}$$

Suppose $X^i_\alpha \neq 0$ and assume that $X^i_\alpha \neq 0$. Then putting $i = 1$ in the first equation of (2.6), we have $X^j X^1_\alpha + X^1 X^j_\alpha = 0$, from which $X^j = \beta X^j_\alpha$ for a certain function β. Substituting this into the first equation of (2.6), we find $2\beta X^j_\alpha X^i_\alpha = 0$, from which, putting $i = j = 1$, we have $\beta = 0$, i.e., $X^i = 0$. Thus we see that the vector field X is zero at a point such that $X^i_\alpha \neq 0$, that is, in $FM - M$. But the vector field X is continuous at every point of FM. So, we have $X = 0$ in FM. Thus proposition (2.1) is proved.

2.2 Vertical Lifts of Vector Fields

Let $x \in FM$ be such that $X\phi^V = 0$ for all $\phi \in M$. Then we say that X is a vertical vector field. Let $\binom{X^{i'}}{X^i}$ be components of X with respect to the induced coordinates. The, from $X\phi^V = 0$, we have $X^i\partial_i\phi = 0$ for all $\phi \in M$, from which $X^i = 0$, i.e.,

$$\begin{pmatrix} X^i \\ X^{i'} \end{pmatrix} = \begin{pmatrix} 0 \\ X^{i'} \end{pmatrix}. \tag{2.7}$$

Thus X is vertical if, and only if, its components in FU satisty (2.7).

Suppose that X is a vector field in M. We define a vector field X^V in FM by

$$X^V(\gamma\tau) = (\tau(X))^V, \tag{2.8}$$

being an arbitrary 1-form in M. We call X^V the vertical lift of X in M to FM. If X^i and τ_i are respectively components of X and τ with respect to the local coordinates in U, and if $\binom{X^i}{X^{i'}}$ are components of X^V with respect to the induced coordinates in FU, then we have from (2.8),

$$X^j(\partial_j \tau_i)X^i_\alpha + X^{j'}\tau_j = \tau_i X^i$$

for any τ_i, from which $X^i = 0$, $X^{i'} = X^i$. Thus the vertical lift X^V of X with components X^i in M to FM has components

$$X^V = \begin{pmatrix} 0 \\ X^i \end{pmatrix} \tag{2.9}$$

with respect to the induced coordinates in FM. Thus the vertical lift X^V of X to FM is a vertical vector field in FM. Consequently, we have by the definition of X^V.

$$X^V \phi^V = 0 \tag{2.10}$$

for any vector field X and ay differentiable function ϕ on M. Using (2.8) or (2.9) and taking account of (2.2), we can easily verify that

$$(X+Y)^V = X^V + Y^V \tag{2.11}$$

$$(\phi X)^V = \phi^V X^V \tag{2.12}$$

for any vector fields X, Y and differentiable function ϕ on M. We now have

Proposition 2.2. *For the Lie product $[X^V, Y^V]$ of X^V and Y^V*

$$[X^V, Y^V] = 0 \tag{2.13}$$

holds, X and Y being arbitrary element of M.

Proof: Let τ be an arbitrary elements of 1-form in M. Then, taking account of (2.8) and (2.10), we have

$$\begin{aligned}[X^V, Y^V](\gamma\tau) &= X^V Y^V(\gamma\tau) - Y^V X^V(\gamma\tau) \\ &= X^V(\tau(Y))^V - Y^V(\tau(X))^V = 0\end{aligned}$$

and consequently $[X^V, Y^V] = 0$ by virtue of proposition (2.1). Thus propositon (2.2) is proved.

We see from (2.9) that the mapping $X \to X^V$ deternimes a linear isomorphism of M into FM with respect to constant coefficents. From (2.9), we find in each open set FU

$$\left(\frac{\partial}{\partial x^i}\right)^V = (X^i_\alpha)^V \tag{2.14}$$

with respect to the induced coordinates in FM.

2.3 Vertical Lifts o 1-Froms

Let τ be a 1-form on M, the 1-form $\tau^V = \pi^*\tau$ on FM is completely defined by the following.

τ^V is the unique 1-form on FM which verifies

$$\tau^V(X^C) = (\tau(X))^V \tag{2.15}$$

for any vector field X on M.

If $\tau = \tau_i dx^i$ is the local expression in U of τ, then $\tau^V = \tau_i dx^i$ in FU with respect to the global coframe θ^γ and ω^ρ_σ in FU. Their expression on FU are

$$\theta^\gamma = X^\gamma_i dx^i$$
$$\omega^\rho_\sigma = X^\rho_h(\Gamma^h_{ji} X^i dx^j + dX^h_\sigma)$$

one has $\tau^V = X^i_\alpha \tau_i \theta^\alpha$.

The vertical lift τ^V of τ with local expression $\tau = \tau_i dx^i$ has components of the form

$$V : (\tau_i, 0), \tag{2.16}$$

with respect to the induced coordinates in FM. Thus the vertical lift τ^V of τ to FM is a vertical 1-form in FM.

Consequently, we have from (2.9) and (2.16)

$$\tau^V(X^V) = 0 \tag{2.17}$$

for any τ on M. Using (2.16) and taking account of (2.2), we can easily verify that

$$(\tau + \eta)^V = \tau^V + \eta^V \tag{2.18}$$
$$(\phi\tau)^V = \phi^V \tau^V \tag{2.19}$$

for any τ, η on M.

We see from (2.16) that the mapping $\tau \rightarrow \tau^V$ deternimes a linear isomorphism of M into FM with respect to constant coefficients, and that in each open set FU the formula

$$(dx^i)^V = dx^i \tag{2.20}$$

holds with respect to the induced coordinates.

References

1. K.P. Mok, *Complete lifts of tensor fields and connections to the frame bundle*, Proc. London Math. Soc. **32** (1979), 72–88.
2. ———, *On the differential geometry of frame bundles of Riemannian manifolds*, J. Reine Angew Math. **302** (1979), 16–31.
3. J.M. Terrier, *Linear connection and almost complex structures*, Proc. Amer. Math. Soc. **49** (1975), 59–65.

4. L.A. Cordero-M.De Leon, *Lifts of tensor fields to the frame bundle*, Rend. Circ. Math. Palermo **32** (1983), 236–271.
5. ———————, *Horizontal lift of connections to the frame bundle*, Boll. Un. Mat. Ital. B(6) **3** (1984), 223–240.
6. R.N. Singh, *On the complex tangent bundle*, Proc. Math. Soc. **2** (1986), 122-132.
7. ————, *Horizontal lifts of tensor fiels to the complex tangent bundle*, Accepted for presentation in the VI International Colloquium on differential Geometry at Santiago de Compostela (SPAIN),.
8. S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. **10** (1958), 338–354.
9. T. Okubo, *On the differential geometry of fram bundles $F(X_n)$, $n = 2m$*, Mem. Defense Acad. **5** (1965), 1–17.
10. ————, *On the differential geometry of frame bundles*, Ann. Mat. Pure Appl. **72** (1966), 29–44.