韓國數學敎育學會誌<數學敎育>
1990. 6, 제 29 권 제 1 호 55-61

Journal of the Korea Society of Mathematical Education
June 1990: Vol. 29, No.1, 55-61

A Generalization of Abel's Theorem on Power Series

W.H Hsiang*

Department of Mathematics, Chinese University of Hong Kong
Shatin, New Territories, Hong Kong

ABSTRACT. There are three objectives of this paper. First, we present an elegant and simple generalization of Abel's theorem (i.e. the Abel summability (on the unit disk of the euclidean plane) is regular). Second, we consider the definition of Abel summability through $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{+\infty} a_n x^n$ which immediately has clear connexctions with Cesaro summability and Cesaro sums $\frac{1}{n+1}\sum_{k=0}^{n} a_k$. This approach examplifies some simple aspects of so-called Tauberian theorems of divergent series. Third, we present the applications of the previous results to find the limits of transition probabilities of homogeneous Markov chain. Finally, we explain why the original Abel's theorem which looks obvious is difficult to be proved, and can not be proved analytically.

A series of real numbers $\sum_{n=0}^{+\infty} a_n$ is said to converge to a in the Abel's sense if

$$\lim_{x\to 1^-}\sum_{n=0}^{+\infty} a_n x^n = a,$$

where the limit is taken from the left of 1.

In this case, we denote

$$(A)\sum_{n=0}^{+\infty} a_n = a.$$

Abel summability of infinite series is regular, i.e. the convergence of the series implies $(A)\sum_{n=0}^{+\infty} a_n = \sum_{n=0}^{+\infty} a_n$.

Lemma 1. *For any power series $\sum_{n=0}^{+\infty} c_n(z-z_0)^n$ of a complex variable, where $z_0 \in \mathbf{C}$ is fixed, then there is a largest $0 \le \mathbf{R} \le +\infty$, which is called the radius of convergence of the given power series, such that the power series converges on $|z-z_0| < \mathbf{R}$ (where the convergence is absolute and uniform on any closed disk in $|z-z_0| < \mathbf{R}$) and represents an analytic function on $|z-z_0| < \mathbf{R}$, diverges on $|z-z_0| > \mathbf{R}$, and no conclusion is on $|z-z_0| = \mathbf{R}$.*

Theorem 1(Generalized Abel's theorem). *If $a_n \in \mathbf{R}$ for any $n \in \mathbf{Z}+\cup\{0\}$ and $0 \le \mathbf{R} \le +\infty$ such that $\sum_{n=0}^{+\infty} a_n R^n$ converges, then*

$$\lim_{x\to R^-}\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n R^n$$

(cf. Theorem 45 *on* p.80,[3], *where the assumption that the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n z^n$ is R is redundant) and $\sum_{n=0}^{+\infty} a_n z^n$ has radius of convergence $\ge R$.*

* Current Address: 273 Spadina Avenue, Toronto, Ontario M5T 2E3, Canada.

The case $R = 1$ is the Abel's theorem (Theorem I.4.3(i),[4]). *Conversely, if $a_n \geq 0$ for any $n \in \mathbf{Z}+\cup\{0\}$, and $0 < R \leq +\infty$ with $\lim_{x\to R^-}\sum_{n=0}^{+\infty} a_n x^n = a \leq +\infty$, then*

$$\sum_{n=0}^{+\infty} a_n R^n = a.$$

(cf. Theorem I.4.3(ii),[4]).

Proof: Let $b_n = a_n R^n$ for any $n \in \mathbf{Z}+\cup\{0\}$, then $\sum_{n=0}^{+\infty} b_n$ converges and $(A)\sum_{n=0}^{+\infty} b_n = \sum_{n=0}^{+\infty} b_n$, i.e.

$$\lim_{x\to R^-}\sum_{n=0}^{+\infty} a_n x^n = \lim_{\frac{x}{R}\to 1^-}\sum_{n=0}^{+\infty} a_n R^n (\frac{x}{R})^n = \lim_{x\to 1^-}\sum_{n=0}^{+\infty} b_n x^n = \sum_{n=0}^{+\infty} b_n = \sum_{n=0}^{+\infty} a_n R^n.$$

Since $\sum_{n=0}^{+\infty} b_n$ converges, $\lim_{n\to+\infty} b_n = 0$ implies that there is an $0 < K < +\infty$ with $|b_n| \leq K$ for any $n \in \mathbf{Z}+\cup\{0\}$.

For any $|z| < 1, \sum_{n=0}^{+\infty} |b_n z^n| \leq k\sum_{n=0}^{+\infty} |z|^n = \frac{K}{1-|z|} < +\infty$ and $\sum_{n=0}^{+\infty} b_n z^n$ converges. If $|z| < R$, then $|\frac{z}{R}| < 1$ and $\sum_{n=0}^{+\infty} b_n \frac{z^n}{R^n} = \sum_{n=0}^{+\infty} a_n z^n$ converges, i.e. the radius of convergence of $\sum_{n=0}^{+\infty} a_n z^n$ is $\geq R$.

If $R = +\infty$ in Theorem 1, then $a_n = 0$ for any $n \in \mathbf{Z}_+$. We assume $0 \cdot (\pm\infty) = 0$. Theorem 1 has a partial, complex case: If $c_n \in \mathbf{C}$ for any $n \in \mathbf{Z}+\cup\{0\}$ and $0 \leq R \leq +\infty$ such that $\sum_{n=0}^{+\infty} c_n R^n$ converges, then $\sum_{n=0}^{+\infty} a_n R^n$ and $\sum_{n=0}^{+\infty} b_n R^n$ converge, where $c_n = a_n + ib_n$ for any $n \in \mathbf{Z}+\cup\{0\}$. Thus $\lim_{x\to 1^-}\sum_{n=0}^{+\infty} c_n x^n = \sum_{n=0}^{+\infty} c_n$. We note that $\sum_{n=0}^{+\infty} c_n z^n$ is analytic function on $|z| < R$. But $\sum_{n=0}^{+\infty} c_n(-R)^n$ may even not converge. Thus we can not prove $\lim_{z\to 1}\sum_{n=0}^{+\infty} c_n z^n = \sum_{n=0}^{+\infty} c_n$. But, by Dirichlet problem, we can have the following similar result.

Lemma 2. *Let $f(z)$ be a function continuous on $|z| \leq R < +\infty$, and analytic on $|z| < R$. If $f(z) = \sum_{n=0}^{+\infty} c_n z^n$ on $|z| < R$, then $f(z) = \sum_{n=0}^{+\infty} c_n z^n$ on $|z| \leq R$, where the former convergence is absolute on $|z| < R$, and the latter convergence is uniform on $|z| \leq R$* (Proposition 3,[1]).

In the latter part of this paper, we consider the connections between Abel summability and Cesaro summability. We also present some applications on Markov chains.

Abel (and also cesaro) summability can also be defined in terms of sequence. For this purpose, we use the following conventional notations: If $\{a_n\}_{n=0}^{+\infty}$ is a sequence of real numbers, then we let $s_n = \sum_{k=0}^{n} a_k$ and $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$ for any $n \in \mathbf{Z}+\cup\{0\}$. The series $\sum_{n=0}^{+\infty} a_n$ (or sequence $\{s_n\}_{n=0}^{+\infty}$ which may not be defined as the partial sums) is said to be c_1-summable (or c_1-limitable) to a if $\lim_{n\to+\infty}\sigma_n = a$(p.4–p.7,[6]).

Lemma 3. *if $a_n \in \mathbf{R}, b_0 = a_0$ and $b_n = a_n - a_{n-1}$ for any $n \in \mathbf{Z}_+$, then $a_n = \sum_{k=0}^{n} b_k$ for any $n \in \mathbf{Z}+\cup\{0\}$. Thus $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n} a_k = a \in \mathbf{R}$ iff $\sum_{n=0}^{+\infty} b_n$ is c_1-summable to a iff* $\lim_{n\to+\infty}\sum_{k=0}^{n}(1-\frac{k}{n+1})b_k = a$ (Example 1 on p.7,[6]). *Also, $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{+\infty} a_n x^n = a'$ exists iff $\sum_{n=0}^{+\infty} b_n$ is A-summable to a'.*

Proof: We can prove by induction that a_n's are the partial sums of the series $\sum_{n=0}^{+\infty} b_n$. Thus $\lim_{n\to+\infty} \frac{1}{n+1}\sum_{k=0}^{n} a_k = a$ iff $\sum_{n=0}^{+\infty} b_n$ is c_1-summable to a. But

$$\frac{1}{n+1}\sum_{k=0}^{n} a_k = \frac{1}{n+1}\sum_{m=0}^{n}(\sum_{k=0}^{m} b_k) = \frac{1}{n+1}\sum_{k=0}^{n}(n+1-k)b_k = \sum_{k=0}^{n}(1-\frac{k}{n+1})b_k$$

for any $n \in \mathbf{Z}_+$.

The second statement is proved.

We note $(1-x)\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} a_k x^k - \sum_{k=1}^{n+1} a_{k-1}x^k = a_0 + \sum_{k=1}^{n}(a_k - a_{k-1})x^k - a_n x^{n+1} = \sum_{k=0}^{n} b_k x^k - (\sum_{k=0}^{n} b_k)x^{n+1}$ for any $n \in \mathbf{Z}_+$ and $0 < x < 1$. The third statement is proved.

From the second statement of Lemma 3, we can easily derive the following consequence.

Theorem 2. *For any series $\sum_{n=0}^{+\infty} a_n$ of real numbers, $\sigma_n - s_n = \frac{1}{n+1}\sum_{k=0}^{n} ka_k$. Thus if $a_n = 0(\frac{1}{n})$ as $n \to +\infty$, and $\lim_{n\to+\infty}\sigma_n = a \in \mathbf{R}$, then $\lim_{n\to+\infty} s_n = a$* (Example 3 on p.7,[6]).

Proof: Let $b_n = \sum_{k=0}^{n} a_k$ for any $n \in \mathbf{Z}+\cup\{0\}$, then $\sigma_n = \sum_{k=0}^{n}(1-\frac{k}{n+1})a_k$. Hence $|s_n - a| \le |\sigma_n - a| + |\sigma_n - s_n| < \frac{1}{2}\varepsilon + |\sigma_n - s_n|$ for any $n \ge n_1$, where $\varepsilon > 0$ is given and $n_1 = n_1(\varepsilon) \in \mathbf{Z}_+$. Let $n_2 \in \mathbf{Z}_+$ with $|na_n| < \frac{1}{4}\varepsilon$ for any $n \ge n_2$. Thus $|\sigma_n - s_n| \le \frac{1}{n+1}(\sum_{k=0}^{n_2}|ka_k| + \sum_{k=n_2+1}^{n}|ka_k|) < \frac{1}{n+1}\sum_{k=0}^{n_2}|ka_k| + \frac{(n-n_2)\varepsilon}{4(n+1)} < \frac{1}{n+1}\sum_{k=0}^{n_2}|ka_k| + \frac{1}{4}\varepsilon$.

Let $n_3 \in \mathbf{Z}_+$ with $\frac{1}{n+1}\sum_{k=0}^{n_2}|ka_k| < \frac{1}{4}\varepsilon$ for $n \ge n_3$. Thus $|s_n - a| < \varepsilon$ for $n \ge n_0 = \max\{n_1, n_2, n_3\}$. There are several important results similar to Theorem 2.

For example, if σ_n is replaced by $\sigma(x) = \sum_{n=0}^{+\infty} a_n x^n = (1-x)\sum_{n=0}^{+\infty}(\sum_{k=0}^{n} a_k)x^n$ for any $0 < x < 1$ and $\lim_{x\to 1^-}\sigma(x) = a$, then $\lim_{n\to+\infty} s_n = a$. This is the first tauberian theorem (Theorem III.3,[6]). If σ_n is replaced by $\sigma(x), \lim_{x\to1^-}\sigma(x) = a$ and $a_n = 0(\frac{1}{n})$ as $n \to +\infty$, then $\lim_{n\to+\infty} s_n = a$. This is so called the J.E.Littlewood theorem (Theorem III.9,[6]). But both results can not be proved as easily as Theorem 2.

By the third statement of Lemma 2, we can reformulize Abel summability through sequence: A sequence $\{s_n\}_{n=0}^{+\infty}$ is said to be A-limitable to $a \in \mathbf{R}$ if

$$(1-x)\sum_{n=0}^{+\infty} s_n x^n = \sum_{n=0}^{+\infty} s^n x^n / \sum_{n=0}^{+\infty} x^n \to a \text{ as } x \to 1^-$$

The series $\sum_{n=0}^{+\infty} a_n$ of real numbers is said to be A-summable to a iff $\{s_n\}_{n=0}^{+\infty}$ is A-limitable to a (Lines 8-9 on p.24,[6]).

We give another expression of the well-known fact that Cesaro summability implies Abel summability in the first statement of the following results.

Theorem 3. *Let $a_n \in \mathbf{R}$ for any $n \in \mathbf{Z}+\cup\{0\}$ and $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be well-defined for $x > 0$ sufficiently closed to 1 in* (ii), (iv), (v) *and* (vi).

(i) *If $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n} a_k = a \in \mathbf{R}$, then $\lim_{x\to1^-}(1-x)\sum_{n=0}^{+\infty} a_n x^n = a$* (cf. Theorem 2.3,[2]).

(ii) *If* $a_n \geq 0$*(or there is an* $K \in \mathbf{R}$ *with* $a_n \geq -K$*) for any* $n \in \mathbf{Z}+\cup\{0\}$*, and* $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{+\infty} a_n x^n = a \in \mathbf{R}$*, then* $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n} a_k = a$ (Theorem 2.3.5,[2] *and* Theorems III.6-7,[6]). *By the previous language (i.e.*a_n*'s are considered as partial sums), any A-summable infinite series* $\sum_{n=0}^{+\infty} a_n$ *with* s_n *to be non-negative (or bounded below) is* c_1*-summable to the same value* (Theorem III.8,[6]). *The extra conditions in the latter statements are stringent.*

(iii) *(*N.H.Abel;1826) *If* $\sum_{n=0}^{+\infty} a_n = a \in \mathbf{R}$*, then* $\lim_{x\to 1^-}\sum_{n=0}^{+\infty} a_n x^n = a$.

(iv) *If* $\sum_{n=0}^{+\infty} a_n$ *is A-summable to* $a \in \mathbf{R}$ *and* $a_n = 0(\frac{1}{n})$ *as* $n \to +\infty$*, then* $\sum_{n=0}^{+\infty} a_n$ *converges to* a (Theorem III.3,[6] and p.104,[7]).

(v) *If* $\sum_{n=0}^{+\infty} a_n$ *is A-summable to* $a \in \mathbf{R}$ *and* $a_n = 0(\frac{1}{n})$ *as* $n \to +\infty$*, then* $\sum_{n=0}^{+\infty} a_n$ *converges to* a (Theorem 13,[7]).

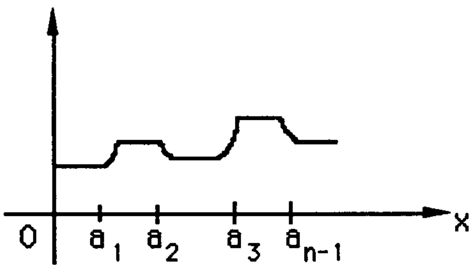
(vi) (E.Landau;1913) *If* $\sum_{n=0}^{+\infty} a_n$ *is A-summable to* $a \in \mathbf{R}$ *and there is an* $K \in \mathbf{R}$ *with* $na_n \geq -K$ *for any* $n \in \mathbf{Z}+\cup\{0\}$*, then* $\sum_{n=0}^{+\infty} a_n$ *converges and is* c_1*-summable to* a (Theorem 14,[7]).

Proof: (i) Let $b_0 = a_0$ and $b_n = a_n - a_{n-1}$ for any $n \in \mathbf{Z}_+$. Since $\sum_{n=0}^{+\infty} b_n$ is C_1-summable to a, it is A-summable to a. Thus $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{+\infty} a_n x^n = \lim_{x\to 1^-}\sum_{n=0}^{+\infty} b_n x^n = a$ by Lemma 3. (ii) We consider a partial case: $a_0 \geq 0$ and $a_n \geq a_{n-1}$ for any $n \in \mathbf{Z}_+$. Thus $b_n \geq 0$ for any $n \in \mathbf{Z}+\cup\{0\}$ and $\lim_{x\to 1^-}\sum_{n=0}^{+\infty} b_n x^n = a$. By the converse of Abel's theorem, $\sum_{n=0}^{+\infty} b_n = a$, i.e. $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\sum_{k=0}^{n} b_k = a$ and $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{k=0}^{n} a_k = a$.
For the proof of the general case of the first statement, we need the following technical lemma.

Lemma 4. *If* $f : [0,1] \to [0,+\infty)$ *is a Riemann integrable function and* $\varepsilon > 0$ *is given, then there exist polynomials* $p(x)$ *and* $P(x)$ *on* $[0,1]$ *with* $p(x) \leq f(x) \leq P(x)$ *for any* $0 \leq x \leq 1$ *and* $\int_0^1 (P(x) - p(x))dx < \varepsilon$ (p.501 of K.Knopp's "Infinite series", 1961).

Proof: We consider the following cases:

(i) f is a finite union of step functions: Let $0 = a_0 < a_1 < a_2 < \ldots a_{n-1} < a_n = 1$ and f be constant on the (open/closed/open-closed/closed-open) interval $< a_{k-1}, a_k >$ for $k = 1, 2, \ldots, n$ (Fig.1).



There is a continuous function $h(x)$ on $[0,1]$ with $f(x) \leq h(x)$ for any $0 \leq x \leq 1$ and $\int_0^1 (h(x) - f(x))dx < \frac{\varepsilon}{6}$. By Weierstrass approximation theorem, there is a polynomial $Q(x)$ on $[0,1]$ with $|Q(x) - h(x)| < \frac{\varepsilon}{6}$ for any $0 \leq x \leq 1$. Let $P(x) = Q(x) + \frac{\varepsilon}{6}$ for any

$0 \le x \le 1$, then $f \le P$ on $[0,1]$ and $\int_0^1 (P(x) - f(x))dx < \frac{1}{2}\varepsilon$. Similarly, we can find a polynomial $p(x)$ on $[0,1]$ with $p \le f$ on $[0,1]$ and $\int_0^1 (f(x) - p(x))dx < \frac{1}{2}\varepsilon$. Thus $\int_0^1 (P(x) - p(x))dx < \varepsilon$.

(ii) General case: By the definition of Riemann integrability, there exist two unions of step functions $S_1(x), S_2(x)$ and polynomials $p_1(x), p_2(x), P_1(x)$ and $P_2(x)$ with

$$0 \le S_1(x) \le f(x) \le S_2(x), \quad p_1(x) \le S_1(x) \le P_1(x), \quad p_2(x) \le S_2(x) \le P_2(x)$$

for any $0 \le x \le 1$, and

$$\int_0^1 (f(x) - S_1(x))dx < \frac{1}{4}\varepsilon, \int_0^1 (S_2(x) - f(x))dx < \frac{1}{4}\varepsilon;$$
$$\int_0^1 (P_1^{(x)} - p_1(x))dx < \frac{1}{4}\varepsilon, \int_0^1 (P_2(x) - p_2(x))dx < \frac{1}{4}\varepsilon.$$

Thus $\int_0^1 (P_2(x) - p_1(x))dx < \frac{1}{4}\varepsilon$.

Abel's theorem is applied to characterize the recurrency and transciency of states in a homogeneous Markov chain (Theorem II.2.1,[4]). But most important results in this subject are based on the limits of transition probabilities: $\lim_{n\to+\infty} p_{kk}^{(n)} = 1/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$ and $\lim_{n\to+\infty} p_{hk}^{(n)} = f_{hk}/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$ for any aperiodic, recurrent state k and any state h, and $\lim_{n\to+\infty} p_{kk}^{(nd)} = d/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$ for any periodic recurrent state k with period d. But the identity $\lim_{n\to+\infty} p_{hk}^{(nd)} = df_{hk}/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$ for any periodic recurrent state k with period d and any state h is not necessarily correct (cf.Theorem III. 2.3 and Exercise III.2.11,[4]).

A general identity can be obtained for all the cases by applying Theorem 3.

We let $\phi_{hk}(z) = \sum_{n=0}^{+\infty} p_{hk}^{(n)} z^n$ and $\mathcal{F}_{hk}(z) = \sum_{n=0}^{+\infty} f_{hk}^{(n)} z^n$ for any states h, k and $|z| < 1$. Thus $\phi_{hk}(z)$ and $\mathcal{F}_{hk}(z)$ are analytic functions on $|z| < 1$ with $\phi_{kk}(z) = \frac{1}{1-\mathcal{F}_{kk}(z)}$ and $\phi_{hk}(z) = \mathcal{F}_{hk}(z)\phi_{kk}(z)$ for $h \ne k$ (Proposition 4(iii) and Lemma 3(ii),[1]).

Since $f_{hk} = \sum_{n=0}^{+\infty} f_{hk}^{(n)}$ and $0 \le f_{hk} \le 1, \mathcal{F}_{hk}(z)$ converges absolutely and uniformly on $|z| \le 1$. This implies $\lim_{x\to1^-} \mathcal{F}_{hk}(x) = f_{hk}$ since $\mathcal{F}_{hk}(z)$ is the uniform limit of continuous function $\sum_{j=0}^{n} f_{hk}^{(j)} z^j$ on $|z| \le 1$.

Theorem 4. *If $\{X_m : m = 0,1,2,\ldots\}$ is a homogeneous Markov chain and h, k are states with k to be positive recurrent and $h \ne k$, then*

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{j=0}^{n} p_{kk}^{(j)} = 1/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$$

and $\lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} p_{hk}^{(j)} = f_{hk}/\sum_{n=0}^{+\infty} nf_{kk}^{(n)}$ (cf. Theorem III.2.4,[4]).

Proof: Since k is positive recurrent, $\sum_{n=1}^{+\infty} nf_{kk}^{(n)} < +\infty$ and $\sum_{n=1}^{+\infty} nf_{kk}^{(n)} z^{n-1}$ is continuous on $|z| \le 1$, and equal to $\frac{d}{dz}\mathcal{F}_{kk}(z)$ on $|z| < 1$. We note

$$\lim_{x\to1^-} (1-x) \sum_{n=0}^{+\infty} x^n \cdot p_{kk}^{(n)} = \lim_{x\to1^-} (1-x)\phi_{kk}(x) = \lim_{x\to1^-} (1-x)/(1-\mathcal{F}_{kk}(x))$$
$$= 1/\frac{d}{dx}\mathcal{F}_{kk}(x)|_{x=1} = 1/\lim_{x\to1^-} \frac{d}{dx}\mathcal{F}_{kk}(x) = 1/\sum_{n=1}^{+\infty} nf_{kk}^{(n)}.$$

Similarly,

$$\lim_{x\to1^-}(1-x)\sum_{n=0}^{+\infty}p_{hk}^{(n)}x^n=\lim_{x\to1^-}(1-x)\phi_{hk}(x)$$

$$=\lim_{x\to1^-}(1-x)\mathcal{F}_{hk}(x)\phi_{kk}(x)=f_{hk}/\sum_{n=1}^{+\infty}nf_{kk}^{(n)}.$$

Since $p_{hk}^{(n)},p_{kk}^{(n)}\geq0$ for any $n\in\mathbf{Z}+\cup\{0\}$, we prove these two identities by the second statement of Theorem 3.

Another application of Theorem 3 is the following.

Lemma 5. *Let $\{X_m:m=0,1,2,\dots\}$ be a homogeneous Markov chain and h,k be states with $h\neq k$. If $\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}p_{kk}^{(j)}=\pi_k$ exists, then $\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}p_{hk}^{(j)}=f_{hk}\pi_k$. Thus if $\lim_{n\to+\infty}p_{kk}^{(n)}=\pi_k$, then $\lim_{n\to\infty}p_{hk}^{(n)}=f_{hk}\pi_k$* ((4D) on p.220,[5]).

Proof: $\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}p_{kk}^{(j)}=\pi_k$ implies $\lim_{x\to1^-}(1-x)\sum_{n=0}^{+\infty}p_{kk}^{(n)}x^n=\lim_{x\to1^-}(1-x)\phi_{kk}(x)=\pi_k$ and $\lim_{x\to1^-}(1-x)\sum_{n=1}^{+\infty}p_{hk}^{(n)}x^n=\lim_{x\to1^-}(1-x)\phi_{hk}(x)=\lim_{x\to1^-}(1-x)\mathcal{F}_{hk}(x)\phi_{kk}(x)=f_{hk}\pi_k$.

Since $p_{kk}^{(n)}\geq0$ for any $n\in\mathbf{Z}_+$, this implies $\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}p_{hk}^{(j)}=f_{hk}\pi_k$.

We note $p_{hk}^{(n)}=\sum_{j=0}^{n}f_{hk}^{(j)}p_{kk}^{(n-j)}$ for any $n\in\mathbf{Z}_+$ ((2.1) on p.51,[4]), and $0\leq\sum_{n=0}^{+\infty}f_{hk}^{(n)}=f_{hk}\leq1$. If $\lim_{n\to+\infty}p_{kk}^{(n)}=\pi_k$, then $\lim_{n\to+\infty}p_{hk}^{(n)}=f_{hk}\pi_k$ by Theorem I.4.4,[4], or $\lim_{n\to+\infty}p_{hk}^{(n)}=\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}p_{hk}^{(j)}=f_{hk}\pi_k$.

Lemma 6. *If $\{X_m:m=0,1,2,\dots\}$ is a homogeneous Markov chain and h,k are states with $h\neq k$ and k to be not positive recurrent, then*

$$\lim_{n\to+\infty}p_{kk}^{(n)}=\lim_{n\to+\infty}p_{hk}^{(n)}=0.$$

Proof: If k is transcient, then $\sum_{n=0}^{+\infty}p_{kk}^{(n)}<+\infty$ implies $\lim_{n\to+\infty}p_{kk}^{(n)}=0$. If k is null recurrent, then we let C be a closed, irreducible set of states containing k. Thus all states in C are null recurrent, and the Markov chain with transition probabilities induced by those among the states of C is an irreducible, homogeneous Markov chain. If k is a periodic, then $\lim_{n\to+\infty}p_{kk}^{(n)}=0$ (Theorem III.2.1,[4]). If k is periodic with period d, then $\lim_{n\to+\infty}p_{kk}^{(nd)}=d/\sum_{n=0}^{+\infty}nf_{kk}^{(n)}=0$. But $p_{kk}^{(n)}=0$ for $d\nmid n$. Thus $\lim_{n\to+\infty}p_{kk}^{(n)}=0$.

For the cases in Lemma 5, we have $\lim_{n\to+\infty}\frac{1}{n+1}\sum_{j=0}^{n}p_{kk}^{(j)}=0$ and $\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}p_{hk}^{(j)}=0$.

At the end of this paper, we explain the difficulty of proving Abel's theorem :

If $a_n\in\mathbf{R}$ for any $n\in\mathbf{Z}+\cup\{0\}$ and $\sum_{n=0}^{+\infty}a_n$ converges, then $\lim_{n\to+\infty}a_n=0$ and there is an $0<k<+\infty$ with $|a_n|\leq K$ for any $n\in\mathbf{Z}+\cup\{0\}$. If $|z|<1$, then $\sum_{n=0}^{+\infty}|a_nz^n|\leq K\sum_{n=0}^{+\infty}|z|^n=\frac{K}{1-|z|}<+\infty$, and $\sum_{n=0}^{+\infty}a_nz^n$ converges. Thus the power series $\sum_{n=0}^{+\infty}a_nz^n$ has radius of convergence $R\geq1$, and converges at $z=1$. If $R>1$, then $\sum_{n=0}^{+\infty}a_nz^n$ represents an analytic function on $|z|<R$ and $\sum_{n=0}^{+\infty}a_n=\lim_{z\to1}\sum_{n=0}^{+\infty}a_nz^n$, in particular, $\sum_{n=0}^{+\infty}a_n=\lim_{x\to1^-}\sum_{n=0}^{+\infty}a_nx^n$. Thus we can not conclude $R>1$, otherwise Able's theorem becomes a trivial consequence for power series of a complex variable.

References

1. Hsiang, W.H., "A Proof of Renewal Theorem on Infinite Series," submission.
2. Hunter, J., "Mathematical Techniques of Applied Probability, vol.1," Academic Press.
3. Hyslop, J., "Infinite Series," Oliver & Boyd, 1959.
4. Issacson, D.and R.Madsen, "Markov Chains - Theory and Applications," Willey, 1976.
5. Parzen, E., "Stochastic Processes," Holden - Day, 1964.
6. Peyerimohov, A., "Lectures on Summability, III," Lecture Notes in Math. v.107, Springer, 1969.
7. Wiener, N., "The Fourier Integral and Certain of Its Applications," Dover, 1959.