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Central Simple Algebras and Invariant Polynomials

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1. Introduction

The following fact is well-known.

If $f : M_n(R) \to R$ is a polynomial function such that $f(xy) = f(yx)$ for all $x, y \in M_n(R)$, then $f = g(trx, trx^2, \ldots, trx^n)$ for some $g \in R[x_1, \ldots, x_n]$, where R is real.

In this paper, we shall generalize this fact. All algebras are finite dimensional in this paper.

2. Main theorem

Theorem. *Let A be an algebra over a field K. Let $C_A = [A, A]$ be the space spanned by $\{xy - yx | x, y \in A\}$. Then A is central simple if and only if* $\dim_K(A/C_A) = 1$.

Proof: Suppose that A is central simple over K. Let $\bar{K}$ be the algebraic closure of K. Then $A \otimes_K \bar{K}$ is a central simple $\bar{K}$-algebra and hence $A \otimes_K \bar{K} = M_n(\bar{K})$ by Wedderburn's theorem. This implies that $\dim_K(A/C_A) = 1$. Conversely suppose that $\dim_K(A/C_A) = 1$. We may assume that K is algebraically closed. Consider the algebra $A/J(A) = B$. where $J(A)$ is the Jacobson radical of A. Since B is a semi-simple algebra with $\dim_K(B/C_B) = 1$,

$$B = M_n(K)$$

by Wedderburn's theorem. Hence there is a local algebra D such that

$$A = M_n(D) = M_n(K) \otimes_K D.$$

Clearly

$$\dim_K(D/C_D) = 1.$$

Now we have $D/J(D) = K$ and hence $\dim_K(J(D)) = \dim_K D - 1 = \dim_K(C_D)$.
Since every element of $J(D)$ is nilpotent, we have $J(D) \subset C_D$. Hence $J(D) = C_D = [D, D]$. Now we have $D = K \oplus J(D)$. Hence $J(D) = [J(D), J(D)]$. Hence the n-th derived Lie-algebra of $J(D)$ is $J(D)^{[n]} = J(D)$. But since $J(D)$ is an ideal, we have $J(D)^{[n]} \subset J(D)^{2^n}$. Since $J(D)$ is a nilpotent ideal, we have $J(D)^{2^n} = 0$ for some n. Hence $J(D) = 0$ and hence $D = K$.

Corollary: Let A be an algebra such that $\dim_K(A/C_A) = 1$. If $f : A \to K$ is a polynomial function such that

$$f(xy) = f(yx)$$

for all $x, y \in A$, then $f(x) = g(\mathrm{trx}, \mathrm{trx}^2, \ldots, \mathrm{trx}^n)$ for some $g \in K[x_1, \ldots, x_n]$, where $tr : A \to K$ is a non-zero linear map, unique up to constant multiple, such that $f(xy) = f(yx)$ for all $x, y \in A$ and $n = \deg A$.

References

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