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# On Common Fixed Points of Expansive Mappings

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ABSTRACT. S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iséki proved some fixed point theorems on expansion mappings, which correspond some contractive mappings. In a recent paper, B. E. Rhoades generalized the results for pairs of mappings.

In this paper, we obtain the following theorem, which generalizes the result of B. E. Rhoades.

THEOREM. *Let A, B, S and T be mappings from a complete metric space $(X,d)$ into itself satisfying the following conditions:*

(1) $\phi(d(Ax,By)) \geq d(Sx,Ty)$ *holds for all $x$ and $y$ in $X$, where* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-decreasing, uppersemicontinuous and* $\phi(t) < t$ *for each* $t > 0$,

(2) *A and B are surjective,*

(3) *one of A, B, S and T is continuous, and*

(4) *the pairs A, S and B, T are compatible.*

*Then A, B, S and T have a unique common fixed point in X.*

## 1. Introduction

B. E. Rhoades [4] summarized contractive mappings of some types and discussed on the fixed points. S. Z. Wang, B. Y. Li, Z. M. Gao and K. Iséki [7] proved some fixed points theorems on expansion mappings, which correspond some contractive mappings in [4]. In a recent paper [5], B. E. Rhoades generalized the results of [7] for pairs of mappings. On the other hand, G. Jungck [3] introduced the concept of compatible mappings, as a generalization of commuting mappings.

In this paper, we give a common fixed point theorem for expansive mappings using compatible mappings, which generalizes the result of B. E. Rhoades [5].

The following are given in G. Jungck [3].

**Definition 1.1:** Let $A$ and $B$ be mappings from a metric space $(X,d)$ into itself. Then $A$ and $B$ are said to be *compatible* if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point $t$ in $X$.

Thus, if $d(ABx_n, BAx_n) \to 0$ as $d(Ax, Bx) \to 0$, then $A$ and $B$ are compatible.

Commuting mappings are clearly compatible, but the converse is not necessarily true.

**Lemma 1.2.** *Let $A$ and $B$ be compatible mappings from a metric space $(X,d)$ into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some point $t$ in $X$. Then $\lim_{n\to\infty} BAx_n = At$ if $A$ is continuous.*

## 2. A Fixed Point Theorem

Throughout this paper, let $\mathbf{R}^+$ be the non-negative real numbers. Following D. W. Boyd and J. S. W. Wong [1], let $\Phi$ denote the family of all real functions $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ with the following condition: (1) $\phi$ is non-decreasing, upper-semicontinuous and $\phi(t) < t$ for each $t > 0$.

**Lemma 2.1** ([3]). *Let $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be a function with the condition* (1). *Then* $\lim_{n\to\infty} \phi^n(t)$
$= 0$, *where $\phi^n(t)$ denotes the composition of $\phi(t)$ with itself n-times.*

Now, let $A, B, S$ and $T$ be mappings from a metric space $(X, d)$ into itself such that

$$\text{(2)} \qquad A \text{ and } B \text{ are surjective}$$

and

$$\text{(3)} \qquad \phi(d(Ax, By)) \geq d(Sx, Ty)$$

for all $x, y \in X$, where $\phi \in \Phi$. Then for arbitrary $x_0$ in $X$, by (2), we choose $x_1$ in $X$ such that $Ax_1 = Tx_0 = y_0$ and, for this a point $x_1$, there exists a point $x_2$ in $X$ such that $Bx_2 = Sx_1 = y_1$. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$$\text{(4)} \qquad Ax_{2n=1} = Tx_{2n} = y_{2n} \text{ and } Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}.$$

We then have

**Lemma 2.2.** *The sequence $\{y_n\}$ in $X$ defined by (4) is a Cauchy sequence.*

**Proof:** By (3) and (4), we have

$$\phi(d(y_0, y_1)) = \phi(d(Ax_1, Bx_2)) \geq d(y_1, y_2).$$

Similarly, we obtain

$$d(y_2, y_3) \leq \phi(d(y_1, y_2)) \leq \phi^2(d(y_0, y_1)).$$

In general, we have

$$d(y_n, y_{n+1}) \leq \phi(d(y_{n-1}, y_n)) \leq \cdots \leq \phi^n(d(y_0, y_1)).$$

By Lemma 2.1, we obtain

$$\text{(5)} \qquad \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

We shall prove that $\{y_n\}$ is a Cauchy sequence. In virtue of (5), it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ such that

$$\text{(6)} \qquad d(y_{2m(k)}, y_{2n(k)}) > \epsilon \quad \text{for} \quad 2m(k) > 2n(k) > 2k.$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (6), that is,

$$d(y_{2m(k)}, y_{2m(k)-2}) \leq \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon.$$

Then for each even integer $2k$,

$$\begin{aligned}\epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})\end{aligned}$$

implies, from (5),

$$\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon.$$

From the triangular inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)-1}, y_{2m(k)}).$$

Thus, we obtain, as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon$$

By (3) and (4), we have

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(y_{2n(k)}, y_{2m(k)-1})).\end{aligned}$$

By upper-semicontinuity of $\phi$, we have

$$\epsilon \leq \phi(\epsilon) \text{ as } k \to \infty,$$

yielding a contradiction.

Using the above lemma, we obtain the following.

**Theorem 2.3.** *Let $A, B, S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions* (2) *and* (3) *holds for all $x, y \in X$, where $\phi \in \Phi$. Further, if*

(7) *one of $A, B, S$ and $T$ is continuous*

*and*

(8) *the pairs $A, S$ and $B, T$ are compatible,*

*then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

**Proof:** By Lemma 2.2, $\{y_n\}$ is a Cauchy sequence and it converges to some point $z$ in $X$. Consequently, the subsequences $\{Ax_{2n+1}\}, \{Bx_{2n}\}, \{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ converges to $z$.

New, suppose that $A$ is continuous. Since $A$ and $S$ are compatible, Lemma 1.2 implies

$$A^2x_{2n+1} \text{ and } SAx_{2n+1} \to Az.$$

By (3), we have

$$\phi(d(A^2x_{2n+1}, Bx_{2n})) \geq d(SAx_{2n+1}, Tx_{2n}).$$

Letting $n \to \infty$, we have

$$\phi(d(Az, z)) \geq d(Az, z),$$

so that $Az = z$. By (3), we also obtain

$$\phi(d(Az, Bx_{2n})) \geq d(Sz, Tx_{2n}),$$

which implies $z = Sz$. Let $z = Bv$ for some $v$ in $X$. Then we have

$$\phi(d(A^2x_{2n+1}, Bv) \geq d(SAx_{2n+1}, Tv),$$

Letting $n \to \infty$, we obtain

$$\phi(d(Az, Bv) \geq d(Az, Tv),$$

so that, $z = Tv$. Since $B$ and $T$ are compatible and $Bv = Tv = z, d(BTv, BTv) = 0$ and have $Bz = BTv = TBv = Tz$. Moreover, by (3), we have

$$\phi(d(Ax_{2n+1}, Bz)) \geq d(Sx_{2n+1}, Tz),$$

which implies that $z = Tz$. Similarly, we can complete the proof in the case of the continuity of $B$.

Next, suppose that $S$ is continuous. Since $A$ and $S$ are compatible. Lemma 1.2 implies

$$S^2x_{2n+1} \text{ and } ASx_{2n+1} \to Sz.$$

By (3), we have

$$\phi(d(ASx_{2n+1}, Bx_{2n})) \geq d(S^2x_{2n+1}, Tx_{2n}).$$

Letting $n \to \infty$, we obtain $z = Sz$. Let $z = Av$ and $z = Bw$ for some $v, w \in X$. Then

$$\phi(d(ASx_{2n+1}, Bw)) \geq d(S^2x_{2n+1}, Tw),$$

which implies that $z = Tw$. Since $B$ and $T$ are compatible and $Bw = Tw = z, d(TBw, BTw) = 0$ and hence $Bz = BTw = TBw = Tz$. Moreover, by (3), we have

$$\phi(d(Ax_{2n+1}, Bz)) \geq d(Sx_{2n+1}, Tz)$$

which implies $z = Tz$. Further, we have

$$\phi(d(Av, Bz) \geq d(Sv, Tz),$$

so that $z = Sv$. Since $A$ and $S$ are compatible and $Av = Sv = z, d(SAv, ASv) = 0$ and hence $Az = ASv = SAv = Sz$. therefore, $z$ is a common fixed point of $A, B, S$ and $T$. Similarly, we can complete the proof in the case of the continuity of $T$.

It follows easily from (3) that $z$ is a unique common fixed point of $A, B, S$ and $T$.

The following corollary follows from Theorem 2.3 by assuming $\phi(t) = \frac{1}{h}t$ for all $t$ in $\mathbf{R}^+$, where $h > 1$.

**Corollary 2.4.** *Let $A, B, S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions* (2), (7), (8) *and* (9);

$$d(Ax, By) \geq hd(Sx, Ty)$$

*for all $x, y \in X$, where $h > 1$. Then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

The following example shows that Theorem 2.3 is stronger result than Corollary 2.4. The idea of this example appears in $S$. Sessa and B. Fisher [6]

**Example 2.5:** Let $X = [0, \frac{1}{2}]$ with the Euclidean metric $d$. Define $A, B, S$ and $T : X \to X$ by

$$Ax = \frac{1}{2}x, \quad Bx = b, \quad Sx = \frac{1}{2}x - \frac{1}{8}x^2 \quad \text{and} \quad Tx = x - \frac{1}{2}x^2$$

for all $x$ in $X$. Then it is easily seen that the pair $A, S$ and $B, T$ are compatible. Consider

$$\phi(t) = \begin{cases} t - \frac{1}{2}t^2 & 0 \leq \ t \leq 1 \\ \frac{1}{2}t & \quad t > 1. \end{cases}$$

for all $t$ in $\mathbf{R}^+$. Then $\phi \in \Phi$. Further, we have

$$\begin{aligned} \phi(d(Ax, By)) &= \left|\frac{1}{2}x - y\right| \left(1 - \frac{1}{2}\left|\frac{1}{2}x - y\right|\right) \\ &\geq \left|\frac{1}{2}x - y\right| \left|1 - \frac{1}{2}\left(\frac{1}{2}x - y\right)\right| \\ &= d(Sx, Ty) \end{aligned}$$

for all $x, y \in X$. All assumptions of Theorem 2.3 are therefore satisfied.

However, the condition (9) is not satisfied. Indeed, for $x = 0, 0 < y \leq \frac{1}{2}$ and $h > 1$,

$$d(A0, By) = y \geq hd(S0, Ty) = h\left(y - \frac{1}{2}y^2\right).$$

This implies that $1 \geq h$, which yields a contradiction.

**Remark 2.6:** If $S$ and $T$ are the identity mapping on $X$, then Corollary 2.4 becomes a result of B. E. Rhoades [5].

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