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Lift and Drag of a Circular Cylinder by the Discrete Vortex Method

by

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이산 보오텍스법에 의한 원주의 양력 및 항력

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Abstract

Expressions for the lift and the drag exerted on a circular cylinder by an unsteady flow of an ideal fluid with embedded discrete vortices are derived. The formulae can be used in the discrete vortex method of flow simulation. These formulae are derived via contour integration on the complex plane. Terms have been produced which are significantly different from those in Sarpkaya's formulae. These are expected to bring a change to the forces obtained so far.

요 약

이산 보오텍스를 내포하고 있는 이상유체의 비정상 유동에 의하여 원주에 작용되는 양력과 항력을 계산하기 위한 수식을 유도하였다. 이 식은 유체유동에 대한 시뮬레이션에서 이산 보오텍스법을 적용할 때 사용될 수 있으며 복소평면 위에서 연결선 적분에 의하여 유도되었다. Sarpkaya의 공식과는 의미에 상당한 차이가 있는 항이 나와 있어 지금까지 얻어진 힘의 크기에 변화를 초래할 것으로 예상된다.

Notations

i : the imaginary unit
: meaning image vortex when used as a subscript

$U(t)$: time-dependent free stream velocity, assumed to be in the positive x-direction

a : radius of the circular cylinder

$N(t)$: number of the discrete vortices shed up to the time t

$z_k(t)$: position of the k-th vortex at the time t

Γ_k : strength of the k-th vortex (positive anticlockwise)

u, v : x- and y- components of velocity respectively

X, Y : x- and y- components of force exerted on a unit length of the cylinder by the fluid respectively

ρ : density of the fluid

n : number of the nascent vortices
: meaning nascent vortex when used as a subscript

Γ_{nk} : strength of the k-th nascent vortex

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z_{ink} : position of the image of the k-th nascent vortex, $z_{ink}=x_{ink}+iy_{ink}$

(r_{nk}, θ_{nk}) : position of the k-th nascent vortex (polar description)

u_k, v_k : x- and y- components of the velocity of the k-th vortex respectively

u_{ik}, v_{ik} : x- and y- components of the velocity of the image of the k-th vortex respectively

D, L : drag and lift exerted on a unit length of the cylinder by the fluid respectively

1. Introduction

In recent decades, simulation of fluid flow by means of discrete vortices has attracted increasing attention, and has been developed in the case of a two-dimensional problem, to a stage of wide application. The region of rotational flow (i.e. the boundary layer and the wake) is approximated by a number of lumped discrete vortices replacing, thereby, the real flow by the flow of an ideal fluid with embedded discrete point vortices. The force exerted on the body by the fluid can be obtained by the application of the Bernoulli's theorem in this flow model.

A specific expression for the force in terms of the relevant parameters can be derived in the case of a circular cylinder placed in a spatially uniform but unsteady flow. Sarpkaya dealt with this problem twice.[1, 2] In his former work, he obtained the formulae by the use of the Blasius' theorem extended by Milne-Thomson. The formulae are elegant and convenient for use. However, one term concerned with the growth rate of vortices is not clear and most users of these formulae seem to ignore it on account of constancy of vortex strength, once it is shed in an ideal fluid. In his latter work, he tried to clarify this term rectifying the concept of growth by confining this concept to the nascent vortices only. In the process of development, however, the integral of complex logarithmic function was not treated properly. The contour integral was integrated along what he called the feeding zone only, leaving, consequently, the parameters concerned with this zone in the final expression for the force. The definition of feeding zone might be useful if one wishes to relate the role of the nascent vortices with the separation process but as far as the integration is concerned one need not rely on such an artifice.

This problem has been once more dealt with in the present paper. The nascent vortices are supposed to represent the generation of vorticity at, or the shedding of vortices from, the surface of the cylinder. How the flow phenomena are modelled by them is not directly relevant although the concept of a multiple, nascent vortex representation of the boundary layer has been suggested. The major term at issue, which is the time-derivative of the integral of the complex potential, has been treated in a coherent context.

2. Derivation of the expression for the force

A circular cylinder of radius a is supposed to be placed in a time-dependent spatially uniform flow of incompressible inviscid fluid. A frame of Cartesian coordinates, with its origin at the centre of the cylinder and the x-axis parallel to the uniform flow, is employed for the description of the flow (Fig. 1). Suppose there are a number of discrete vortices around the cylinder whose strengths, instantaneous positions and instantaneous velocities are assumed to be known. The number should be taken as a function of time, increasing by a fixed number, say n, at a time interval (however small this interval may be). This increment denotes the number of so-called nascent vortices introduced on every time step as usual in the discrete vortex method. The normal boundary condition is satisfied if the image vortices, whose strengths and positions are determined by Milne-Thomson's circle theorem,[3] are imposed. Then the complex potential describing the flow field will be

$$w(z,t)=U(t)\left(z+\frac{a^2}{z}\right)-\frac{i}{2\pi}\sum_{k=1}^{N(t)}\Gamma_k(\log[z-z_k(t)]-\log[z-z_{ik}(t)])-\frac{i}{2\pi}\Gamma_\Sigma(t)\log z \qquad (1)$$

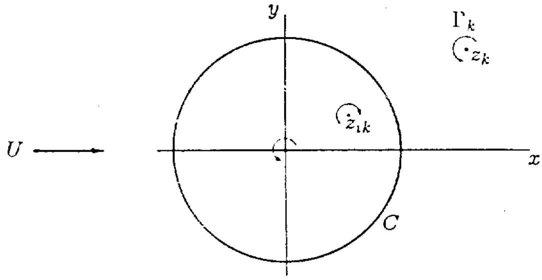


Fig. 1 A circular cylinder in a uniform flow

where

$$z_{ik}(t)=\frac{a^2}{\bar{z}_k(t)} \text{ : position of the image of the } k\text{-th vortex} \quad (2)$$

$$\Gamma_\Sigma(t)=\sum_{k=1}^{N(t)}\Gamma_k \quad (3)$$

The velocity at an arbitrary, nonsingular field point is given in conjugate form by the derivative of this complex potential

$$u(z,t)-iv(z,t)=\frac{dw}{dz}=U\Big(1-\frac{a^2}{z^2}\Big) -\frac{i}{2\pi}\sum_{k=1}^{N}\Gamma_k\Big(\frac{1}{z-z_k}-\frac{1}{z-z_{ik}}\Big)-\frac{i\Gamma_\Sigma}{2\pi z} \quad (4)$$

The velocity by which the m-th vortex moves is

$$u_m(t)-iv_m(t)=U\Big(1-\frac{a^2}{z_m^{\,2}}\Big) -\frac{i}{2\pi}\Big[\sum_{\substack{k=1\\k\neq m}}^{N}\Gamma_k\Big(\frac{1}{z_m-z_k}-\frac{1}{z_m-z_{ik}}\Big) -\frac{\Gamma_m}{z_m-z_{im}}\Big]-\frac{i\Gamma_\Sigma}{2\pi z_m} \quad (5)$$

The extended Blasius theorem appears, in our particular problem, as

$$X-iY=\frac{1}{2}i\rho\oint_C\Big(\frac{dw}{dz}\Big)^2dz+i\rho\frac{\partial}{\partial t}\oint_C\bar{w}d\bar{z} \quad (6)$$

where C : contour of the circle

The first integral of eq. (6) can, according to the Cauchy-Goursat theorem, be evaluated from

$$X_1-iY_1=\frac{1}{2}i\rho\oint_C\Big(\frac{dw}{dz}\Big)^2dz =\frac{1}{2}i\rho\Big[\oint_{C_\infty}\Big(\frac{dw}{dz}\Big)^2dz-\sum_{k=1}^{N}\oint_{C_k}\Big(\frac{dw}{dz}\Big)^2dz\Big] \quad (7)$$

where

C_∞ : contour of large radius enclosing all the singularities

C_k : contour of small radius enclosing the k-th vortex only

Since,

$$\frac{1}{z-z_k}-\frac{1}{z-z_{ik}}=\frac{z_k-z_{ik}}{(z-z_k)(z-z_{ik})}\to O(|z|^{-2}) \quad \text{as} |z|\to\infty \quad (8)$$

$$\Big(\frac{dw}{dz}\Big)^2=U^2-\frac{iU\Gamma_\Sigma}{\pi z}+O(|z|^{-2}) \quad \text{as} |z|\to\infty \quad (9)$$

we obtain

$$\oint_{C_\infty}\Big(\frac{dw}{dz}\Big)^2dz=2U\Gamma_\Sigma \quad (10)$$

To deal with the second term in the brackets of eq. (7), let us consider the contour C_m enclosing the m-th vortex. For any z at the neighbourhood of z_m, we can write,

$$\Big(\frac{dw}{dz}\Big)^2=-\frac{\Gamma_m^{\,2}}{4\pi^2(z-z_m)^2}-\frac{i\Gamma_m}{\pi}\Big\{U\Big(1-\frac{a^2}{z^2}\Big) -\frac{i}{2\pi}\Big[\sum_{\substack{k=1\\k\neq m}}^{N}\Gamma_k\Big(\frac{1}{z-z_k}-\frac{1}{z-z_{ik}}\Big)-\frac{\Gamma_m}{z-z_{im}}\Big] -\frac{i\Gamma_\Sigma}{2\pi z}\Big\}\frac{1}{z-z_m}+h(z) \quad (11)$$

where $h(z)$ is an analytic function on, and inside, the contour C_m. Note that the expression within the curly brackets tends to the velocity at the position of the m-th vortex (i.e. u_m-iv_m) as $z\to z_m$. The use of the Cauchy integral formula then yields

$$\oint_{C_m}\Big(\frac{dw}{dz}\Big)^2dz=2\Gamma_m(u_m-iv_m) \quad (12)$$

Collecting these results, we have

$$X_1-iY_1=-\rho\sum_{m=1}^{N}\Gamma_m v_m+i\rho\sum_{m=1}^{N}\Gamma_m(U-u_m) \quad (13)$$

Since the integrand of the second term on the right hand side of eq. (6) does not possess the property shown in eq. (8), the choice of contour should be sought on, or within, the circle as appropriate for the individual case. Let us denote this term as X_2-iY_2, but instead, for convenience, evaluate the conjugate of this term as follows:

$$X_2+iY_2=-i\rho\frac{\partial}{\partial t}\oint_C w dz \quad (14)$$

In the process of manipulating this expression, the following points should be borne in mind:

1. The total number of vortices changes with time to take into account the newly shed vortices in the time interval, Δt, the increase being equal to the number of nascent vortices;

2. The strength of each vortex shed previously is

an invariant quantity;

3. The contour surrounding an image vortex moves with it so that the contour encloses the concerned image vortex only all the time.

Then the terms in eq. (1) will appear as the following when they are inserted in eq. (14).

$$\frac{\partial}{\partial t}\oint_C U(t)\left(z+\frac{a^2}{z}\right)dz=2\pi i a^2\frac{dU}{dt} \tag{15}$$

$$\frac{\partial}{\partial t}\oint_C \sum_{k=1}^{N(t)}\Gamma_k \log[z-z_k(t)]dz=0 \tag{16}$$

$$\frac{\partial}{\partial t}\oint_C \sum_{k=1}^{N(t)}\Gamma_k \log[z-z_{ik}(t)]dz$$

$$=-2\pi i\sum_{k=1}^{N}\Gamma_k(u_{ik}+iv_{ik})$$

$$+2\pi i\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}(ae^{i\theta_{nk}}-z_{ink}) \tag{17}†$$

$$\frac{\partial}{\partial t}\oint_C \Gamma_\Sigma(t)\log z dz=2\pi i\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}ae^{i\theta_{nk}} \tag{18}†$$

where

$$\frac{dz_{ik}}{dt}=u_{ik}+iv_{ik}=-\frac{a^2}{\bar{z}_k^2}(u_k-iv_k) \tag{19}$$

in which $d\Gamma_{nk}/dt$ denotes the rate of growth in strength of the k-th nascent vortex Involvement of this growth rate makes the present force formulae different from those obtained by Sarpkaya. For a vortex outside the cylinder, there are two corresponding vortices, one the image vortex (determined by the circle theorem) and the other the vortex of the same strength at the origin (to satisfy the far field condition). These three vortices constitute a set. The same branch cut for the three vortices in any one set should be used for the evaluation of the integrals of logarithmic functions in eq.(17) and eq. (18). The simplest choice is the radial line stretching to infinity from the branch points, that is, $\theta=\theta_{nk}$ for the k-th nascent vortex set (Fig. 2).

Inserting the expressions eq. (15) - eq. (14), we have

$$X_2+iY_2=\rho\Big((2\pi a^2\frac{dU}{dt}+\sum_{k=1}^{N}\Gamma_k v_{ik}$$

$$+\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}y_{ink}\Big)-i\rho\Big(\sum_{k=1}^{N}\Gamma_k u_{ik}$$

$$+\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}x_{ink}\Big) \tag{20}$$

and finally

$$D=X_1+X_2=\rho\Big[2\pi a^2\frac{dU}{dt}$$

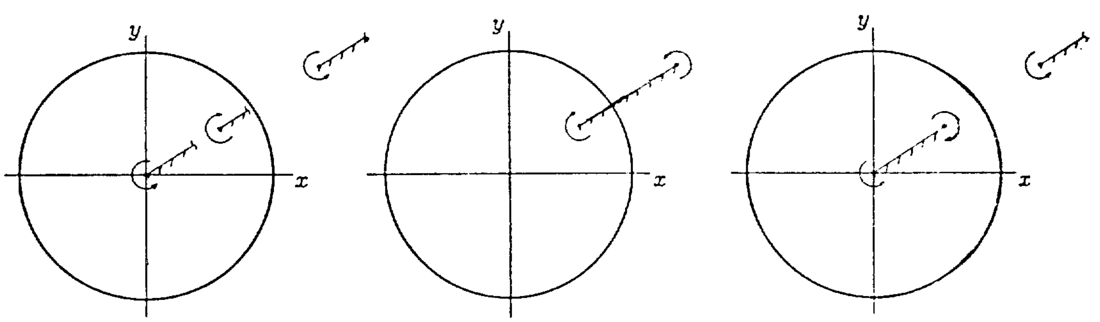
$$+\sum_{k=1}^{N}\Gamma_k(v_{ik}-v_k)+\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}y_{ink}\Big] \tag{21}$$

$$L=Y_1+Y_2=-\rho\Big[\sum_{k=1}^{N}\Gamma_k(U+u_{ik}-u_k)$$

$$+\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}x_{ink}\Big] \tag{22}$$

It is to be noted that N represents the number of all the vortices introduced up to the concerned time, hat is, the number of vortices shed already plus the number of nascent vortices just introduced.

The strength of nascent vortices is determined



(i) individual branch cut (ii) effective branch cut for the vortex & its image only (relevant to eq.(17)) (iii) effective branch cut for the set

Fig. 2 The branch cuts

† see the Appendix.

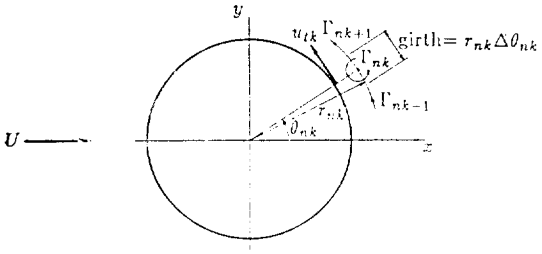


Fig. 3 The tangential velocity and the strength of a nascent vortex

frequently so that the no slip condition is satisfied on the cylinder surface. However, the way of satisfying the condition can differ depending on the flow simulation model and the analysis technique employed by the investigator to tackle a particular problem. The model[4] of fixed separation point with one nascent vortex on each side of the cylinder and the model[5] of multi-nascent vortices equally spaced around the cylinder are typical examples. An alternative way of determining the strength of nascent vortex is use of the concept of vorticity shedding rate as can be found in, for instance, Clements' work[6]. In either of the ways, it should be possible to define the growth rate of the strength of nascent vortex, which can be expressed usually in terms of the rate of change of the tangential velocity on the cylinder surface. As an example, take the multi-nascent vortex model in which each strength is set equal to the local instantaneous tangential velocity on the cylinder surface multiplied by the girth of the segment which the nascent vortex is to represent, i.e.

$$\Gamma_{nk}=u_{tk}r_{nk}\Delta\theta_{nk} \text{ (see Fig. 3)} \tag{23}$$

Incidentally, it can be shown[7] that this way of setting the strength of nascent vortices can satisfy the no slip condition exactly for a distribution of tangential velocity of the form $u_{tk}\propto\sin\,\theta_k$ (i.e. the free stream over a circular cylinder), and approximately for a distribution of arbitrary from[8]. Now if eq.(23) is accepted for determination of the strength of nascent vortices,

$$\frac{d\Gamma_{nk}}{dt}=r_{nk}\Delta\theta_{nk}\frac{du_{tk}}{dt} \tag{24}$$

where du_{tk}/dt is to be interpreted as the rate of variation of the tangential velocity at the point corresponding to the k-th nascent vortex due to the motion of vortices shed already and due to the change of the velocity of the free stream. It can be calculated from

$$\begin{aligned}\frac{du_{tk}}{dt}&=-Im\left(\left[\frac{d}{dt}(u-iv)\right]_{z=ae^{i\theta_{nk}}}e^{i\theta_{nk}}\right)\\&=-2\frac{dU}{dt}\sin\theta_{nk}\\&\quad+Im\left(\frac{i}{2\pi}e^{i\theta_{nk}}\sum_{m=1}^{N}\Gamma_m\left[\frac{u_m+iv_m}{(ae^{i\theta_{nk}}-z_m)^2}\right.\right.\\&\quad\left.\left.-\frac{u_{im}+iv_{im}}{(ae^{i\theta_{nk}}-z_{im})^2}\right]\right)\end{aligned} \tag{25}$$

Eq.(24) in connection with eq.(25) can be used for the rate of change of the nascent vortex in eq.(21) and eq.(22) to produce the drag and the lift.

3. Conclusion

In Sarpkaya's work[1], a term in the expression for the drag or the lift is given as the sum of image vortex coordinates multiplied by the rate of change of their strengths. This term can have a non zero value 1) when the strength of the vortex changes with time due to, for instance, diffusion or 2) if some vortices are artificially removed from the flow field which can be taken accounted of as the rate of vortex decay.

It is frequently taken for granted that the fluid is an ideal one in the application of the discrete vortex method to simulate a real flow. The reason is that this assumption is logically in harmony with the existing set of rules about vortex kinematics. Although incorporation of the diffusion effect into the analysis by the method is sometimes tried, the strength of any individual vortex is rarely allowed to change. Hence, the frequent neglect of the term by the users of the formulae may be quite excusable due to the constancy of vortex strength.

The fact that the meaning of this term should be sought from the nascent vortices only, was admitted in Sarpkaya's later work[2], though this interpretation was not implied in the former. This development was an improvement but unfortunately the derivation was again hampered by incorrect evaluation of the contour integral producing a formula not readily acceptable by the users. It appears to be quite fair to say that his former formulae are more frequently used than his latter one just because of the complicated appearance of his feeding zone parameters.

It has been shown in the present paper that a term to account for the effect of continuous creation of vorticity at the cylinder surface should be included in the force calculation. This term is missing in Sarpkaya's former study and is given, but with an unnecessary extra term, in his latter study. The term can produce a significant change to the values of the force obtained by the use of his former, or latter, formulae. To be consistent with the usual postulation in the discrete vortex method, variation of vortex strength, once the vortex is shed, is precluded in the present study.

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Appendix

A1. Time-derivative of the contour integral of the complex logarithmic function

$$\frac{\partial}{\partial t}\oint_C \sum_{k=1}^{N(t)} \Gamma_k \log[z-z_{ik}(t)]dz = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\oint_C \sum_{k=1}^{N}\Gamma_k\log[z-z_{ik}(t+\Delta t)]dz\right.$$

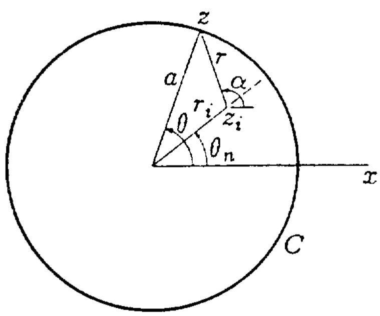
$$\left. + \oint_C\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}\Delta t\log(z-z_{ink})dz - \oint_C\sum_{k=1}^{N}\Gamma_k\log[z-z_{ik}(t)]dz\right\}$$

$$=\sum_{k=1}^{N}\Gamma_k\left(-\frac{dz_{ik}}{dt}\oint_C\frac{dz}{z-z_{ik}}+\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}\oint_C\log(z-z_{ink})dz\right.$$

$$=-2\pi i\sum_{k=1}^{N}\Gamma_k(u_{ik}+iv_{ik})+2\pi i\sum_{k=1}^{n}\frac{d\Gamma_{nk}}{dt}(ae^{i\theta_{nk}}-z_{ink})$$

which is eq.(17).

A2. The contour integral of the complex logarithmic function



From this diagram

$$z-z_i=re^{i\alpha}\quad\text{where } r(\theta)=[a^2+r_i^2-2ar_i\cos(\theta-\theta_n)]^{1/2}$$

$$\alpha(\theta)=\sin^{-1}\left[\frac{a}{r}\sin(\theta-\theta_n)\right]+\theta_n,\qquad 0\leqq\alpha<2\pi$$

It is to be noted that α and θ are of one-to-one correspondence and that, when θ increases by 2π, so does α.

Since a complex logarithmic function is multi-valued, let us take a branch cut at $\alpha=\alpha_0$ and let $\theta=\theta_0$ correspond to $\alpha=\alpha_0$. Then,

$$\oint_C\log(z-z_i)dz=\Big[(z-z_i)\log(z-z_i)-(z-z_i)\Big]_{ae^{i\theta_0}}^{ae^{i(\theta_0+2\pi)}}$$

$$=\Big[(ae^{i\theta}-z_i)(\log r(\theta)+i\alpha(\theta)+2k\pi i)-(ae^{i\theta}-z_i)\Big]_{\theta_0,}^{\theta_0+2\pi}$$

where k: arbitrary integer

$$=(ae^{i\theta_0}-z_i)[\log r(\theta_0+2\pi)-\log r(\theta_0)+i\alpha(\theta_0+2\pi)-i\alpha(\theta_0)]$$

$$=2\pi i(ae^{i\theta_0}-z_i)$$

since(by examination of the diagram),

$$r\theta_0+2\pi)=r(\theta_0)$$

$$\alpha(\theta_0+2\pi)=\alpha(\theta_0)+2\pi$$

If the branch cut is taken at $\alpha=\theta=\theta_n$, by putting $\theta_0=\theta_n$ in the above equations, the right hand side of eq.(18) and the second term of the right hand side of eq.(17) are obtained.