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특이값 접근방법에 의한 다단 정현파 수의 결정에 관한 연구

Determination of the Number of Multiple Sinusoids by a Singular Value Approach

安 泰 天* · 柳 創 善** · 李 祥 宰***
(Tae-Chon Ahn · Chang-Sun Ryu · Sang-Jae Lee)

요 약

유한 잡음 측정값으로 부터 다단 정현파의 수를 결정하기 위한 특이값 접근 방법이 제시된다. 여러가지의 다른 SNR의 보기와 주파수 추정 방법에서 Akaike의 AIC, Rissanen의 MDL 및 특이값 접근 방법에 대한 시뮬레이션을 수행한다. 그리고 성능을 비교한다. 특이값 접근 방법이 FBLP, HOYW 및 공분산행렬 방법에서 AIC와 MDL보다 우수하다는 시뮬레이션 결과가 조사 된다.

이 방법은 유한 잡음 측정값으로 부터 다단 정현파의 주파수 추정에 공헌할 것이고, 더 나아가 통신 및 의용생체 공학의 DSP에도 응용될 것이다.

Abstract- A singular value approach is presented in order to determine the number of multiple sinusoids from the finite noisy data. Simulations are conducted for Akaike's information criterion(AIC), Rissanen's shortest data description(MDL) and a singular value approach, for various examples with different SNR's and methods of estimating frequencies. And then the performances are compared. Simulation results that the singular value approach is superior to AIC and MDL for FBLP, HOYW and covariance matrix based methods are investigated.

The approach will contribute to the frequency estimation of multiple sinusoids from the finite noisy data. Furthermore, this will be applied to the DSPs of communication and bio-medical engineering.

1. Introduction

Estimation of multiple sinusoidal frequencies from finite noisy data is very interisting and practical problem. It has been studied and used in many fields[8][13]. With the rapid development of modern technology, the need for estimation of frequencies becomes ever increasing and therefore motivates more and more researchers on this issue.

It is important to determine the number of multiple sinusoids before estimating the fre-

*正 會 員 : 圓光大 工大 制御計測工學科 副敎授 · 工博
**正 會 員 : 圓光大 大學院 電氣工學科 碩士課程
***正 會 員 : 金鳴工大 電子制御工學科 助敎授 · 工博
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quencies. Generally it should be assumed that the number of multiple sinusoids in the signal was known. In practice, however, since this information is often unknown, it must be determined before estimating the frequencies. This is a difficult problem, particularly for short data.

A popular approach for selection of model order is, of course, the information theoretic criteria, introduced by Akaike[1][2] and Rissanen.[3] Wax and Kalaith[5] have extended them to the sinusoidal case. Chatterjee et al.[6] have recently proposed a test rule for a generalized autoregressive(GAR) model, based on Bayesian approach, while Satorius and Alexander[7] used the determinant test on the covariance matrix. These methods are complex and don't exactly select the model order and the number of frequencies. A simple and powerful method is needed. A new idea is a singular value approach to examine the singular values of certain matrices by distinguished the signal singular value from noise singular values.

In the paper, a singular value approach applied to the sinusoidal case is presented and investigated. Using the approach, the number of multiple sinusoids is to be determined from finite noisy data. Simulations are conducted for Akaike's information criterion(AIC), Rissanen's shortest data description(MDL) and a singular value approach, in various examples of different SNR's and methods of estimating frequencies. And then performances are compared.

2. Problem Formulation

Consider the following sinusoidal signal

$$x(t)=\sum_{i=1}^{m}\alpha_i\sin(\omega_i t+\varphi_i) \tag{1}$$

where α_i, $\varphi_i \epsilon R$, $\omega_i \epsilon (0, \pi)$ and $\omega_i \neq \omega_j$ for $_i \neq _j$.
Let $y(t)$ denote the noise-corrupted measurements of $x(t)$

$$y(t)=x(t)+e(t) \tag{2}$$

where $e(t)$ is a sequence of independent and identically distributed random variable of zero mean and variance λ^2, It is assumed that $x(t)$ and $e(s)$ are uncorrelated for any t and s.
As is well-known, $x(t)$ obeys the following autoregressive(AR) process

$$A(q^{-1})x(t)=0 \tag{3}$$

where q^{-1} denotes the unit delay operator and $A(q^{-1})$ is a polynomial of degress 2m defined by

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_{2m}q^{-2m}=\prod_{i=1}^{m}(1-2\cos\omega_i q^{-1}+q^{-2}) \tag{4}$$

It follows from (2)-(4) that $y(t)$ obeys the following autoregressive moving average(ARMA) process

$$A(q^{-1})y(t)=A(q^{-1})e(t) \tag{5}$$

It is easy to show that the roots of $A(q^{-1})$ appear on the unit circle at $e^{\pm j\omega_i}$, $i=1, 2\cdots m$.
Next multiplying both sides of equation (5) by a nonzero polynomial in q^{-1}, say $B(q^{-1})$, we obtain

$$C(q^{-1})y(t)=C(q^{-1})e(t) \tag{6}$$

where

$$C(q^{-1})=B(q^{-1})A(q^{-1}) \tag{7}$$

It will be assumed that $C(q^{-1})$ is a polynomial degree $L(L>2m)$ given by

$$C(q^{-1})=c_0+c_1q^{-1}+\cdots c_Lq^{-L} \tag{8}$$

The problem is to determine the number of multiple sinusoids from the matrices using the available data $y(1)$, $y(2)$,$\cdots\cdots y(N)$.

The number of multiple sinusoids are usually obtained from the following procedure:

(i) Construct a data[13] covariance[9]or [10] Gramian matrix[11], using (5)or (6).

(ii) Perform the singular value decomposition (SVD) on the matrices.

(iii) Find the number of multiple sinusoids from the low effective rank of the matrices that is reconstructed using SVD.

Specially in the model reduction(MR) method, the first step is to find the reachability and observability Gramians[11]. The number of multiple sinusoids is determined from the singular values through the square roots of the multiplication of two Gramians.

3. A Singular value approach(Per)

First consider a covariance matrix determining of the number of multiple sinusoids from N noisy data. Then Akaike's information criterion (AIC) [1][2] and Rissanen's shortest data description (MDL) [3][5] using the covariance matrix of dimension p×p is discussed.

AIC(n) and MDL(n) were given by the follows.

$$AIC(n) = -2\log\left(\frac{\prod_{i=n+1}^{p} \sigma_i^{1/(p-n)}}{\frac{1}{p-n}\sum_{i=n+1}^{p}\sigma_i}\right)^{(p-n)N} + 2n(2p-n) \quad (9)$$

$$MDL(n) = -\log\left(\frac{\prod_{i=n+1}^{p} \sigma_i^{1/(p-n)}}{\frac{1}{p-n}\sum_{i=n+1}^{p}\sigma_i}\right)^{(p-n)N} + 1/2n(2p-n)\log N \quad (10)$$

Where σ_i is the i-th singular values of matrix and n is the possible number of complex sinusoids, or twice the number of real sinusoids. The correct number of n should be chosen as the one which makes AIC(n) or MDL(n) each its minimum. It is interisting to note that the term in the parenthesis of (9) and (10) is the ratio of geometric mean of noise singular values to the arithmatic mean.

Next a singular value approach (Per) is presented and investigated analytically to determine the number of multiple sinusoids. It is known in the noiseless case that data, covariance or Gramian matices[10] are of rank 2m, again m being the number of multiple sinusoids. In such case, these matrices have 2m non-zero (greater than zero) signal singular values. In nosiy data case, all the singular values will be non-zero, but the signal singular values are usually much greater than the noise singular values. This information is much valuable when determining the number of multiple sinusoids. Decisions can be made, for example, by inspecting the "gaps" between two succesive singular values. Here we will examine the singular values by observing how "close" a low rank matrix is to the originally full rank matrix. To do this, we will introduce the concept of low "effective rank" [8][4].

Let $\boldsymbol{R}$ be a data, covariance or Gramian matrix of dimension MxL, which contains the information about the sinusoids-in-noise process, in order to define a singular value approach to determine the number of multiple sinusoids from the nosiy data. Perform a SVD for the matrix $\boldsymbol{R}$[12]

$$\boldsymbol{R} = \boldsymbol{U}\Sigma \boldsymbol{V}^T \quad (11)$$

where

$$\boldsymbol{U} = (u_1, u_2, \cdots, u_M) \quad (12)$$

$$\boldsymbol{V} = (v_1, v_2, \cdots, v_L) \quad (13)$$

$$\Sigma \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_q\}, \; q = \min(M, L) \quad (14)$$

The diagonal elements σ_i are commonly referred to as the singular values of matrix $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ are the eigenvector matrices of $\boldsymbol{RR}^T$ and $\boldsymbol{R}^T\boldsymbol{R}$, repectively. It is well known that the nonzero singular values will correspond to the positive square roots of the eigenvalues of the nonnegative Hermitian matrices $\boldsymbol{RR}^T$ and $\boldsymbol{R}^T\boldsymbol{R}$.

The singular values σ_i convey the signal and noise informations concerning the rank characterization of matrix $\boldsymbol{R}$. This is readily demonstrated upon considering the problem of finding that $M\times L$ matrix of rank n which will best approximate $\boldsymbol{R}$ in the Frobenious norm sense (this assumes that n $\le$rank[$\boldsymbol{R}$]).
The Frobenious norm of the $M\times L$ matrix difference $\boldsymbol{R}$-$\boldsymbol{S}$ is defined to be

$$\|\boldsymbol{R}-\boldsymbol{S}\| = \left[\sum_{i=1}^{M}\sum_{j=1}^{L}|r_{ij}-s_{ij}|^2\right]^{1/2} \quad (15)$$

where r_{ij} and s_{ij} are elements of matrices $\boldsymbol{R}$ and $\boldsymbol{S}$.

We now seek to find that $M\times L$ matrix $\boldsymbol{S}$ of rank n which will render this criterion a minimum. The solution to this approximation problem is gotten from the follows. Namely, the unique $M\times L$ matrix of rank $n\le$ rank[$\boldsymbol{R}$] which best approximates the $M\times L$ matrix $\boldsymbol{R}$ in the Frobenious norm sense is given by[12]

$$\boldsymbol{S} = \boldsymbol{R}^{(n)} = \boldsymbol{U}\Sigma_n \boldsymbol{V}^T \quad (16)$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are as in (11) while Σ_n is obtained from Σ by setting to zero all but its n signal singular values.

The quality of this optimum approximation is given by

$$\|\boldsymbol{R}-\boldsymbol{R}^{(n)}\| = \left[\sum_{j=1}^{n}(\sigma_{jj}^2-\sigma_{jj}^2) + \sum_{j=n+1}^{q}(\sigma_{jj}^2-0)\right]^{1/2} = \left[\sum_{j=n+1}^{q}\sigma_{jj}^2\right]^{1/2}, \quad 0\le n\le q \quad (17)$$

where $\boldsymbol{R}$ is all singular values and $\boldsymbol{R}^{(n)}$ is only the signal singular values.

The degree to which $R^{(n)}$ approximates R is seen to be dependent on the sum of the $(q-n)$ noise singular values squared. As n approaches q

this sum will become progressively smaller and will eventually go to zero at $n=q$. In order to provide a convenient measure for this behavior which does not depend on the size of matrix R, define the normalized square norm in Frobenious norm sense as

$$Per(n)=\frac{\|R^{(n)}\|_F^2}{\|R\|_F^2}=\frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{q}\sigma_i^2} \qquad (18)$$

where

$$R^{(n)}=\sum_{i=1}^{n}\sigma_i u_i v_i^T \qquad (19)$$

Obviously Per(n) is a nondecreasing function of n. Since matrix R in our case is generally of full rank, Per(n) will monotonically increasing to one as n approaches min(M, L). $\|R^{(n)}\|_F^2$ is the sum of the signal singular values and $\|R\|_F^2$ is the sum of all the singular values. Because the signal singular values of R are much larger than the noise singular values, Per(n), namely, the ratio signal singular values to all singular values, is close to one for some small number of n.

The low effective rank of R is then defined as the number which is much smaller than min(M, L) and which makes Per(n) very close to one. Any larger number than this low effective rank makes no significant contribution to Per(n).

In the section IV two examples are given to demonstrate how the low effective rank should be chosen and how the design parameters affect Per

for determination of the number of multiple sinusoids of a sinusoids-in-noise process, using covariance matrices for MUSIC[10] and HOYW [9], a data matrix for FBLP[12] and a Gramian

matrix for MR[11]. In all the examples, the signal was assumed to consist of two sinusoids. The SVD has been performed for different matrices and different design parameters like the order L of filter, the number M of equation and the number n of multiple sinusoids. The P is (L+1).

Example 1. The data simulated is given by

$$y(t)=\sqrt{2}\sin(0.7226t)+\sqrt{2}\sin(1.367t)+e(t) \qquad (20)$$

where $e(t)$ is a white Gaussian process with zero mean and variance $\lambda^2=0.1$ and the amplitude is $\alpha_i=\sqrt{2}$ for $i=1, 2$.

The data length is $N=64$ and the SNR is 10 dB, i. e., $SNR_i=10\log_{10}(\alpha_i^2/2\lambda^2)$ for i=1, 2. AIC(n), MDL(n) and Per(n) are calculated with a num-

Table 1 Performances of AIC(n) for MUSIC, Example 1

P \ n	2	4	6	8
12	1972.8	383.4	300.0	285.5
16	3899.5	911.1	705.7	677.2
20	4945.4	810.0	705.4	670.5
24	6096.0	1346.3	1301.8	1277.6

Table 2 Performance of MDL(n) for MUSIC, Example 1

P \ n	2	4	6	8
12	1033.9	278.1	266.6	280.9
16	2014.7	576.5	521.2	545.9

Table 4 Performances of Per(n) for for FBLP, Example 1

L \ n	2	4	6	8
12	0.8665	0.9997	0.9998	0.9999
14	0.6768	0.9991	0.9997	0.9998
20	0.5878	0.9988	0.9994	0.9998
24	0.6148	0.9981	0.9992	0.9996

Table 5 Performances of Per(n) for HOYW (L=12), Example 1

M \ n	2	4	6	8
12	0.7251	0.9999	1.0000	1.0000
14	0.7431	0.9999	1.0000	1.0000
16	0.7646	0.9999	1.0000	1.0000
18	0.7613	0.9998	1.0000	1.0000

Table 6 Performances of Per(n) for MUSIC, Example 1

L \ n	2	4	6	8
12	0.8805	0.9994	0.9998	0.9999
16	0.6820	0.9992	0.9998	0.9999
20	0.5688	0.9993	0.9997	0.9998
24	0.6789	0.9993	0.9996	0.9998

Table 7 Performances of AIC(n) for MUSIC, Example 2

P \ n	2	4	6	8
12	498.4	240.7	253.8	280.2
16	1144.3	450.7	484.9	479.3
20	1938.1	758.7	811.5	821.5
24	2790.1	1445.4	1495.6	1533.7

Table 8 Performances of MDL(n) for MUSIC, Example 2

P \ n	2	4	6	8
12	296.7	206.7	243.5	278.3
16	636.9	346.2	410.8	446.9
20	1051.1	534.8	626.0	687.1
24	1494.4	912.7	1019.8	1112.3

ber of values of n. Results are given in table 1 through 6.

Table 9 Performances of Per(n) for MR, Example 2

L \ n	2	4	6	8
12	0.9517	0.9929	0.9986	0.9999
16	0.9966	0.9998	0.9999	0.9999
20	0.9974	0.9991	0.9997	0.9998
24	0.9386	0.9906	0.9964	0.9989

Table 10 Performances of Per(n) for FBLP, Example 2

L \ n	2	4	6	8
12	0.8260	0.9767	0.9878	0.9944
16	0.6614	0.9706	0.9824	0.9901
20	0.5695	0.9717	0.9804	0.9877
24	0.5842	0.9708	0.9798	0.9858

Table 11 Performances of Per(n) for HOYW (L=12), Example 2

M \ n	2	4	6	8
12	0.7056	0.9944	0.9974	0.9941
14	0.7187	0.9931	0.9973	0.9989
16	0.7402	0.9926	0.9974	0.9990
18	0.7436	0.9916	0.9975	0.9990

Table 12 Performances of Per(n) for MUSIC, Example 2

L \ n	2	4	6	8
12	0.8434	0.9751	0.9881	0.9941
16	0.6680	9.9719	0.9824	0.9914
20	0.5522	0.9741	0.9819	0.9891
24	0.6444	0.9761	0.9833	0.9887

Example 2.

The same signal as the example 1 has been used but the SNR has been changed to 0 dB. Computations are repeated as done in example 1, and performance are listed in Table 7 through 12.

From the above tables, AIC and MDL methods chose the number of n, when AIC(n) and MDL(n) decrease steeply and Per method choses the number of n, when Per(n) is close to one.

The following comments are drawn from the above two examples:

(i) The AIC(n) and MDL(n) for MUSIC tend to overestimate the number of multiple sinusoids at relatively high SNR. However, MDL(n) may provide correct results for large design parameters. For low SNR, both AIC(n) and MDL(n) give consistent results. This may be argued from the following observations. At high SNR, the noise singular values diverse very much (the ratio of the largest noise singular value to the smallest one is small). The ratio of the geometric mean to the arithmatic mean is small (note it is less than one), and the first term in (9) and (10) will be very large. Therefore the second term, i.e. the penalty term due to parsimony principle, does not play important role in correction of overestimation. For low SNR, the noise singular values are closer to each other. The first term in (9) and (10) decreases, and the second term can give proper penalty for overestimating the number of signal (note also the second term of (9) and (10) is constant with respect to SNR).
(ii) The test of Per(n) for MR method gives satisfactory estimates at high SNR, but unsatisfactory estimates at low SNR. The reason for the latter case may be explained as follows. At low SNR, i.e. heavy noise, the dominant poles (or modes) may be located out of the unit circle. In this case, the high order model will be unstable, and the reachability and observability Gramians will not converge. Therefore reachability and observability Gramians obtained from Lyapunov equation will give false information.
(iii) The tests of Per(n) for FBLP, HOYW and MUSIC methods show that the low effective rank is 4, in both example, which means that there are two sinusoids in the measurement data. But for small design parameters, the low effective rank may sometimes underestimate the number of multiple sinusoids. This can be explained by observations on the singular values. For small design parameters and two equal SNR's (SNR1 = SNR2), the first two signal singular values are much greater than the other two. These small two signal singular values grow very fast (faster than the first two) with the increase of design parameters. Thus the first two signal singular values become less dominant. On the other hand, if the design parameters are too large, the signal singular values may become less dominant since there are too many noise singular values in this case. Finally the value of the SNR also affects the dominancy of the signal singular values.

5. Conclusions.

In the paper, a singular value approach (Per) that determined the number of multiple sinusoids from the finite noisy data was presented.

Simulations were conducted for AIC(n), MDL(n) and Per(n) in MUSIC, HOYW, FBLP and MR methods.

The following conclusions are drawn:
(i) The AIC(n) and MDL(n) for MUSIC tend to overestimate the number of multiple sinusoids at relatively high SNR. However, MDL(n) may provide correct results for large design parameters. For low SNR, both AIC(n) and MDL(n) give consistent results.
(ii) The test of Per(n) for MR method gives satisfactory estimates at high SNR, but fails at low SNR.
(iii) The tests of Per(n) for FBLP, HOYW and MUSIC show that the low effective rank is 4. But for small design parameters, the low effective rank may sometimes underestimate the number of multiple sinusoids.

Finally, Per will need to estimate the frequencies of multiple sinusoids from the noisy measurements. Furthermore, this approach will be applied to the digital signal processing of communication and bio-medical engineering.

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