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상태궤환에 의한 2차원 F-MMII의 비간섭화

Decoupling of 2-D Systems by State Feedback in F-MM II

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요 약

비간섭화문제는 다 입출력계의 제어문제를 동일한 수의 단일 입출력계로 변환할 수 있다는 점에서 중요하다. 이 논문에서는 상태궤환에 의한 2차원 F-MM II의 비간섭화를 위한 필요조건과 충분조건을 유도하였다. 일반적인 경우, 궤환행열 F를 구하는데 비선형 대수방정식의 해를 포함하게 되나, 어떠한 조건하에서는 F가 쉽게 구해질 수 있음을 보였다. 이 방법은 F-MM II뿐 아니라 RM에도 적용될 수 있으며 알고리즘도 RM을 사용한 방법보다 간단한 것으로 보인다.

Abstract- The decoupling problem has a great practical importance in that it simplifies greatly the control of a given system by reducing the multi-input multi-output systems. In this paper, we derive the necessary and sufficient conditions for decoupling 2-D F-MM II via state feedback. For the general case, the problem of determining the feedback matrix F involves the solution of nonlinear algebraic equations. Under certain conditions, however, it is shown that an explicit formula for F may be derived. In comparsion with the method for RM, it appears that this method for F-MM II is more general and the algorithm is simpler.

1. Introduction

In recent years the area of two-dimensional (2-D) systems has attracted considerable attention. This area has been studied in relation to several modern engineering fields such as 2-D processing, X-ray enhancement, image deblurring, weather prediction, seismic analysis, radar and sonar array processing etc.

On the other hand, the decoupling problem has great practical importance, since it makes it possible to simplify the control of the given Systems as it reduces the multi-input multi-output Systems under control to a number of single-input single-output. This motivation has led to a great deal of research into the decoupling of one dimensional systems[1][2][3][4][5]and it has been extended to 2-D Systems[6][7][8]in similar way for 1-D

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systems. But all of these approaches for 2-D systems deal with Rosser's Model (RM) [9].

This paper refers to the problem of designing a state feedback controller for decoupling of 2-D Fornasini-Marcheini's 2nd Model (F-MM Ⅱ) [10] Although the algorithm in this paper is similar to reference [6] and [7], the advantage of this paper is that F-MM Ⅱ is the general from which can be converted into RM and the algorithm is simpler than the case of RM.

2. Definitions and Statement of the Ploblem

Consider the linear time-invariant multivariable discrete-time 2-D Systems in F-MM Ⅱ [10] described in statespace as follows:

$$\boldsymbol{x}(i+1,\ j+1)=\boldsymbol{A}_1\boldsymbol{x}(i,\ j+1)+\boldsymbol{A}_2\boldsymbol{x}(i+1,\ j) + \boldsymbol{B}_1\boldsymbol{u}(i,\ j+1)+\boldsymbol{B}_2\boldsymbol{u}(i+1,\ j) \quad (1a)$$

$$\boldsymbol{y}(i,\ j)=\boldsymbol{C}\boldsymbol{x}(i,\ j) \quad (1b)$$

where i, j are integer-valued vertical and horizontal coordinates, respectively,

$\boldsymbol{x}(i,\ j)\in R^n$ is the local state vector at $(i,\ j)$,

$\boldsymbol{u}(i,\ j)\in R^m$ is the input vector,

$\boldsymbol{y}(i,\ j)\in R^m$ is the output vector,

$\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}$ are real matrices of appropriate dimensions. It is assumed that $m\leq n$.

The controller applied to systems (1) is of the linear state feedback type having the form

$$\boldsymbol{u}(i,\ j)=\boldsymbol{F}\boldsymbol{x}(i,\ j)+\boldsymbol{G}\boldsymbol{v}(i,\ j) \quad (2)$$

where $\boldsymbol{F}$ is an $m\times n$ constant matrix, $\boldsymbol{G}$ is an $m\times m$ nonsingular matrix and $v\in R_m$ is the new control input. Substituting (2) into (1) yieds the closed-loop system.

$$\boldsymbol{x}(i+j,\ j+1)=(\boldsymbol{A}_1+\boldsymbol{B}_1\boldsymbol{F})\boldsymbol{x}(i,\ j+1)+(\boldsymbol{A}_2+\boldsymbol{B}_2\boldsymbol{F})\boldsymbol{x}(i+1\ \ j)+\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i,\ j+1)+\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i+1,\ j). \quad (3)$$

The following defintions will be applied in the sequel:The state transition matrix of systme (1) is $\boldsymbol{A}^{i,j}$ and is defined as follows[10]

$$\boldsymbol{A}^{0,0}=\boldsymbol{I}\ \text{(the identity matrix)} \quad (4a)$$

$$\boldsymbol{A}^{i,j}=\boldsymbol{A}_1\boldsymbol{A}^{i-1,j}+\boldsymbol{A}_2\boldsymbol{A}^{i,j}$$
$$for(i,\ j)>(0,\ 0) \quad (4b)$$

$$\boldsymbol{A}^{-i,j}=\boldsymbol{A}^{i\ -j}=0\ \text{(the zero matrix)}$$
$$for\ i>0\ \text{or}\ j>0 \quad (4c)$$

$$\boldsymbol{A}_1\boldsymbol{A}^{i-1,j}+\boldsymbol{A}_2\boldsymbol{A}^{i,j-1}=\boldsymbol{A}^{i-1,j}\boldsymbol{A}_1+\boldsymbol{A}^{i,j-1}\boldsymbol{A}_2$$
$$for(i,\ j)\geq(0,\ 0) \quad (4d)$$

The following partial ordering is used for integer pairs:

$$(h,\ k)\leq(i,\ j)\ if\ h\leq i\ \text{and}\ k\leq j \quad (5a)$$

$$(h,\ k)=(i,\ j)\ if\ h=i\ \text{and}\ k=j, \quad (5b)$$

$$(h,\ k)<(i,\ j)\ if(h,\ k)\leq(i,\ j)\ \text{and}\ (h,\ k)\neq(i,\ j), \quad (5c)$$

$$(h,\ k)>(i,\ j)\ if(h,\ k)\geq(i,\ j)\ \text{and}\ (h,\ k)\neq(i,\ j). \quad (5d)$$

Considering $\boldsymbol{x}(i-1,\ j)=\boldsymbol{x}(i,j-1)=0$ for i, $j\geq0$, $\boldsymbol{v}(i,\ j-1)=\boldsymbol{v}(i-1,\ j)=0$ for i, $j\neq0$, eq. (3) may be written as follows:

$$\boldsymbol{x}(i,\ j+1)=(\boldsymbol{A}_2+\boldsymbol{B}_2\boldsymbol{F})\boldsymbol{x}(i,\ j) + \boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j) \quad (6a)$$

$$\boldsymbol{x}(i+1,\ j)=(\boldsymbol{A}_1+\boldsymbol{B}_1\boldsymbol{F})\boldsymbol{x}(i,\ j) + \boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i,\ j) \quad (6b)$$

Using (6) recursively, the output eq. (1b) may be written as follows:

$$\boldsymbol{y}(i,\ j)=\boldsymbol{C}\boldsymbol{x}(i,\ j) \quad (7a)$$

$$\boldsymbol{y}(i+1,\ j)=\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{1,0}\boldsymbol{x}(i,\ j) + \boldsymbol{C}\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i,\ j) \quad (7b)$$

$$\boldsymbol{y}(i,\ j+1)=\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{0,1}\boldsymbol{x}(i,\ j) + \boldsymbol{C}\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j) \quad (7c)$$

$$\boldsymbol{y}(i+2,\ j)=\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{2,0}\boldsymbol{x}(i,\ j) + \boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{1,0}\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i,\ j) + \boldsymbol{C}\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i+1,j) \quad (7d)$$

$$\boldsymbol{y}(i+1,\ j+1)=\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{1,1}\boldsymbol{x}(i,\ j) + \boldsymbol{C}\{(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{1,0}\boldsymbol{B}_1+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{1,0}\boldsymbol{B}_2\}\boldsymbol{G}\boldsymbol{v}(i,\ j)+\boldsymbol{C}\boldsymbol{B}_1\boldsymbol{G}\boldsymbol{v}(i+1,\ j)+\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j+1) \quad (7e)$$

$$\boldsymbol{y}(i,\ j+2)=\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{0,2}\boldsymbol{x}(i,\ j)+\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F})^{0,1}\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j)+\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j)+\boldsymbol{C}\boldsymbol{B}_2\boldsymbol{G}\boldsymbol{v}(i,\ j+1) \quad (7f)$$

$$y(i+n,\ j+n)=C(A+BF)^{n,n}\ x(i,\ j)+C[(A+BF)^{n-1,n}B_1+(A+BF)^{n,n-1}B_2]Gv(i,\ j)+$$
$$C[(A+BF)^{n-1,n-1}B_1+(A+BF)^{n,n-2}B_2]Gv(i,\ j+1)+$$
$$C[(A+BF)^{n-2,n}B_1+(A+BF)^{n-1,n-1}B_2]Gv(i+1,\ j)+$$
$$\cdots$$
$$+C[(A+BF)^{n-\mu-1,n-\nu}B_1+(A+BF)^{n-\mu,n-\nu-1}B_2]Gv(i+\mu,\ j+\nu)+\cdots$$
$$+CB^1Gv(i+n-1,\ j+n)+CB_2Gv(i+n,j+n-1) \tag{7g}$$

where the identifier $(A+BF)^{1,0}=(A_1+B_1F)$ and $(A+BF)^{0,1}=(A_2+B_2F)$ are used.

The closed-loop characteristic polynomial $P(z_1,\ z_2)$ is given by

$$P(z_1,\ z_2)=det[I_nz_1\ z_2-(A_1+B_1F)z_2-(A_2+B_2F)z_1]=\sum_{i=0}^{n}\sum_{j=0}^{n}P_{i,j}(F)z_1^iz_2^j,\ \ P_{n,n}(F)=1. \tag{8}$$

The 2-D Cayley-Hamilton theorm[9] is given by

$$\sum_{i=o}^{n}\sum_{i=o}^{n}P_{i,j}(A+BF)^{i,j}=0. \tag{9}$$

Now, if we multiply both sides of (7a) by $P_{0,0}$, of (7b) by $P_{1,0},\cdots$, and of $(7g)$ by $P_{n,n}$ and if we add all equations and making use of (9), we get

$$y(i+n,j+n)+P_{n-1,n}y(i+n-1,j+n)+P_{n,n-1}y(i+n,j+n-1)+\cdots+P_{k,l}y(i+k,j+l)+\cdots+$$
$$P_{1,0}y(i+1,\ j)+P_{0,1}y(i,\ j+1)+P_{0,0}y(i,j)=$$
$$C[\sum_{k=o}^{n-1}\sum_{i=0}^{n}P_{k+1,l}B_1Gv(i+k,\ j+l)+\sum_{k=0}^{n}\sum_{l=0}^{n-1}P_{k,l+1}B_2Gv(i+k,\ j+l)+$$
$$\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}P_{k+1,\ l+1}\{(A+BF)^{0,1}B_1+(A+BF)^{0,1}B_2\}Gv(i+k,\ j+l)+\cdots+$$
$$\sum_{k=0}^{\mu}\sum_{l=0}^{\nu}P_{k+n-\mu,l1n-\nu}\{(A+BF)^{n-\mu-1,n-\nu}B_1+(A+BF)^{n-\mu,n-\nu-1}B_2\}Gv(i+k,\ j+l)+\cdots+$$
$$\{(A+BF)^{n-1,n}B_1+(A+BF)^{n,n-1}B_2\}Gv(i,\ j) \tag{10}$$

Consider the qth output y_q. Then (10) may be written compactly as follows:

$$\sum_{k=0}^{n}\sum_{l=0}^{n}P_{k,\ l}y_q(i+k,\ \ j+l)=tr[L^q(F,\ \ G)\ \Omega]\ ;\ q=1,\ 2,\cdots,\ m. \tag{11}$$

where $L^q(F,G)$ (or simply L^q) and Ω are $\tau\times m$ and $m\times\tau$ matrices, with $\tau=(n+1)(n+1)-1=n^2+2n$, defined as follows:

$$L^q=\begin{bmatrix} L_{0,0}^q \\ L_{1,0}^q \\ \cdot \\ \cdot \\ L_{w,v}^q \\ \cdot \\ \cdot \\ L_{n^q-1,n} \\ L_{n^q,n-1} \end{bmatrix}$$

where

$$L_0{}^q{}_{,0}=C_q[P_{1,0}B_1+P_{0,1}B_2+\cdots+P_{k,l}\{(A+BF)^{k-1,l}B_1+(A+BF)^{k,2-1}B_2\}+\cdots$$
$$+(A+BF)^{n-1,n}B_1+(A+BF)^{n,n-1}B_2]G \tag{12a}$$
$$L_1{}^q{}_{,0}=C_q[P_{2,0}B_1+P_{1,1}B_2+\cdots\cdots+P_{k+1,l}\{((A+BF)^{k-1,l}B_1+(A+BF)^{k,l-1}B_2\}+\cdots$$
$$+(A+BF)^{n-2,n}B_1+(A+BF)^{n-1,n-1}B_2]G \tag{12b}$$
$$L_0{}^q{}_{,1}=C_q[P_{1,1}B_1+P_{0,2}B_2+\cdots+P_{k,l+1}\{(A+BF)^{k-1,l}B_1+(A+BF)^{k,l-1}B_2\}+\cdots$$
$$+(A+BF)^{n-1,n-1}B_1+(A+BF)^{n,n-2}B_2]G \tag{12c}$$
$$L_\mu{}^q{}_{,\nu}=C_q[P_{u+1,\nu}B_1+P_{\mu,\nu+1}B_2+\cdots$$
$$+P_{u+k,\nu-l}\{(A+BF)^{k-1,l}B_1+(A+BF)^{k,l-1}B_2]+\cdots$$
$$+(A+BF)^{n-\mu-1,\ n-\nu}B_1+(A+BF)^{n-\mu,n-\nu-1}B_2]G \tag{12d}$$
$$L_{n}{}^q{}_{-1,n}=C_qB_1G \tag{12e}$$
$$L_{n}{}^q{}_{,n-1}=C_qB_2G \tag{12f}$$

and

$$\Omega=[v(i,\ j),\ v(i+1,\ j),\ v(i,\ j+1),\cdots,\ v(i+\mu,\ j+\nu),\cdots,v(i+n-1,\ j+n),\ v(i+n,\ j+n-1)] \tag{13}$$

Let $E_{i,j}$ denote the $m\times m$ matrix with 1 as ij th entry and zeros elsewhere. Then $E_{i,j}\Omega$ is an $m\times m$ matrix with the ith row identical to the jth row of Ω and all other rows zero. The matrix $E_{q,q}\ \Omega$ will be denoted by Ω^q. Then precise definition of the decoupling problem may now be stated as follows:The matrices F and G, with G nonsingular, decouple system (1) if

$$\sum_{k=0}^{n}\sum_{i=0}^{n}P_{k,l}y_q(i+k,\ j+l)\equiv\mathrm{tr}[L^q(F,\ G)\Omega]\equiv\mathrm{tr}[L^q(F,\ G)\Omega^q] \tag{14a}$$

and if

$$\mathrm{tr}[L^q(F,\ G)\Omega^q]\neq 0,\quad q=1,\ 2,\cdots\cdots,m. \tag{14b}$$

3. Main Results

(Definition 1)

For each output q, let Γ_q denote the set of all points $(\mu_q,\ \nu_q)$ for which the following relationship hold:

$$C_q[A^{s-1,t}B_1+A^{s,t-1}B_2]$$
$$\begin{cases}\neq 0, & \forall(s,\ t)=(n_q,\ \nu_q)\leq(n,\ n) \quad (15a)\\ =0, & \forall(s,\ t)<(\mu_q,\ \nu_q)\leq(n,\ n) \quad (15b)\end{cases}$$

And let $(\hat{\mu}_q,\ \hat{\nu}_q)$ denote the points of Γ_q which is the nearest to the i-axis and $(\bar{\mu}_q,\ \bar{\nu}_q)$ denote the nearest point to the j-axis. Then, one may define a set points $(s,\ t)$ for which the follwing lelationship holds:

$$C_q[A^{s-1,t}B_1+A^{s,t-1}B_2]=0 \tag{15c}$$

$$\forall(s,\ t):(\hat{\mu}_q+1,0)\leq(s,\ t)\leq(n,\ \hat{\mu}_q-1),\quad (0,\ \bar{\nu}_q+1)\leq(s,\ t)\leq(\bar{\mu}_q-1,\ n)$$

(Defintion 2)

For each output q, let $\bar{\Gamma}_q$ donote the set of all points $(s,\ t)$ for which the relationship (15b) and (15c) hold.

(Definition 3)

For each output q, let Δ_q denote the set of vectors

$$B^*_q=C_q[A^{\mu_q-1,\nu_q}B_1+A^{\mu g,v g-1}B_2] \tag{16}$$

for all points of Γ_q.

[Theorem 1]

The decoupling problem has a solution if and only if

(i) all rows B_q^* in Δ_q are proportional,

(ii) $det\ B^*\neq 0$

where

$$B^*=\begin{bmatrix}B_1{}^*{}_d\\ \cdot\\ \cdot\\ B_q^*\\ \cdot\\ \cdot\\ B_m^*\end{bmatrix}=\begin{bmatrix}C_1[A^{\mu_1-1,\nu_1}B_1+A^{\mu_1,\nu_1-1}B_2]\\ \cdot\\ \cdot\\ C_q[A^{\mu_q-1,\nu_q}B_1+A^{\mu_q,\nu_q-1}B_2]\\ \cdot\\ \cdot\\ C_m[A^{\mu_m-1,\nu_m}B_1+A^{\mu_m,\nu_m-1}B_2]\end{bmatrix} \tag{17}$$

(iii) there is an F such that

$$C_q[(A+BF)^{s-1,t}B_1+(A+BF)^{s,t-1}B_2]=K_q(s,\ t)B^*_q \tag{18}$$

for $q=1,\ 2,\cdots,\ m$ and for all $(s,\ t)\leq(n,\ n)$ which do not belong to $\bar{\Gamma}_q$ and Γ_q. The factor $K_q(s,\ t)$ is a proportionality constant.

(Proof)

(Necessary conditions) Suppose that there is a pair of matrices F and G which decouple (1). Then it follows from (12d) and the defintion 1 that for any point $(\mu_q,\ \nu_q)$ that belongs to the set Γ_q, the row $L^q{}_{n-\mu_q,n-\nu_q}$ reduces to its last 2 terms, i.e.

$$L^q{}_{n-\mu_q,n-\nu_q}=C_q[(A+BF)^{\mu_q-1,\nu_q}B_1+(A+BF)^{\mu_q,\nu_q-1}B_2]G \tag{19}$$

Using the identity

$$(P+Q)^{i,j}=P^{i,j}+P^{i-1,j}Q^{1,0}+P^{i,j-1}+Q^{0,1}\cdots+Q^{i,j} \tag{20}$$

where P and Q are square matrices and defintion 1, relation (19) becomes

$$\begin{aligned}L_n{}^q{}_{-\mu_q,n-\nu_q}&=C_q[(A+BF)^{\mu_q-1,\nu_q}B_1+(A+BF)^{\mu_q,\nu_q-1}B_2]G\\&=C_q[\{A^{\mu_q-1,\nu_q}+A^{\mu_q-2,\nu_q}(BF)^{1,0}+A^{\mu_q-1,\nu_q-1}(BF)^{0,1}+\cdots+(BF)^{\mu_q-1,\nu_q}\}B_1\\&\quad+\{A^{\mu_q,\nu_q-1,}+A^{\mu_q-1,\nu_q-1}(BF)^{1,0}+A^{\mu_q\nu_q-2}(BF)^{0,1}+\cdots+(BF)^{\mu_q,\nu_q-1}\}B_2]G\end{aligned}$$

$$= C_q[A^{\mu_{q-1},\nu_q}B_1 + A^{\mu_q,\nu_{q-1}}B_2]G \quad (21)$$
$$= B_q^* G$$

Now if

$$C_q[A^{s-1,t}B_1 + A^{s,t-1}B_2] = 0, \text{ for all} (s,\ t) \le (n,\ n)$$

that would imply that $\mathrm{tr}[L^q(F, G)\Omega] = 0$, which, according to (14), would contradict the fact that F and G decouple(1). Hencse, it is clear that $B_q^* \neq 0$ for $q = 1, 2, \cdots, m$. As G is non-singular, $B_q^* G \neq 0$ for all q. Since (14a) is satisfied it follows that $B_q^* G = a_q e_q$, with $a_q \neq 0$ and $e_q = (0, 0, \cdots, 0, 1, 0 \cdots 0)$ with the unity element in the qth position. Hence, for each row B_q^*, which belongs to the set $\triangle_q$, it must hold that

$$B_q^* G = B_q^*[g_1^T,\ g_2^T, \cdots, g_m^T] = [0, 0, \cdots, a_q, \cdots, 0] \quad (22)$$

that is $B_q^*(g_i)^T = 0$, $i = 1, \cdots, q-1, q+1, \cdots,\ m$, where $g_i^T$is the i th column of G. Consequently all B_q^* that belongs to $\triangle_q$are orthogonal to the m-1 independent vectors g_i, $i = 1, \cdots, q-1, \cdots, m$ and hence all B_q^* that belong to $\triangle_q$are proportional to each other. This last property can be expressed as in relation (23) that follows

To this end we let $(\tilde{\mu}_q,\ \tilde{\nu}_q)$ denote a certain point of the set Γ_q. For this point the corresponding vector B_q^* will be denoted as $\tilde{B}_q^*$, that is

Then, if all vectors B_q^* of the set $\triangle_q$ are proportional to each other then the following rela tionship must hold

$$B_q^* = C_q[A^{\mu_{q-1},\nu_q}B_1 + A^{\mu_q,\nu_{q-1}}B_2] = \lambda_q(\mu_q,\ \nu_q)\tilde{B}_q^* \quad (23)$$

for all $(\mu_q,\ \nu_q)$ of the set Γ_q, where $\lambda_q(\mu_q,\ \nu_q)$ is a proportionality constant that can readily be computed.

From (22) we have that

$$B^* G = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & a_m \end{bmatrix} \ ; \ \prod_{q=1}^{m} a_q \neq 0.$$

Since G is assumed nonsigular it follows that B^* is also nonsingular. Here, the second requirement of the theorem is satisfied. To prove the last requirement of the theorem, we introduce the following definitions : Let $Q^q(F)$ be the $\tau \times m$ matrix given by

$$Q^q(F) = \begin{bmatrix} C_q[(A+BF)^{n-1,n}B_1 + (A+BF)^{n,n-1}B_2] \\ \cdot \\ \cdot \\ C_q[(A+BF)^{s-1,t}B_1 + (A+BF)^{S,t-1}B_2] \\ \cdot \\ \cdot \\ C_q[(A+BF)^{\mu_{q-1},\nu_q}B_1 + (A+BF)^{\mu_q,\nu_{q-1}}B_2] \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Also let $P^q(F)$ be the $\tau \times \tau$ matrix given bv

$$P^q(F) = \begin{bmatrix} 1 & P_{n-1,n} & P_{n,n-1} & P_{n-1,n-1} & \cdots & P_{k,l} & \cdots & P_{\mu_q,\nu_q} & \cdot \\ & 1 & P_{n-1,n} & P_{n,n-1} & \cdots & P_{k+1,l} & \cdots & P_{\mu_{q+1},\nu_q} & \cdot \\ & & 1 & P_{n-1,n} & \cdots & P_{k,l+1} & \cdots & P_{\mu_q,\nu_{q+1}} & 0 \\ & 0 & & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & I \end{bmatrix}$$

Since $\det P^q(F) = 1$, it follows that the rank of $P^q(F)\ \ Q^q(F)$ is the same as the rank of $Q^q(F)$. The matrix $L^q(F, G)$ defined in (11) may now be written as

$$L^q(F, G) = P^q(F)Q^q(F)G \quad (24)$$

Hence

$$\text{rank } L^q(F, G) = \text{rank } Q^q(F) \quad (25)$$

Since Ω is arbitrary, we conclude from (14) that the qth coulmn of $L^q(F, G)$ is a nonzero vector, while every other column of $L^q(F, G)$ is ia zero vector. It follows that $L^q(F, G)$ has rank one and hence, by (25), that rank $Q^q(F) = 1$. Now since at least one row of $Q^q(F)$ is identical to B^*_q, it follows that for $Q^q(F)$ to have nuity rank. all its rows must be analogous to B_q^*. This leads directly to the last condition of the theorem.

(Sufficient conditions). Let all conditions of the theorem hold, i.e. let (23) hold, $\det B^* \neq 0$ and rank $Q^q(F) = 1$.

Then $Q^q(F)$ takes on the form

$$Q^q(F)=\begin{bmatrix} K_q(n,n)B_q \\ \cdot \\ \cdot \\ K_q(s,t)B_q \\ d\ ^* \\ \cdot \\ \cdot \\ B_q^* \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

If we post-multiply $Q^q(F)$ by $(B^*)^{-1}$ we have

$$Q^q(F)(B^*)^{-1}=\begin{bmatrix} & \cdot & \cdot & K_q(n,n) & \cdot & \cdot & \\ & & & \cdot & & & \\ 0 & \cdot & \cdot & K_q(s,t) & \cdot & \cdot & \\ & & & \cdot & & & 0 \\ & & & 1 & & & \\ & & & 0 & & & \end{bmatrix}$$

Thus for $G=(B^*)^{-1}$ we have

$$\begin{aligned}&\mathrm{tr}[L^q[F,\ (B^*)^{-1}]\Omega]\\ &=\mathrm{tr}[P^q(F)Q^q(F)(B^*)^{-1}\Omega]\\ &=\mathrm{tr}[P^q(F)Q^q(F)(B^*)^{-1}\Omega^q]\neq 0\end{aligned}$$

Since relations(14) are satisfied it follows that the pair F and $(B^*)^{-1}$ decouples(1).

4. Special cases

In section 3, the third condition i.e rank $Q^q(F)=1$ is a set of non-linear algebraic equations in the elements of F. Therefore, it is difficult to use this condition to derive the feedback controller matrix F. Under certain special conditios, however, an explicit expression for F can be derived. To this end let $(\alpha_q,\ \beta_q)$ and $(\bar{\alpha}_q, \bar{\beta}_q)$ be the pairs of integers defined by

$$(\alpha_q,\ \beta_q)=\min\{(s,\ t) : C_qA^{s-1,t}B_1 \neq 0 ;\ (0,\ 0)\leq(s,\ t)\leq(n,\ n)\} \quad (26a)$$

$$(\bar{\alpha}_q,\ \bar{\beta}_q)=\min\{(s,\ t) : C_qA^{S,t-1}B_2 \neq 0 ;\ (0,\ 0)\leq(s,\ t)\leq(n,\ n)\} \quad (26b)$$

where by minimum over the set (s,t) we refer to the minimum order ρ of the pair (s,t) which is defined by $\rho=s+t$, The pair (α_q,β_q) is assumed here to be unique in the sense that, if a pair (α_q,β_q) is found such that $C_qA^{\alpha_q-1,\beta_q}B_1\neq 0$, then all other vectors $C_qA^{s-1,t}B_1$, for which $s+t=\alpha_q+\beta_q$ are zero. Similary for the pair $(\bar{\alpha}_q,\ \bar{\beta}_q)$. From (26), it immediately follows that

if

$$\alpha_q+\beta_q<\bar{\alpha}_q+\bar{\beta}_q$$

then

$$C_qA^{s,t-1}B_2=0,\ \text{for}(0,\ 0)\leq(s,\ t)\leq(\alpha_q,\beta_q)\} \quad (27a)$$

and if

$$\bar{\alpha}_q+\bar{\beta}_q<\alpha_q+\beta_q$$

then

$$C_qA^{s-1,t}B_1=0,\ \text{for}(0,0)\leq(s,\ t)\leq(\bar{\alpha}_q,\ \bar{\beta}_q)\} \quad (27b)$$

[Theorem 2]

The system(1) cna be decoupled via the state feedback laww of eq.(2) if the $m\times m$ matrix $\tilde{B}^*$, where its qth row $\tilde{B}_q^*$ is given by

$$\tilde{B}_q^*\begin{cases}=C_qA^{\alpha_q-1,\beta_q}B_1\ \ if\ \ \alpha_q+\beta_q<\bar{\alpha}_q+\bar{\beta}_q & (28a)\\ =C_qA^{\bar{\alpha}p,\bar{\beta}q-1}B_2 if \bar{\alpha}_q+\bar{\beta}_q<\alpha_q+\beta_q & (28b)\end{cases}$$

is nonsigular and if the following conditions are satisfied. If $\alpha_q+\beta_q<\bar{\alpha}_q+\bar{\beta}_q$, then

$$\begin{aligned}C_qA^{i-1,j}&=0 i+j\\ &=\alpha_q+\beta_q+1\ \text{execpt for}(i,\ j)\\ &=(\alpha_q+1,\ \beta_q)\end{aligned} \quad (29a)$$

and if $\bar{\alpha}_q+\bar{\beta}_q<\alpha_q+\beta_q$, then

$$\begin{aligned}C_qA^{i,j-1}&=0 i+j\\ &=\bar{\alpha}_q+\bar{\beta}_q+1\ \text{except for}(i,\ j)\\ &=(\bar{\alpha}_q,\ \bar{\beta}_q+1)\end{aligned} \quad (29b)$$

(Proof)

Assume that $\tilde{B}^*$is nonsigular. Then choose

$$G^*=(\tilde{B}^*)^{-1} \quad (30)$$

and

$$F^*=-(\tilde{B}^*)^{-1}A^* \quad (31)$$

where the q th row A_q^* of the martix A^* is given by

$$A_q^*\begin{cases}=C_qA^{\alpha_q,\beta_q}\ \ if\ \ \alpha_q+\beta_q<\bar{\alpha}_q+\bar{\beta}_q & (32a)\\ =C_qA^{\bar{\alpha}_q,\bar{\beta}_q}\ \ if\ \ \bar{\alpha}_q+\bar{\beta}_q<\alpha_q+\beta_q & (32b)\end{cases}$$

If $\alpha_q+\beta_q<\bar{\alpha}_q+\bar{\beta}_q$, then

$$C_q(A+BF)^{s-1,t}B_1$$

$$\begin{cases} = 0 \; (0,0) \le (s,t) < (\alpha_q, \beta_q) & (33a) \\ = 0 \; (s+t) = \alpha_q + \beta_q \\ \quad \text{except for} (s,t) = (\alpha_q, \beta_q) & (33b) \\ = \tilde{B}_q^* \; (s,t) = (\alpha_q, \beta_q) & (33c) \\ = 0 \; (s,t) > (\alpha_q, \beta_q) & (33d) \end{cases}$$

$$C_q(A+BF)^{S,t-1}B_2 = 0 \; (s,t) \ge (0,0) \qquad (33e)$$

and if $\bar{\alpha}_q + \bar{\beta}_q < \alpha_q + \beta_q$, then

$$C_q(A+BF)^{s-1,t}B_2$$

$$\begin{cases} = 0 \; (0,0) \le (s,t) < (\bar{\alpha}_q, \bar{\beta}_q) & (34a) \\ = 0 \; (s+t) = \bar{\alpha}_q + \bar{\beta}_q \\ \quad \text{except for} (s,t) = (\bar{\alpha}_q, \bar{\beta}_q) & (34b) \\ = \tilde{B}_q^* \; (s,t) = (\bar{\alpha}_q, \bar{\beta}_q) & (34c) \\ = 0 \; (s,t) > (\bar{\alpha}_q, \bar{\beta}_q) & (34d) \end{cases}$$

$$C_q(A+BF)^{s-1,t}B_1 = 0 \; (s,t) \ge (0,0) \qquad (34e)$$

To p rove eqs.(33), using eq.(20)

$$\begin{aligned} &C_q(A+BF)^{s-1,t}B_1 \\ &= C_q[A^{s-1,t} + A^{s-2,t}(BF)^{1,0} \\ &\quad + A^{s-1,t-1}(BF)^{0,1} + \cdots]B_1 \\ &= C_q[A^{s-1,t} + A^{s-2,t}B_1F \\ &\quad + A^{s-1,t-1}B_2F + \cdots]B_1 \end{aligned} \qquad (35)$$

where the identities $(BF)^{1,0} = B_1F$ and $(BF)^{0,1} = B_2F$ are used. For $(s,t) < (\alpha_q, \beta_q)$ and according to (26a), all terms in the right-hand side of (35) become zero. Thus (33a) is established. For $s+t = \alpha_q + \beta_q$ and according to (26a), all terms in the right-hand side of (35) become zero, except for the case where $(s,t) = (\alpha_q, \beta_q)$, in which case

$$C_q(A+BF)^{s-1,t}B_1 = C_qA^{\alpha_q-1,\beta_q}B_1 = \tilde{B}_q^* \qquad (36)$$

and thus eqs.(33b) and (33c) are established. For $(s,t) > (\alpha_q, \beta_q)$, we have : Let $s+t = \alpha_q + \beta_q + 1$. Then, according to (26$a$) and (29$a$) all terms in the right-hand side of (35) become zero, except for the case where $(s,t) = (\alpha_q + 1, \beta_q)$ in which case

$$\begin{aligned} &C_q(A+BF)^{\alpha_q,\beta_q}B_1 \\ &= C_q[A^{\alpha_q,\beta_q}B_1F]B_1 \end{aligned} \qquad (37)$$

Introducing (28a) and (31), the second term in above eq. become

$$\begin{aligned} &C_qA^{\alpha_q-1,\beta_q}B_1[-(\tilde{B}^*)^{-1}A^*]B_1 \\ &= -\tilde{B}_q^*(\tilde{B}^*)^{-1}A^*B_1 = -A^*_qB_1 = -C_qA^{\alpha_q,\beta_q}B_1 \end{aligned} \qquad (38)$$

In relation (37) and (36), eq (33d) is established. For $s+t = \alpha_q + \beta_q + 2$, using (4$b$), we have:

$$\begin{aligned} &C_q(A+BF)^{s-1,t}B_1 \\ &= C_q[(A+BF)^{s-2,t}(A+BF)^{1,0} \\ &\quad + (A+BF)^{s-1,t-1}(A+BF)^{0,1}]B_1 \end{aligned} \qquad (39)$$

In the above eq, the terms $C_q(A+BF)^{s-2,t}$ and $C_q(A+BF)^{s-1,t-1}$ are both zero.

This is due to the fact that, since the indices $(s-2, t)$ and $(s-1, t-1)$ are of order $(s+t)-2 = \alpha_q + \beta_q$, it follows from the previous case. Continuing in this manner, eq.(33d) may be established for all $(s,t) > (\alpha_q, \beta_q)$ Equation (33e) readily results from the procedures used to prove eqs.(33a)-(33d) and by making use of (26b). Finally, eqs. (34a)-(34e) may be proved analogously to eqs. (33a)-(33e).

Thus the proof of the theorem 2 has been completed.

5. Illustrative Examples

(Example 1)

A system is described as in (1), with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Checking the conditions of theorem 2, we have: First output: $q=1$

$$(\alpha_1, \beta_1) = (1, 0) \Rightarrow C_1B_1 = (0, 0)$$
$$(\bar{\alpha}_1, \bar{\beta}_1) = (0, 1) \Rightarrow C_1B_2 = (1, 0)$$

Hence $(\bar{\alpha}_1, \bar{\beta}_1) = (0,1)$ and $\tilde{B}_1^* = C_1A^{\bar{\alpha}_1,\bar{\beta}-1}B_2 = (1, 0)$

Second output : $q=2$

$$(\alpha_2, \beta_2) = (1, 0) \Rightarrow C_2B_1 = (1, 1)$$
$$(\bar{\alpha}_2, \bar{\beta}_2) = (0, 1) \Rightarrow C_2B_2 \Rightarrow (0, 0)$$

Hence $(\alpha_2, \beta_2) = (1,0)$ and $\tilde{B}_2{}^* = C_2A^{\alpha_2-1,\beta_2}B_1 = (1, 1)$

Thus the matrix $\tilde{B}^*$ will be

$$\tilde{B}^* = \begin{bmatrix} \tilde{B}_1^* \\ \tilde{B}_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ here, } |\tilde{B}^*| \ne 0$$

Examing the rest of the conditions of theorem 2, we have:

$q=1$

$C_1A^{1,0}=(0,\ 0,\ 0),\quad C_1A^{0,1}=(1,\ 0,\ 0)$

$q=2$

$C_2A^{1,0}=(1\ -3,\ 0),\quad C_2A^{0,1}=(0,\ 0,\ 0)$

Thus the condition of theorem 2 is all satisfied. Therefore the matrix A^* will be

$$A^*=\begin{bmatrix} A_1 \\ \cdot \\ A_2^* \end{bmatrix}=\begin{bmatrix} C_1\bar{A}^{\alpha_1,\beta_1} \\ C_2A^{\alpha_2,\beta_2} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

Hence the given system is decouplable, with

$$G^*=(\tilde{B}^*)^{-1}=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } F^*$$

$$=-(\tilde{B}^*)^{-1}A^*=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Checking, we have

$$A_1+B_1F^*=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 3 & -1 \end{bmatrix},$$

$$A_2+B_2F^*=\begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 1 \\ -1 & 4 & -1 \end{bmatrix}$$

$$[Inz_1z_2-(A_1+B_1F^*)z_2-(A_2+B_2F^*)z_1]$$

$$=\begin{bmatrix} z_1z_2 & 0 & 0 \\ -z_1 & z_1\ z_2+4z_1+3z_2 & -(z_1+z_2) \\ z_1 & -(4z_1+3z_2) & z_1z_2+z_1+z_2 \end{bmatrix}$$

$$H(z_1,\ z_2)=C[I_nz_1z_2-(A_1+B_1F^*)z_2-(A_2+B_2F^*)z_1]^{-1}(B_1z_2+B_2z_1)G^*$$

$$=\frac{1}{\triangle}\begin{bmatrix} z_1^2z_2(z_1z_2+5z_1+4z_2) & 0 \\ 0 & z_1z_2^2(z_1z_2+5z_1+4z_2) \end{bmatrix},$$

$$\triangle=z_1^2z_2^2(z_1z_2+5z_1+4z_2)$$

[Example 2]

$$A_1=\begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix},\quad A_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Checking the conditions of therem 2, we have.

$q=1$,

$(1,\ 0)\quad C_1B_1=(0,\ 0)$

$(0,\ 1)\quad C_1B_2=(0,\ 0)$

$(1,\ 1)\quad C_1A^{0,1}B_1=(0,\ 0)\quad C_1A^{0,1}B_1=(0,\ 0)$

$(2,\ 0)\ C_1A^{0,1}B_1=(1,\ 0)$

$(0,\ 2)\ C_1A^{0,1}B_1=(0,\ 0)$

Hence $(\alpha_1,\ \beta_1)=(2,\ 0)$ and $\tilde{B}_1^*=C_1A^{\alpha_1-1,\beta_1}B_1=(1,\ 0)$

$q=2$,

$(1,\ 0)\quad C_2B_1=(0,\ 0)$

$(0,\ 1)\quad C_2B_2=(0,\ 0)$

Hence $(\bar{\alpha}_2,\ \tilde{\beta}_2)=(0, 1)$ and $\tilde{B}_2^*=C_1A^{\bar{\alpha}_2,\beta_2-1}B_2=(0,\ 1)$

Thus the matrix $\tilde{B}^*$ will be

$$\tilde{B}^*=\begin{bmatrix} \tilde{B}^*_1 \\ \tilde{B}^*_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

here, $|\tilde{B}^*|\neq 0$

Examing the rest of the conditions, we have:

$q=1\quad C_1A^{2,0}=(1,\ -3,\ 1)$

$C_1A^{1,1}=(0,\ 0,\ 0)$

$C_1A^{0,2}=(0,\ 0,\ 0)$

$q=2\quad C_2A^{1,0}=(0,\ 0,\ 0)$

$C_2A^{0,1}=(-1,\ 0,\ -1)$

Thus the condition (29) is satisfied. Therefore the matrix A^* will be

$$A^*=\begin{bmatrix} A^*_1 \\ A^*_2 \end{bmatrix}=\begin{bmatrix} C_1A^{\alpha_1,\beta_1} \\ C_2A^{\bar{\alpha}_2,\beta_2} \end{bmatrix}=\begin{bmatrix} 1 & -3 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

Hence the given system is decouplable, with

$$G^*=(\tilde{B}^*)^{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and}$$

$$F^*=-(\tilde{B}^*)^{-1}A^*=\begin{bmatrix} -1 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Checking, we have

$$H(z_1,\ z_2)=C[I_nz_1z_2-(A_1+B_1F^*)z_2-(A_2+B_2F^*)z_1]^{-1}(B_1z_2+B_2z_1)G^*$$

$$= \begin{bmatrix} \frac{1}{z_1 z_2} & 0 \\ & \frac{1}{z_1} \\ 0 & \end{bmatrix}$$

The system of example 2 is equivalent to the following RM:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 1 \\ -1 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, $G^*$and F^* are equal to F-MM II except the following transfer function.

$$H(z_1, z_2) = C[zI - A - BF]^{-1}BG$$
$$= \begin{bmatrix} \frac{1}{z_1^2} & 0 \\ 0 & \frac{1}{z_2} \end{bmatrix}$$

6. Conclusions

The problem of decoupling 2-D F-MM Ⅱ using state variable feedback has been considered. The necessary and sufficient conditions for decoupling are established. For the general case, the problem of determining the feedback matrix $\boldsymbol{F}$ involves the solution of a non-linear system of algebraic equations. Uneder certain conditions, however, it is shown that an explicit formular for $\boldsymbol{F}$ may be derived. In comparision with the method for $\boldsymbol{RM}$, it appears that this method for F-MM Ⅱ is more general and algorithm is simpler. But the problem for excluding non-liner equations in general case is required for future research.

REFERENCES

[1] P.L. Falb and W.A. Wolovich, "Decoupling in the design and synthesis of multivariable control systems"IEEE Tr., AC-12, pp. 651-659, 1967.

[2] E.G. Gilbert, "The decoupling of multivariable system by state feedback"SIAM J. Control, 7, pp. 50-63, 1969.

[3] I.H. Mufti. "A note on the decoupling of multivariale systems" IEEE Tr., AC-14, pp. 415-416, 1969.

[4] W.M. Wonham and A.S. Morse, "Decoupling and pole assignment in linear multivariable systems: A geometric approach,"SIAM J. Control, 8, pp. 1-18, 1970

[5] P.N. Paraskevopoulos, "Decoupling controller design via exact model-matching techiques,"Proc. IEEE, 125, (11), pp. 1285-1289, 1978.

[6] P.N. Paraskevopoulos and P. Stavroulakis, "Decoupling of linear multivariable two-dimensional system via state feedback," Porc. EEE, 129, Pt. D, No.1, pp. 1163-1175, Jan. 1982.

[7] B. Mertzios and P.N. Paraskevopoulos, "On the input-output decoupling of 2-D Systems by state feedback," J. of the Franklin Institute, vol. 314, No.1, pp. 56-76, July 1982.

[8] B. Mertzios, "Decoupling of 2-D Systems by dynamic state feedback controllers," Reprint of 9th IFAC World Congress, Budapest, vol. VIII, pp. 144-149, July, 2-6, 1984.

[9] R.(. Rosser, "A discrete state-space model for linear image processing, IEEE Tr., AC. vol. 20, pp. 1-10, 1975.

[10] E. Fornasini and G. Marchesini, "Doubly-indexed dynamical systems:state space models and structural properties", Mathematical systems Theory vol. 12. no. 1. pp. 59-72. 1979.