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Relative Risk Aversion and Stochastic-Statistical Dominance*

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相對的 危險과 推計的-統計的 優勢法則

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Abstract

This paper presents stochastic-statistical dominance rules which eliminate dominated alternatives thereby reduce the number of satisficing alternatives to a manageable size so that the decision maker can choose the best alternative among them when neither the utility function nor the probability distribution of outcomes is exactly known. Specifically, it is assumed that only the characteristics of the utility function and the value function are known. Also, it is assumed that prior probabilities of the mutually exclusive states of nature are not known, but their relative bounds are known. First, the notion of relative risk aversion is used to describe the decision maker's attitude toward risk, which is defined with the acknowledgement that the utility function of the decision maker is a composite function of a cardinal value function and a utility function with respect to the value function. Then, stochastic-statistical dominance rules are developed to screen out dominated alternatives according to the decision maker's attitude toward risk represented in the form of the measure of relative risk aversion.

1. Introduction

There has been extensive work on what risk means and how we can measure attitude toward risk of a decision maker(DM). The first measure of risk aversion was suggested by Arrow[1] and Pratt[18], independently, on a risky choice under

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uncertainty. This measure deals with situations where alternatives have a single attribute. Next, Duncan[3] suggested a measure of risk aversion for multiattribute decision making under uncertainty.

These measures, however, are represented in terms of the utility function of a DM with the assumption that the utility function reflects the DM's attitude toward risk. This is only partially true because the utility function is to represent not only attitude toward risk but also strength of preference. That is, the utility function is a mixture of the cardinal value function for strength of preference and the utility function with respect to the value function for attitude toward risk. Thus, the measures mentioned above do not exactly reflect the DM's risk attitude.

Dyer and Sarin[5] proposed a measure for relative risk attitude of a DM, measure of relative risk aversion. This measure solely depends on the DM's risk attitude by separating the utility function into two parts: the cardinal value function for strength of preference under certainty and the utility function with respect to the value function for risk attitude under uncertainty. Lee, Fraser, and Miller[15] extended the result of Dyer and Sarin to multiattribute decision making under uncertainty.

Meanwhile, the idea of dominance of one alternative over another has been developed in economics and finance fields. The main results are stochastic dominance[2], [8], [9], [10], [16], [17], [22] and statistical dominance[6], [7], [12]. Stochastic dominance is used to choose the nondominated alternatives out of available alternatives with the assumption that the probability distribution is known exactly but the utility function is not known precisely. On the other hand, statistical dominance is used to choose the nondominated alternatives with the assumption that the probability distribution is not known exactly but the utility function is known precisely.

It is quite awkward to realize that there has been a big gap between research in measures of risk aversion and dominance principles. Quite recently, Lee[14] proposed a bridge between the measure of risk aversion and stochastic dominance. The purpose of this paper is to extend the result of Lee[14] in such a way to develop stochastic-statistical dominance using the measure of relative risk aversion.

2. Measures of Risk Attitude

What is risk? How does a DM feel about risk involved in his decision making situation? How can we express a DM's risk attitude in quantifiable form? These have been important questions hanging around research fields in economics, finance, organizational behavior, psychology, and decision sciences. The answer to these may be the following several measures of risk aversion.

Pratt[18] and Arrow[1] asserted that a measure of risk aversion can be expressed in terms of the utility function of a DM. They independently developed a measure of risk aversion, r(x),

$$r(x) = -u''(x)/u'(x), \quad \cdots\cdots (1)$$

given the single attribute utility function u(x) of a DM. Duncan[3] introduced a multiattribute measure of risk aversion in a matrix form, $R(\mathbf{x})$,

$$R(\mathbf{x}) = [-u_{ij}(\mathbf{x})/u_i(\mathbf{x})], \quad \cdots\cdots (2)$$

where $u(\mathbf{x})$ is the multiattribute utility function of DM and u_i, and u_{ij}, are the first and the second partial derivatives with respect to i and(i, j), respectively.

Dyer and Sarin[4] proposed a measurable value function which provides an interval scale of measurement for preferences under centainty. Later they[5] proposed that the utility function of a DM, which provides an interval scale of measurement for preferences under uncertainty, can be divided into the cardinal value function and the utility function with respect to the value function. They defined a measure of relative risk aversion,

$$r_v(v(x)) = -u''_v(v(x))/u'_v(v(x)). \quad \cdots\cdots (3)$$

where v(x) is the univariate cardinal value function and $u_v(v)$ is DM's utility function with respect to v.

Lee, Fraser, and Miller[15] extended the results of Dyer and Sarin to multiattribute decision making and proposed that a multiattribute utility function is a mixture of multivariate cardinal value function and a utility function with respect to the value function. They further developed a measure of risk attitude of a DM in a multiattribute setting, measure of multiattribute relative risk aversion, $r_v(v(\mathbf{x}))$,

$$r_v(v(\mathbf{x})) = -u''_v(v(\mathbf{x}))/u'_v(v(\mathbf{x})). \quad \cdots\cdots (4)$$

Both measures of relative risk attitude developed by Dyer and Sarin and Lee, Fraser, and Miller were shown to exist without their uniqueness. Krzysztofowicz [13] did experiments to explore the relationship between the value function and the utility function and found that in certain circumstances relative risk attitude of DMs is invariant.

3. Dominance Principles

3.1. Stochastic Dominance

When a DM is faced with a situation to choose the best one among several alternatives, he calculates the expected utility for each alternative and then chooses the one with the maximum expected utility[11], [19], [20]. To do so he has to know the utility function and the probability distribution of outcomes. Stochastic dominance is very useful when he does not know the utility function exactly.

When the utility function is not known exactly, he cannot calculate the expected utility(EU) for each alternative so that he cannot compare them in terms of their EUs. But if there is a little bit of information about characteristics of the utility function, then he can somehow screen out dominated alternatives and reduce the number of nondominated alternatives. Then, the DM can choose the best one from the remaining nondominated alternatives.

Specifically, suppose that there are two alternatives X and Y such that their cumulative distribution function(cdf) s are F_X and F_Y, respectively. Further suppose that the DM's utility function is u(x). Then, EUs of X and Y are

$$E[u, X]=\int_{-\infty}^{+\infty} u(x)\ dF_X(x),$$
$$E[u, Y]=\int_{-\infty}^{+\infty} u(x)\ dF_Y(x). \quad \cdots\cdots\cdots\ (5)$$

If EU of X is greater than or equal to that of Y, that is, $E[u, X] \geq E[u, Y]$, then it is said that X is preferred to Y. Usually, we compare E[u, X] with E[u, Y] by calculating them. But in some cases, we can compare them without knowing u(x) in its exact form. That is, from equation(5),

$$E[u, X]-E[u, Y] = \int_{-\infty}^{+\infty} u(x)\cdot d(F_X-F_Y)(x)$$
$$=[u(x)\{(F_X-F_Y)(x)\}]$$
$$-\int_{-\infty}^{+\infty} u'(x)(F_X-F_Y)(x)dx$$
$$=\int_{-\infty}^{+\infty} u'(x)\ (F_Y-F_X)\ (x)dx \quad \cdots\cdots\cdots\ (6)$$

So if $u'(x)\geq 0$ and $F_Y(x)\geq F_X(x)\ \forall x$, then $E[u, X]\geq E[u, Y]$. That is, if the utility function is nondecreasing and the cdf of $Y(F_Y)$ is always greater than or equal to that of $X(F_X)$, then X is preferred to Y, denoted as $X\geq_1 Y$. This is called 1st-degree stochastic dominance.

Second-degree stochastic dominance can be derived with the similar idea by expanding equation(6).

$$E[u, X]-E[u, Y]$$
$$=\int_{-\infty}^{+\infty} u'(x)\ (F_Y-F_X)\ (x)dx$$
$$=[u'(x)\ \{(F_Y^1-F_X^1)\ (x)\}]$$
$$-\int_{-\infty}^{+\infty} u''(x)\ (F_Y^1-F_X^1)\ (x)dx$$
$$=\int_{-\infty}^{+\infty}\{-u''(x)\}\ (F_Y^1-F_X^1)\ (x)dx \quad \cdots\cdots\ (7)$$

where $F_Y^1(x)=\int_{-\infty}^{x} F_Y(t)dt,$
$$F_X^1(x)=\int_{-\infty}^{x} F_X(t)dt. \quad \cdots\cdots\cdots\ (8)$$

So it is easily seen from(7) that if $u''(x)\leq 0$ and $F_Y^1\geq F_X^1\ \forall x$, then $E[u, X]\geq E[u, Y]$. That is, if the utility function is concave and the integral of cdf of Y is always greater than or equal to that of X, then X is preferred to Y, denoted as $X\geq_2 Y$. This is called 2nd-degree stochastic dominance.

Lee[14] developed stochastic dominance when the utility function of a DM can be represented using a cardinal value function, v(·), to represent the strength of preference and utility function with respect to v, $u_V(\cdot)$, to represent his risk attitude. Then, the DM's utility function u(x) is denoted as $u_v(v(x))$ where $u_V(v)$ is a univariate function which exclusively represents his attitude toward risk.

When possible outcomes of an alternative are represented as single attribute consequences, we can think of a class of utility function U_1 such that

$$U_1=\{u_v \mid \frac{du_v(v)}{dv} = u'_v\geq 0,\ x\in X\}$$
$$V_1=\{v \mid \frac{dv(x)}{dx} = v'(x)\geq 0,\ x\in X\}$$

where $u(x)=u_v(v(x))$.

According to Lee[14], lst-degree stochastic dominance can be defined as follows:

Definition 1. Let X and Y be two alternatives under consideration and F_X and F_Y be the cumulative distribution functions for X and Y, respectively. Then, X is preferred to $Y(X \geq_1 Y)$ in the sense of 1st-degree stochastic dominance if $E[u, X] \geq E[u, Y]$ for all $u_v \in U_1$ and $v \in V_1$.

Also from Lee[14], we can state 1st-degree stochastic dominance based on Definition 1 as follows:

Theorem 1. 1st-degree stochastic dominance.

For all x, $X \geq_1 Y$ if $F_Y(x) \geq F_X(x)$ where DM's utility function $u(\cdot)=u_v(v(\cdot))$ is such that $u_v \in U_1$, $v \in V_1$.

In Theorem 1, it was assumed that the decision maker's utility function u_V and the value function v are nondecreasing. Thus, no specific attitude toward risk in inferred from the assumption.

Next, we can think of another class of utility functions U_2 which is a subset of U_1 such that

$$U_2=\{u_v \mid u_v \in U_1, \frac{d^2u_v(v)}{dv^2}=u''_v \leq 0, x \in X\}$$

or $U_2=\{u_v \mid u_v \in U_1, \ r_v(v) \geq 0, \ x \in X\}$

and a subset of V_1 such that

$$V_{12}=\{v \mid v \in V_1, \ \frac{d^2v}{dx^2} \leq 0, \ x \in X\}.$$

Again, according to Lee[14], 2nd-degree stochastic dominance can be defined as follows:

Definition 2. Lex X and Y be two alternatives under consideration and F_x and F_Y be the cumulative distribution functions for X and Y, respectively. Then, X is preferred to $Y(X \geq_2 Y)$ in the sense of 2nd-degree stochastic dominance if $E[u, X] \geq E[u, Y]$ for all $u_v \in U_2$ and $v \in V_{12}$.

It is important to note that a utility function which belongs to U_2 represents relatively risk averse(RRA) behavior of the decision maker. That is, a function u_v that belongs to U_2 satisfies $u''_v \leq 0$ and $u'_v > 0$ and thus, $r_v(v) = -\frac{u''_v}{u'_v} \geq 0$. Then, by definition, the decision maker is relatively risk averse(RRA). Now we can state second-degree stochastic dominance as follows:

Theorem 2.[14] 2nd-degree stochastic dominance

Let $H_1(x)=F_Y(x)-F_X(x)$ and $H_2(x)=\int_{-\infty}^{x} H_1(t)dt$. For all x, $X \geq_2 Y$ if $H_2(x) \geq 0$ where DM's utility function $u(\cdot)=u_v(v(\cdot))$ is such that $u_v \in U_2$ and $v \in V_{12}$.

3.2. Statistical Dominance

Statistical dominance is used to select the most preferred alternative given a set of alternatives under consideration by partially ordering the alternatives in terms of the EU criterion which is also used in stochastic dominance. It is assumed that there exists a finite number of mutually exclusive and exhaustive states of nature(K) and the DM has precise knowledge about the form of the utility function depending on the selected alternative and the state of nature.

However, it is assumed that he does not know exactly the probabilities of the states of nature. Rather he can only rank the order of states of nature in terms of probability attached to each state of nature which can be denoted as

$$P_k - P_{k+1} \geq 0 \text{ for all } k=1, 2, \cdots, K-1, \quad \cdots\cdots (9)$$

where P_k is the probability that state k is the true state and K is the number of the states of nature. Now suppose that there are two alternatives X and Y and the utility function of DM is u_{ik} given i^{th} anternative and the true state is k. Then expected utilities for alternatives X and Y are

$$E[u, X] = \Sigma_{j=1}^{K} P_j \cdot u_{Xj},$$
$$E[u, Y] = \Sigma_{j=1}^{K} P_j \cdot u_{Yj}. \quad \cdots\cdots (10)$$

If $E[u, X] \geq E[u, Y]$, then we say that X is preferred to Y. From equation(10), using Abel's summation identity[6],

$$\begin{aligned} &E[u, X] - E[u, Y] \\ &= \Sigma_{j=1}^{K} P_j \cdot u_{Xj} - \Sigma_{j=1}^{K} P_j \cdot u_{Yj} \\ &= \Sigma_{j=1}^{K} P_j \cdot (u_{Xj} - u_{Yj}) \\ &= \Sigma_{j=1}^{K} [\Sigma_{s=1}^{j} (u_{Xs} - u_{Ys})] (P_j - P_{j+1}), \\ &\quad (P_{K+1} = 0) \end{aligned}$$

From equation(9), each$(P_j - P_{j+1})$ is nonnegative. If

$$\Sigma_{s=1}^{j} (u_{Xs} - u_{Ys}) \geq 0 \text{ for all } j=1, 2, \cdots, K, \quad \cdots\cdots (11)$$

then $E[u, X] \geq E[u, Y]$. That is, X dominates Y if equations(9) and(11) hold true.

If the DM can identify probability ratios more specifically, then the order of probabilities of states of nature can be represented as follows:

$$H_k \geq \frac{P_k}{P_{k+1}} \geq L_k \text{ for all } k=1, 2, \cdots, K-1, \quad (12)$$

where H_k and L_k are nonnegative constants which have the meaning of maximum and minimum proportions of P_k to P_{k+1}, respectively.

Under these assumptions, statistical dominance is stated as follows[6], [21]:

Theorem 3. statistical dominance

Let X and Y be two alternatives under consideration. Given maximum occurrence ratios and minimum occurrence ratios H_i, L_i, $i=1, 2, \cdots, K-1$, alternative X dominates Y in the sense of statistical dominance, denoted as

$X \overset{H}{\underset{L}{\geq}} Y$, if $C_{H,L=1}[f(U: H, L, K) \geq 0]$ where

$$f(U: H, L, K) = \sum_{k=1}^{K} [\{\prod_{m=k}^{K-1} L_m / \prod_{m=1}^{k-1} H_m\} \{U_{Xk} - U_{Yk}\}]$$

and $\prod_{m=1}^{0} H_m = \prod_{m=K}^{K-1} L_m = 1.$

Here, f(U: H, L, K) is actually a function of K variables$(U_1, U_2, \cdots, U_K)$ with 2(K-1) parameters and $C_{H,L=1}[f(U: H, L, K) \geq 0]$ denotes a set of $2^{(K-1)}$ inequalities. And U_{Xk} is defined to be a conditional expected utility for the alternative X given that the true state is k^{th} state, $k=1, 2, \cdots, K$.

3.3. Stochastic-Statistical Dominance

Similar to stochastic dominance and statistical dominance, stochastic-statistical dominance is used to select the subset of nondominated alternatives which is guaranteed to include the most preferred alternative, given a set of alternatives. The basic assumptions of stochastic-statistical dominance are as follows: The utility function of a DM is not known precisely. And he does not have the exact knowledge about the true state of nature, i.e., the prior probability of each state of nature is not known except its relative ranking and/or maximum and minimum occurrence ratios. However, he can assess the conditional probability distributions given each state of nature for each alternative.

As before, suppose that there exists a finite number(K) of mutually exclusive and exhaustive states of nature to represent the real world. Under the assumption and with the definitions used in stochastic dominance and statistical dominance, stochastic-statistical dominance is stated as follows: For h=1, 2, alternative X dominates Y, with the conditional cdfs F_{Xk} and F_{Yk} (k=1, 2,⋯, K), respectively, given that the true state is k, in the sense of h^{th}-degree stochastic-statistical dominance for maximum occurrence ratios($H_1, H_2, \cdots, H_{K-1}$) and minimum occurrence ratios($L_1, L_2, \cdots, H_{K-1}$), denoted as $X^H_L \geq_h Y$, if $E[u, X] \geq E[u, Y]$ for all $u \in S_h$, h=1, 2, where

$$S_1 = \{u \mid u'(x) \geq 0 \ \forall x\},$$
$$S_2 = \{u \mid u \in S_1, \ u''(x) \leq 0 \ \forall x\}.$$

The main theorem for stochastic-statistical dominance is stated below[21]: Given maximum occurrence ratios H_k and minimum occurrence ratios L_k, for k=1, 2,⋯, K-1,

1) $X^H_L \geq_1 Y$ if $C_{H,L=1}[f(D^1(x): H, L, K] \geq 0]$ for all x where DM's utility function $u(x) \in S_1$,

2) $X^H_L \geq_2 Y$ if $C_{H,L=1}[f(D^2(x): H, L, K) \geq 0]$ for all x where DM's utility function $u(x) \in S_2$

where $f(D^h(x) : H, L, K)$

$$= \sum_{k=1}^{K} [\{\prod_{m=k}^{K-1} L_m / \prod_{m=1}^{k-1} H_m\} \{D^h_k(x)\}],$$

$$D^1_k(x) = F_{Yk}(x) - F_{Xk}(x),$$

$$D^2_k(x) = \int_{-\infty}^{x} \{F_{Yk}(t) - F_{Xk}(t)\} dt \quad \text{and}$$

$$\prod_{m=1}^{0} H_m = \prod_{m=K}^{K-1} L_m = 1.$$

Here, $F_{Xk}(x)[F_{Yk}(x)]$ is a conditional cumulative distribution function for the alternative X[Y] given that k^{th} state is the true state.

4. Relative Risk Aversion and Stochastic-Statistical Dominance

Lee[14] showed that if the DM is conservative under certainty and relatively risk averse, stochastic dominance can be used to screen out dominated alternatives when the utility function is not exactly known but the probability distribution of outcomes is known precisely. In this section, we will extend the result of Lee[14] to stochastic-statistical dominance when neither the utility function nor the probability distribution is known exactly.

Now let the utility function be $u(x) = u_v(v(x))$ where v(x) is a cardinal value function and $u_v(v)$ is a utility function with respect to v. Suppose that there are m mutually exclusive and exhaustive states of nature and the probability of each state of nature being true is p_i (i=1, 2,⋯, m) where p_i's are such that H_i and L_i are maximum and minimum occurrence ratios of p_i/p_{i+1} which satisfy equation(12).

Let $f^i(x), g^i(x) (i=1, 2, \cdots, m)$ be conditional probability density functions for alternatives X and Y, respectively, given that true state is the i^{th} state. Then, expected utilities of X and Y are, respectively,

$$E(u, X)=\sum_{i=1}^{m} p_i \int u(x) f^i(x)\,dx,$$
$$E(u, Y)=\sum_{i=1}^{m} p_i \int u(x) g^i(x)\,dx, \qquad \cdots\cdots (13)$$

Now we define lst-degree stochastic-statistical dominance as follows:

Definition 3. Let F_{Xi} and F_{Yi} be the conditional cumulative distribution functions for two alternatives X and Y, respectively. Then, alternative X is preferred to Y in the sense of 1st-degree stochastic-statistical dominance, denoted as $X^H_L \geq_1 Y$, if $E[u, X] \geq E[u, Y]$ for all $u_v \in U_1$ and $v \in V_1$.

Theorem 4. Given maximum occurrence ratios and minimum occurrence ratios H_i, L_i, $i=1, 2, \cdots, m-1$, which satisfy equation(12), $X^H_L \geq_1 Y$ if

$$\sum_{i=1}^{m} \frac{\Pi_{h=i}^{m-1} L_h}{\Pi_{h=1}^{i-1} H_h} D_1^i(x) \geq 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots (14)$$

where $\Pi_{h=m}^{m-1} L_h = \Pi_{h=1}^{0} H_h = 1$, $D_1^i(x) = F_{Yi}(x) - F_{Xi}(x) = \int_{-\infty}^{x} D_0^i(t)dt$ and DM's utility function $u(v(\cdot))$ is such that $u'_v \geq 0$, $v'(x) \geq 0$.

〈Proof〉 Since the utility function u is a composite function,

$$u'(x) = \frac{du(v)}{dv} \cdot \frac{dv(x)}{dx} = u'_v(v) \cdot v'(x)$$

From(13),

$E(u, F) - E(u, G) = \sum_{i=1}^{m} p_i \int u(x)[f^i(x) - g^i(x)]\,dx.$

Let $Df = E(u, F) - E(u, G)$ and

$D_0^i(x) = -[f^i(x) - g^i(x)]$. Also, let

$D_1^i(x) = [G^i(x) - F^i(x)] = \int_{-\infty}^{x} D_0^i(t)\,dt$. Then,

$$\begin{aligned} Df &= -\sum_{i=1}^{m} p_i \int u(x)\, D_0^i(x)\,dx \\ &= \sum_{i=1}^{m} p_i \int u'(x)\, D_1^i(x)\,dx \\ &= \sum_{i=1}^{m} p_i \int u'_v v'(x)\, D_1^i(x)\,dx \\ &= \int u'_v \cdot v'(x) \cdot \sum_{i=1}^{m} p_i D_1^i(x)\,dx \\ &= \int u'_v \cdot v'(x)\, [p_1 D_1^1(x) + p_2 D_1^2(x) + \\ &\quad \cdots p_m D_1^m(x)]\,dx \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (15) \end{aligned}$$

If $u'_v \geq 0$, $v'(x) \geq 0$, and $Q \geq 0$, then $E(u, F) \geq E(u, G)$ where $Q = p_1 D_1^1(x) + p_2 D_1^2(x) + \cdots + p_m D_1^m(x) = \Sigma_{i=1}^{m} p_i D_1^i(x)$. We can assume without loss of generality that all p_i's are positive. We now employ mathematical induction for proof.

(1) Let $m=2$. $Q(2) = p_1 D_1^1(x) + p_2 D_1^2(x)$,

$$H_1 \geq \frac{p_1}{p_2} \geq L_1$$

i) $H_1 = 1 \rightarrow 1 \geq \frac{p_1}{p_2} \geq L_1 \rightarrow p_1 \geq L_1 p_2$

$$\begin{aligned} Q(2) &= p_1 D_1^1(x) + p_2 D_1^2(x) \\ &\geq L_1 p_2 D_1^1(x) + p_2 D_1^2(x) \\ &= p_2(L_1 D_1^1(x) + D_1^2(x)) \geq 0. \end{aligned}$$

ii) $L_1 = 1 \rightarrow H_1 \geq \frac{p_1}{p_2} \geq 1 \rightarrow p_2 \geq \frac{p_1}{H_1}$

$$\begin{aligned} Q(2) &= p_1 D_1^1(x) + p_2 D_1^2(x) \\ &\geq p_1 D_1^1(x) + p_2\,(p_1/H_1) D_1^2(x) \\ &= p_1(D_1^1(x) + \frac{1}{H_1} D_1^2(x)) \geq 0 \end{aligned}$$

From i) and ii),

$$Q(2) \geq Q^*(2) = \sum_{i=1}^{2} \frac{\Pi_{h=i}^{1} L_h}{\Pi_{h=1}^{i-1} H_h} D_1^i(x) \geq 0$$

(2) Let m=

3. $Q(3) = p_1 D_1^1(x) + p_2 D_1^2(x) + p_3 D_1^3(x)$

$$H_1 \geq \frac{p_1}{p_2} \geq L_1, \qquad H_2 \geq \frac{p_2}{p_3} \geq L_2.$$

i) $H_1 = H_2 = 1 \rightarrow 1 \geq \frac{p_1}{p_2} \geq L_1,\ 1 \geq \frac{p_2}{p_3} \geq L_2$

$\rightarrow p_1 \geq L_1 p_2,\ p_2 \geq L_2 p_3$

$\rightarrow p_1 \geq L_1 L_2 p_3,\ p_2 \geq L_2 p_3$

$$Q(3)=p_1D_1^1(x)+p_2D_1^2(x)+p_3D_1^3(x)$$
$$\geq L_1L_2p_3D_1^1(x)+L_2p_3D_1^2(x)+p_3D_1^3(x)$$
$$=p_3[L_1L_2D_1^1(x)+L_2D_1^2(x)+D_1^3(x)]\geq 0$$

ii) $H_1=L_2=1 \rightarrow 1\geq \frac{p_1}{p_2}\geq L_1,\ H_2\geq \frac{p_2}{p_3}\geq 1$

$$\rightarrow p_1\geq L_1p_2,\ p_3\geq \frac{p_2}{H_2}$$

$$Q(3)=p_1D_1^1(x)+p_2D_1^2(x)+p_3D_1^3(x)$$
$$\geq L_1p_2D_1^1(x)+p_2D_1^2(x)+(p_2/H_2)D_1^3(x)$$
$$=p_2[L_1D_1^1(x)+D_1^2(x)+\frac{1}{H_2}D_1^3(x)]\geq 0$$

iii) $L_1=H_2=1$

$$\rightarrow H_1\geq \frac{p_1}{p_2}\geq 1,\ 1\geq \frac{p_2}{p_3}\geq L_2$$
$$\rightarrow \frac{p_1}{p_2}\geq L_2,\ \frac{p_2}{p_3}\geq L_2,\ H_1\geq \frac{p_2}{p_3}$$
$$\rightarrow p_1\geq L_2p_2,\ p_2\geq L_2p_3,\ p_3\geq \frac{p_2}{H_1}$$

$$Q(3)=p_1D_1^1(x)+p_2D_1^2(x)+p_3D_1^3(x)$$
$$\geq L_2p_2D_1^1(x)+L_2p_3D_1^2(x)+p_3D_1^3(x)$$
$$\geq L_2p_2D_1^1(x)+(L_2p_2/H_1)D_1^2(x)+(p_2/H_1)D_1^3(x)$$
$$=p_2[L_2D_1^1(x)+\frac{L_2}{H_1}D_1^2(x)+\frac{1}{H_1}D_1^3(x)]\geq 0.$$

iv) $L_1=L_2=1$

$$\rightarrow H_1\geq \frac{p_1}{p_2}\geq 1,\quad H_2\geq \frac{p_2}{p_3}\geq 1$$
$$\rightarrow p_2\geq \frac{p_1}{H_1},\ p_3\geq \frac{p_2}{H_2}$$
$$\rightarrow p_2\geq \frac{p_1}{H_1},\ p_3\geq \frac{p_1}{H_1H_2}$$

$$Q(3)=p_1D_1^1(x)+p_2D_1^2(x)+p_3D_1^3(x)$$
$$\geq p_1D_1^1(x)+p_1/(p_1/H_1)D_1^2(x)+(p_1/H_1H_2)D_1^3(x)$$
$$=p_1[D_1^1(x)+\frac{1}{H_1}D_1^2(x)+\frac{1}{H_1H_2}D_1^3(x)]\geq 0.$$

From i), ii), iii), and iv),

$$Q(3)\geq Q^*(3)=\sum_{i=1}^{3}\frac{\Pi_{h=i}^{2}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)\geq 0.$$

(3) Suppose that equation(14) holds true for m=k.

Then $Q^*(k)=\sum_{i=1}^{k}\frac{\Pi_{h=i}^{k-1}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)\geq 0.$

Now, let's examine what happens if m= k+1,

$$Q^*(k+1)=\sum_{i=1}^{k+1}\frac{\Pi_{h=i}^{k}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)$$

$$=\sum_{i=1}^{k}\frac{\Pi_{h=i}^{k}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)+\frac{\Pi_{h=k+1}^{k}L_h}{\Pi_{h=1}^{k}H_h}D^{k+1}{}_1(x)$$
$$=\sum_{i=1}^{k}\frac{L_k\Pi_{h=i}^{k-1}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)+\frac{\Pi_{h=k+1}^{k}L_h}{\Pi_{h=1}^{k}H_h}D^{k+1}{}_1(x)$$
$$=L_k\sum_{i=1}^{k}\frac{\Pi_{h=i}^{k-1}L_h}{\Pi_{h=1}^{i-1}H_h}D_1^i(x)+\frac{1}{\Pi_{h=1}^{k}H_h}D^{k+1}{}_1(x)$$
$$=L_kQ^*(k)+\frac{1}{H_1\cdot H_2\cdots H_k}D^{k+1}{}_1(x)\geq 0.$$

Since L_k is positive, equation(14) holds for k+1. Thus, equation(14) holds for any positive integer greater than 1. ///

Similar to Definition 2(2nd-degree stochastic dominance) which is an extension to Definition 1(1st-degree stochastic dominance), we can think of a second degree stochastic-statistical dominance if the decision maker is relatively risk averse. So we now extend Definition 3 to 2nd-degree stochastic-statistical dominance as follows:

Definition 4. Let $F_{X|}$ and $F_{Y|}$ be the conditional cumulative distribution functions for two alternatives X and Y, respectively. Then, alternative X is preferred to Y in the sense of 2nd-degree stochastic-statistical dominance, denoted as $X^H_L\geq_2 Y$, if E[u, X] $\geq$E[u, Y] for all $u_v \in U_2$ and $v\in V_{12}$.

Theorem 5. Given maximum occurrence ratios and minimum occurrence ratios H_i, L_i, i=1, 2,$\cdots$, m-1, which satisfy equation(12), $X^{H}_{L} \geq_2 Y$ if $\sum_{i=1}^{m} \frac{\Pi_{h=i}^{m-1} L_h}{\Pi_{h=1}^{i-1} H_h} D_2^i(x) \geq 0$

where DM's utility function is such that $u''_v \leq 0$, $v''(x) \leq 0$, $\Pi_{h=m}^{m-1} L_h = \Pi_{h=1}^{0} H_h = 1$ and

$$D_2^i(x) = \int_{-\infty}^{x} [F_{Y_i}(t) - F_{X_i}(t)]dt$$
$$= \int_{-\infty}^{x} D_1^i(t)dt.$$

⟨Proof⟩ Since the utility function u is a composite function,

$$u''(x) = \frac{d}{dx} u'(x)$$
$$= \frac{d^2u(v)}{dv^2} \cdot [\frac{dv(x)}{dx}]^2 + \frac{du(v)}{dv} \cdot \frac{dv^2(x)}{dx^2}$$
$$= u''_v(v) \cdot [v'(x)]^2 + u'_v \cdot v''(x).$$

From equation(15),

$$Df = \int u'_v v'(x) \ [p_1 D_1^1(x) + p_2 D_1^2(x) + \cdots + p_m D_1^m(x)]dx$$
$$= [u'_v v'(x) \ \{p_1 D_2^1(x) + p_2 D_2^2(x) + \cdots + p_m D_2^m(x)\}]_{-\infty}^{+\infty} - \int \{u''_v \cdot (v'(x))^2 + u'_v \cdot v''(x)\} \ [p_1 D_2^1(x) + p_2 D_2^2(x) + \cdots + p_m D_2^m(x)]dx$$
$$= \int \{(-u''_v) \ (v'(x))^2 + u'_v(-v''(x))\} \ [p_1 D_2^1(x) + p_2 D_2^2(x) + \cdots + p_m D_2^m(x)]dx.$$

If $u'_v \geq 0$, $v'(x) \geq 0$, $u''_v \leq 0$, $v''(x) \leq 0$, and $Q^2 \geq 0$, then $E(u, F) \geq E(u, G)$ where

$$Q^2 = p_1 D_2^1(x) + p_2 D_2^2(x) + \cdots + p_m D_2^m(x)$$
$$= \Sigma^m_{i=1} \cdot p_i D_2^i(x).$$

The proof procedure is similar to the previous one by replacing $D_2^i(x)$ to $D_1^i(x)$.///

5. Conclusion and Summary

In this paper, the expected utility principle is employed as a decision criterion to choose the "bese" one given the utility function of a decision maker and the probability distribution of consequences where the utility function $u_v(v)$ is used instead of u. By doing so, we can not only accurately describe the decision maker's risk attitude but also prescribe how they should behave in multiattribute decision making under uncertainty. Therefore, the expected utility theory is sound in both descriptive and prescriptive sense.

The following assumptions are made in developing stochastic-statistical dominance. Neither the utility function of a decision maker is axactly known but only the functional form is known. Nor the probability distribution of consequences for each alternative is exactly known. Given these assumptions, stochastic-statistical dominance is developed using the measure of relative risk aversion.

Therefore, if the decision maker is conservative under certainty, his attitude toward risk is relatively risk averse and if we can rank and give relative bounds of the probabilities of the states of nature, stochastic-statistical dominance can be used to eliminate dominated alternatives from the set of feasible alternatives in order to come up with a set which includes a smaller number of nondominated alternatives. For further research, we can extend stochastic-statistical dominance developed here to the situation when alternatives have multiple attributes.

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