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A Detection Procedure of a Parameter Change Point in AR(1) Models by Bayesian Approach+

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ABSTRACT

We investigate a procedure which detects the parameter change point in AR(1) by Bayesian Approach using Jeffrey prior, for example, coefficient change point, variance change point, coefficient and variance change point, etc. And we apply our procedure to the simulated data.

1. Introduction

Change point problem has received much attention since Page(1954) proposed an approach how to detect mean level change in the independent normal sequences. Many authors has considered the problem of making inferences for parameter change point problem since then, for example, Quandt(1958, 1960), Chernoff and Zacks(1964), Hsu(1977) and Lombard(1983).

In time series analysis, Box and Tiao(1965, 1975) studied tests for changes in dependent series where the possible change point is known and Bagshaw and Johnson(1977) proposed a method for detecting step changes in AR(1) models. For a detection of step changes in the variance, Wichern et al. (1976) proposed a two-stage method in AR(1) models.

A Bayesian approach to making inferences about change points has been studied by Broemeling(1974), Smith(1975) and Menzefricke(1981). Smith and Spiegelhalter(1980, 1981, 1982) gave

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a unified Bayesian study for linear models in terms of Bayes factor. Booth and Smith(1982) suggested a Bayesian approach for the detection of level changes in time series.

In this paper, we propose a Bayesian procedure which can detect parameter change points in AR(1) models. A Bayesian approach of change point is briefly surveyed in section 2. In section 3, 4 and 5 detection procedure for the changes in the variance and the coefficient in AR(1) models are examined. Finally, in section 6, we apply our procedure to the simulated data.

2. Bayesian approach in change point problem

Let $Z_1, Z_2, \cdots, Z_n$ be realizations of a random process with the joint density $P(Z_1, \cdots, Z_n|\underset{\sim}{\theta})$, where $\underset{\sim}{\theta}$ is a vector of parameters. If there is a change in the underlying structure, the joint density of $Z_1, \cdots, Z_n$ conditional on the change point r is

$$(Z_1, \cdots, Z_n|r, \underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}) = P(Z_1, \cdots, Z_r|\underset{\sim}{\theta_1}) P(Z_{r+1}, \cdots, Z_n|\underset{\sim}{\theta_2})$$

if the process is independent, where $\underset{\sim}{\theta_2}(\underset{\sim}{\theta_1} \neq \underset{\sim}{\theta_2})$ is a vector of parameters after the change.

We shall denote by M_r the model which assumes a change point at r following Booth and Smith. If $\underset{\sim}{\theta_1}$ and $\underset{\sim}{\theta_2}$ are known, we have

$$P(Z_1, \cdots, Z_n|M_r) = P(Z_1, \cdots, Z_n|r, \underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}).$$

On the other hand, if $\underset{\sim}{\theta_1}$ and $\underset{\sim}{\theta_2}$ are unknown,

$$P(Z_1, \cdots, Z_n|M_r) = \iint P(Z_1, \cdots, Z_n|r, \underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}) P(\underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}|r) \ d\theta_1 \ d\theta_1, \qquad (2.1)$$

where $P(\underset{\sim}{\theta_1}, \underset{\sim}{\theta_2}|r)$ is a prior density for $\underset{\sim}{\theta_1}$ and $\underset{\sim}{\theta_2}$.

For convenience, we denote by M_0 the model with no change in the underlying structure. If $\underset{\sim}{\theta_1}$ is unknown, we have

$$P(Z_1, \cdots, Z_n|M_0) = \int P(Z_1, \cdots, Z_n|\underset{\sim}{\theta_1}) P(\underset{\sim}{\theta_1}) d\underset{\sim}{\theta_1},$$

where $P(\underset{\sim}{\theta_1})$ is a prior density for $\underset{\sim}{\theta_1}$. Therefore, given $Z_1, Z_2, \cdots, Z_n$, inferences about the change point are equivalent to inference about the alternative model $M_1, \cdots, M_{n-1}$.

Let $P(M_r)$ and $P(M_r|Z_1, \cdots, Z_n)$ be the prior and posterior probabilities of a change point occurring at r. The comparison of two alternative models, M_r and M_s, is given by

$$B_{rs} = \frac{P(M_r|Z_1, \cdots, Z_n)}{P(M_s|Z_1, \cdots, Z_n)} \Big/ \frac{P(M_r)}{P(M_s)} = \frac{P(Z_1, \cdots, Z_n|M_r)}{P(Z_1, \cdots, Z_n|M_s)}$$

and B_{rs} is called a Bayes factor for M_r against M_s. If $B_{rs} > 1$, it is more likely that a change occurrs at r rather than at s. If $s=0$, B_{r0} provides an indicator of a change point at r.

If we are interested in an overall assessment of change versus no change, the appropriate ratio of posterior to prior odds in given by

$$\frac{1-P(M_0|Z_1,\cdots,Z_n)}{P(M_0|Z_1,\cdots,Z_n)} \Big/ \frac{1-P(M_0)}{P(M_0)} = \sum_r B_{r0}\frac{P(M_r)}{1-P(M_0)} \tag{2.2}$$

If the $P(M_r)/(1-P(M_0))$ terms are equal over the range of r for which they are non-zero, (2.2) is just an average of the Bayes factors B_{r0}, taken over all the possible change points. If ΣB_{r0} is larger than or equal to c, a critical value, then we conclude that parameter change has occurred. Furthermore, if $\max(B_{r0} : 1 \le r < n) = B_{u0}$, u is considered to be a change point.

In the following sections, we will employ a Jeffrey prior for parameters which is the square root of determinant of information matrix and assume the equal prior for M_r.

3. Change of variance in AR(1) models

Consider the following model

$$Z_t-\mu=\phi(Z_{t-1}-\mu)+\varepsilon_t, \quad t=1,\cdots,n,$$

where ε_t's are independently and normally distributed with mean zero. We denote by M_r the model with variance σ^2 for $t=1,\cdots,r$ and τ^2 for $t=r+1,\cdots,n$ and by M_0 the model with σ^2 for $t=1,\cdots,n$.

For convenience, we assume $\mu=0$ as in Wichern et al. (1976) and Ryu and Cho(1987). Assuming ϕ is known, since the likelihood function of $Z_1,\cdots,Z_n$ under M_r is

$$P(Z_1,\cdots,Z_n|\sigma,\tau,r)=(2\pi)^{-\frac{n}{2}}(\sigma^2)^{-\frac{r}{2}}(\tau^2)^{-\frac{n-r}{2}}\sqrt{1-\phi^2}$$

$$\exp\{-\frac{1}{2\sigma^2}[(1-\phi^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2]-\frac{1}{2\tau^2}\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2\},$$

the Jeffrey prior for σ and τ given r is

$$P(\sigma,\tau|r)\propto\frac{1}{\sigma\tau}.$$

It then follows from (2.3)

$$P(Z_1,\cdots,Z_n|M_r)=\int_0^\infty\int_0^\infty P(Z_1,\cdots,Z_n|\sigma,\tau,r)P(\sigma,\tau|r)\,d\sigma d\tau$$

$$\propto 2^{\frac{n-4}{2}}(2\pi)^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-r}{2}\right)\sqrt{1-\phi^2}$$

$$\times[(1-\phi^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2]^{-\frac{r}{2}}[\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n-r}{2}} \quad (3.1)$$

In the case of M_0, by taking $P(\sigma)\propto\frac{1}{\sigma}$, we obtain

$$P(Z_1,\cdots,Z_n|M_0)=\int_0^\infty P(Z_1,\cdots,Z_n|\sigma)P(\sigma)\,d\sigma$$

$$\propto 2^{\frac{n-2}{2}}(2\pi)^{-\frac{n}{2}}\Gamma(\frac{n}{2})\ \sqrt{1-\phi^2}[(1-\phi^2)Z_1^2+\sum_2^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n}{2}} \quad (3.2)$$

If we take the ratio of (3.1) and (3.2), we get

$$B_{r0}=k\frac{\Gamma(\frac{r}{2})\Gamma(\frac{n-r}{2})\ [(1-\phi^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2]^{-\frac{r}{2}}[\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n-r}{2}}}{2\Gamma(\frac{n}{2})[(1-\phi^2)Z_1^2+\sum_2^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n}{2}}} \quad (3.3)$$

However the unpleasant and unavoidable feature of (3.3) is that it depends on the unspecified constant k. When we compare M_r and M_0, we are dealing with models of different dimensionality (one unknown parameter under M_0 versus two under M_r).

In order to motivate a choice for the value of k in (3.3), we use the idea of a "thought experiment", introduced and developed in detail in Smith and Spiegelhalter (1981) and Spiegelhalter and Smith (1982). We imagine that we have observed a sample of minimal size to enable us to compare M_0 and M_r, and that this sample has provided maximal evidence in favor of M_r. This corresponds to assuming that $n=r+1$ and we should wish to have $B_{k0}<1(k=1,\cdots,r-1)$ (evidence favors of M_0), but since we have only one observation following r, we should not wish B_{r0} to differ much from 1.

We also require the smallest possible "imaginary sample" in order for $B_{r0}=1$ to be a reasonable reflection of minimal evidence corresponding to $\sigma^2=\tau^2$. Combining these two elements, we see that an appropriate choice for a minimal sample size would be to take $r=2$, $n=r+1$, which just provides sufficient information about the two unknown parameters σ, τ.

Entering these values with $\sigma^2=\tau^2$, $B_{20}\approx1$, into (3.3) we obtain $k=\sqrt{2}/3$ and hence an explicit form for (3.3) is

$$B_{r0}=\frac{\Gamma(\frac{r}{2})\Gamma(\frac{n-r}{2})[(1-\phi^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2]^{-\frac{r}{2}}[\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n-r}{2}}}{3\sqrt{2}\Gamma(\frac{n}{2})[(1-\phi^2)Z_1^2+\sum_2^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n}{2}}} \quad (3.4)$$

If ϕ is unknown, we will employ the following Jeffrey prior for σ, τ and ϕ given r

$$P(\sigma, \tau, \phi | r) \propto \frac{1}{\sigma\tau\sqrt{1-\phi^2}}$$

for model M_r and

$$P(\sigma, \phi) \propto \frac{1}{\sigma\sqrt{1-\phi^2}}$$

for model M_0 Then

$$P(Z_1, \cdots, Z_n | M_r) \propto (2\pi)^{-\frac{n}{2}} 2^{\frac{n-4}{2}} \Gamma(\frac{r}{2}) \Gamma(\frac{n-r}{2}) \times$$

$$\int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^r (Z_i - \phi Z_{i-1})^2]^{-\frac{r}{2}} [\sum_{r+1}^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n-r}{2}} d\phi.$$

and

$$P(Z_1, \cdots, Z_n | M_0) \propto 2^{\frac{n-2}{2}} (2\pi)^{-\frac{n}{2}} \Gamma(\frac{n}{2}) \int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n}{2}} d\phi.$$

Therefore we obtain

$$B_{r0} = k \frac{\Gamma(\frac{r}{2}) \Gamma(\frac{n-r}{2}) \int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^r (Z_i - \phi Z_{i-1})^2]^{-\frac{r}{2}} [\sum_{r+1}^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n-r}{2}} d\phi}{2\Gamma(\frac{n}{2}) \int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n}{2}} d\phi} \tag{3.5}$$

If we take $k=\sqrt{2}/3$, explicit form for (3.5) is

$$B_{r0} = \frac{\Gamma(\frac{r}{2}) \Gamma(\frac{n-r}{2}) \int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^r (Z_i - \phi Z_{i-1})^2]^{-\frac{r}{2}} [\sum_{r+1}^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n-r}{2}} d\phi}{3\sqrt{2}\Gamma(\frac{n}{2}) \int_{-1}^{1} [(1-\phi^2) Z_1^2 + \sum_2^n (Z_i - \phi Z_{i-1})^2]^{-\frac{n}{2}} d\phi},$$

If μ is unknown using the similar procedure we obtain

$$B_{r0} = k \frac{\int_{-1}^{1}\int_0^{\infty}\int_0^{\infty} (\sigma^2)^{-\frac{r+1}{2}} (\tau^2)^{-\frac{n-r+1}{2}} \exp\{-\frac{1}{2} S(\phi, \sigma, \tau)\} d\sigma d\tau d\phi}{\int_{-1}^{1}\int_0^{\infty} (\sigma^2)^{-\frac{n+1}{2}} \exp\{\frac{1}{2} S(\phi, \sigma)\} d\sigma d\phi},$$

where

$$S(\phi,\sigma,\tau)=\frac{(1-\phi^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2}{\sigma^2}+\frac{\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2}{\tau^2}$$

$$-\frac{\left[\frac{1}{2}\{(1-\phi^2)Z_1+(1-\phi)\sum_2^r(Z_i-\phi Z_{i-1})\}+\frac{1}{\tau^2}(1-\phi)\sum_{r+1}^n(Z_i-\phi Z_{i-1})\right]^2}{\frac{(1-\phi^2)+(r-1)(1-\phi)^2}{\sigma^2}+\frac{(n-r)(1-\phi)^2}{\tau^2}}$$

and

$$S(\phi,\sigma)=\frac{1}{\sigma^2}\left[(1-\phi^2)Z_1^2+\sum_2^n(Z_i-\phi Z_{i-1})^2-\frac{[(1-\phi^2)Z_1+(1-\phi)\sum_2^n(Z_i-\phi Z_{i-1})^2]}{(1-\phi^2)+(n-1)(1-\phi)^2}\right].$$

To determine the value of k in this case, we have to use the numerical integration.

As we mentioned in section 2, by choosing a value of r which maximizes B_{r0}, we can identify the change point. For the overall assessment of change *vs* no change, we have to obtain the distributional property of $\sum B_{r0}$. In the simplest case, however, we have not been able to obtain it. Thus, we compute only empirical distributions of $\sum B_{r0}$. To derive them, we use the Scientistic Subrontine Package, Figure 1 shows the empirical distribution of $-2\log\sum^{19} B_{r0}$ when ϕ and μ are known. Now we can test the hypothesis $H_0 : \sigma^2=\tau^2$ *vs* $H_1 : \sigma^2\neq\tau^2$. To test above hypothesis, we use the Table 1, *i.e.* we can reject H_0 iff $-2\log\sum_{r=2}^{19}B_{r0}<c$.

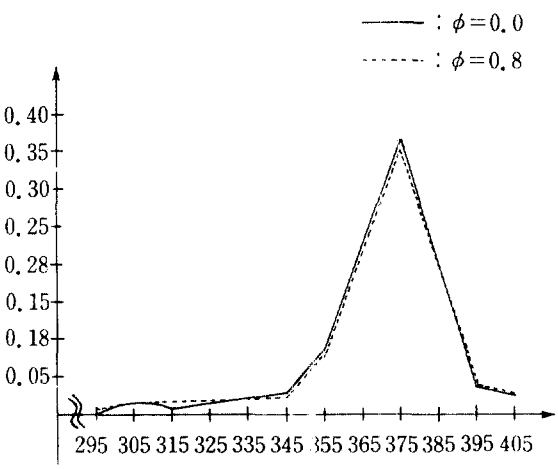

Figure 1. Empirical distribution of $-2\log\sum_{r=2}^{19}B_{r0}$ when ϕ,μ are known.

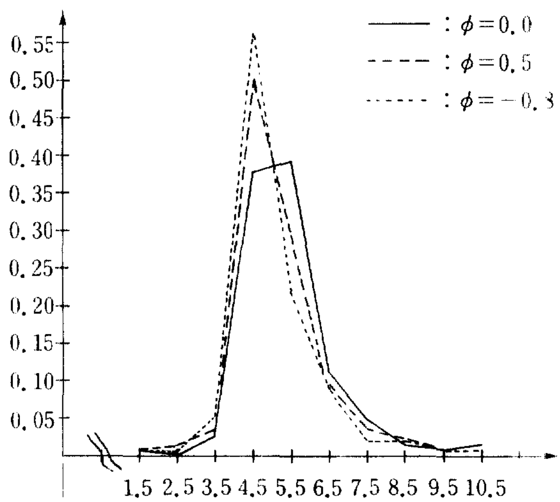

Figure 2. Empirical distribution of $\log\sum_{r=2}^{19}B_{r0}$ when μ is known and ϕ are unknown.

Likewise, Figure 2 shows the empirical distribution of $\log \sum_{r=2}^{19} B_{r0}$ when ϕ is unknown, μ is known. To test the same hypothesis, we should reject the null hypothesis iff $\log \sum_{r=2}^{19} B_{r0} > c$.

Table 1. Critical value of the empirical distribution of $-2\log \sum_{r=2}^{19} B_{r0}$ when μ, ϕ are known

ϕ \ p	0.1	0.05	0.025	0.01
−0.5	349	342	333	305
0.0	347	341	334	305
0.8	347	339	331	304

Table 2. Critical value of the empirical distribution of $\log \sum_{r=2}^{19} B_{r0}$ when μ is known but ϕ is unknown

ϕ \ p	0.1	0.05	0.025	0.01
−0.8	7.32	8.48	9.51	10.70
0.0	7.47	8.47	9.53	10.80
0.5	7.21	8.42	9.35	10.60
0.9	7.63	8.67	9.48	11.20

4. Change of coefficient ϕ in AR(1) models

Consider the model

$$Z_t - \mu = \phi(Z_{t-1} - \mu) + \varepsilon_t, \quad t = 1, \cdots, n, \qquad (4.1)$$

where ε_t's are independently and normally distributed with mean 0 and variance σ^2. If $\phi = \phi_1$ for $t = 1, \cdots, r$ and $\phi = \phi_2$ for $t = r+1, \cdots, n$, we call it by model M_r and otherwise by M_0.

As before, if we assume $\mu = 0$,

$$P(Z_1, \cdots, Z_n | \sigma, \phi_1, \phi_2, r) = (2\pi\sigma^2)^{-\frac{n}{2}} \sqrt{1-\phi_1^2}$$

$$\exp\{-\frac{1}{2\sigma^2}[(1-\phi_1^2)Z_1^2 + \sum_2^r (Z_i - \phi Z_{i-1})^2 + \sum_{r+1}^n (Z_i - \phi Z_{i-1})^2]\}$$

The Jeffrey prior for σ, ϕ_1, and ϕ_2 given r is

$$P(\sigma, \phi_1, \phi_2|r) \propto \frac{1}{\sigma\sqrt{1-\phi_1^2}\ \sqrt{1-\phi_2^2}} \cdot$$

After the integration with respect to σ,

$$P(Z_1, \cdots, Z_n|M_r) \propto \int_{-1}^{1}\int_{-1}^{1}(2\pi)^{-\frac{n}{2}}\, 2^{\frac{n-2}{2}}\, \Gamma\left(\frac{n}{2}\right)\frac{1}{\sqrt{1-\phi_1^2}} \times$$

$$\left[(1-\phi_1^2)Z_1^2 + \sum_2^r (Z_i - \phi_1 Z_{i-1})^2 + \sum_{r+1}^n (Z_i - \phi_2 Z_{i-1})^2\right]^{-\frac{n}{2}} d\phi_1 d\phi_2. \tag{4.2}$$

For model M_0, using $P(\sigma, \phi_1) \propto \frac{1}{\sigma\sqrt{1-\phi_1^2}}$, we obtain

$$P(Z_1, \cdots, Z_n|M_0) \propto \int_{-1}^{1}(2\pi)^{-\frac{n}{2}}\, 2^{-\frac{n-2}{2}}\, \Gamma\left(\frac{n}{2}\right)$$

$$\left[(1-\phi_1^2)Z^2 + \sum_2^n (Z_i - \phi_1 Z_{i-1})^2\right]^{-\frac{n}{2}} d\phi_1. \tag{4.3}$$

By taking the ratio of (4.2) and (4.3)

$$B_{r0} = k\frac{\int_{-1}^{1}\int_{-1}^{1}\frac{1}{\sqrt{1-\phi_2^2}}\left[(1-\phi_1^2)Z_1^2 + \sum_2^r (Z_i - \phi_1 Z_{i-1})^2 + \sum_{r+1}^n (Z_i - \phi_2 Z_{i-1})^2\right]^{-\frac{n}{2}} d\phi_1 d\phi_2}{\int_{-1}^{1}\left[(1-\phi_1^2)Z_1^2 + \sum_2^n (Z_i - \phi_1 Z_{i-1})^2\right]^{-\frac{n}{2}} d\phi_1}$$

In the case of unknown μ, after a tedious calculation we obtain

$$B_{r0} = k\frac{\int_{-1}^{1}\int_{-1}^{1}[S(\phi_1, \phi_2)]^{-\frac{n}{2}} d\phi_1 d\phi_2}{\int_{-1}^{1}[S(\phi_1)]^{-\frac{n}{2}} d\phi_1}$$

where

$$S(\phi_1, \phi_2) = (1-\phi_1^2)Z_1^2 + \sum_2^r (Z_i - \phi_1 Z_{i-1})^2 + \sum_{r+1}^n (Z_i - \phi_2 Z_{i-1})^2$$

$$\frac{\left[(1-\phi_1^2)Z_1 + (1-\phi_1)\sum_2^r (Z_i - \phi Z_{i-1}) + (1-\phi_2)\sum_{r+1}^n (Z_i - \phi_2 Z_{i-1})\right]^2}{(1-\phi_1^2) + (r-1)(1-\phi_1)^2 + (n-r)(1-\phi_2)^2}$$

and

$$S(\phi_1)=(1-\phi_1^2)Z_1^2+\sum_2^n(Z_i-\phi_1 Z_{i-1})^2-\frac{[(1-\phi_1^2)Z_1^2+(1-\phi_1)\sum_2^n(Z_i-\phi_1 Z_{i-1})]^2}{(1-\phi_1^2)+(n-1)(1-\phi_1)^2}$$

As we mentioned in section 3, the value of k can be determined using a numerical integration.

5. Change in the variance and the coefficient in AR(1) models

Up to this point we considered the case where the variance σ^2 or the coefficient ϕ is subject to change but not both. It may be more realistic to consider that both of them are subject to change, see Wichern et al. (1976) and Ryu and Cho (1987), for example. In this section we will consider the situation where both of ϕ and σ^2 are subject to change simultaneously.

Consider the mdoel (4.1). As before we will assume $\mu=0$ for convenience. We denote by M_r the model with $\phi=\phi_1$ and $\text{Var}(\varepsilon_t)=\sigma^2$ for $t=1,\cdots,r$ and $\phi=\phi_2$ and $\text{Var}(\varepsilon_t)=\tau^2$ for $t=r+1,\cdots,n$. Under M_0, $\phi_1=\phi_2$ and $\sigma^2=\tau^2$.

The likelihood function of $Z_1,\cdots,Z_n$ under M_r is

$$P(Z_1,\cdots,Z_n|\phi_1,\phi_2,\sigma,\tau,r)=(2\pi)^{-\frac{n}{2}}(\sigma^2)^{-\frac{r}{2}}(\tau^2)^{-\frac{n-r}{2}}\sqrt{1-\phi_1^2}$$

$$\exp\{-\frac{1}{2\sigma^2}[(1-\phi_1^2)Z_1^2+\sum_2^r(Z_i-\phi Z_{i-1})^2]-\frac{1}{2\tau^2}\sum_{r+1}^n(Z_i-\phi Z_{i-1})^2\}.$$

Using the Jeffrey prior for ϕ_1,ϕ_2, σ and τ given r

$$P(\phi_1,\phi_2,\sigma,\tau|r)\propto\frac{1}{\sigma\tau\sqrt{1-\phi_1^2}\sqrt{1-\phi_2^2}}$$

we obtain

$$P(Z_1,\cdots,Z_n|M_r)=\int_{-1}^1\int_{-1}^1\int_0^\infty\int_0^\infty P(Z_1,\cdots,Z_n|\phi_1,\phi_2,\sigma,\tau,r)\ P(\phi_1,\phi_2,\sigma,\tau|r)\,d\sigma d\tau d\phi_1 d\phi_2$$

$$\propto\int_{-1}^1\int_{-1}^1[(1-\phi_1^2)Z_1^2+\sum_2^r(Z_i-\phi_1 Z_{i-1})^2]^{-\frac{r}{2}}[\sum_{r+2}^n(Z_i-\phi_2 Z_{i-1})^2]^{-\frac{n-r}{2}}d\phi_1 d\phi_2$$

In the case of M_0, we take $P(\phi,0)\propto\frac{1}{\sigma\sqrt{1-\phi^2}}$ then

$$P(Z_1,Z_2,\cdots,Z_n|M_0)\propto\int_{-1}^1[(1-\phi^2)Z_1+\sum_2^n(Z_i-\phi Z_{i-1})^2]^{-\frac{n}{2}}d\phi$$

By taking the ratios of $P(Z_1,\cdots,Z_n|M_r)$ and $P(Z_1,\cdots,Z_n|M_0)$,

$$B_{r0}=k\frac{\int_{-1}^{1}[(1-\phi_1^2)Z_1^2+\sum_{2}^{r}(Z_i-\phi_1 Z_{i-1})^2]^{-\frac{r}{2}}d\phi_1\int_{-1}^{1}[\sum_{r+2}^{n}(Z_i-\phi_2 Z_{i-1})^2]^{-\frac{n-r}{2}}d\phi_2}{\int_{-1}^{1}[(1-\phi^2)Z_1+\sum_{2}^{n}(Z_i-\phi Z_{i-1})^2]^{-\frac{n}{2}}d\phi}$$

The value of k can be determined as in section 4.

6. Numerical example and extensions

In this section we apply our procedure to the simulated data. For the simulation, we generate normal random numbers using GGNML is the IMSL subroutine and use DQG32 in SSP for the integration. Since the integrand is $n/2$ −th order, the integration or other computations are unstable if n is large. Thus we use $n=20$ in our simulation. The simulation results are summarized in table 3 and 4. We restrict our simulation to the case where the variance is subject to change. In other cases, B_{r0}'s are too complicated, hence is not considerd. The point r which maximize B_{r0} is selected as a change point as we explained in section 2. Simulation results show that when μ and ϕ are known our procedure detects the change point very well. In the case of unknown ϕ, it does not perform as well.

For the overall assessment of change *vs* no change we need a knowledge the distributional property of $\sum B_{r0}$. We can use the critical value of the empirical distribution in section 3.

Table 3. Estimates of parameters (ϕ, μ : known)

true values				estimates		
ϕ	r	σ^2	τ^2	$\hat{r}$	$\hat{\sigma}^2$	$\hat{\tau}^2$
0.5	10	1	9	10	0.567	6.966
0.5	10	1	5	10	0.567	3.870
0.5	10	1	0.25	10	0.567	0.193
0.2	8	1	9	8	0.596	6.403
0.2	8	1	5	8	0.596	3.557
0.2	8	1	0.25	8	0.596	0.177
−0.5	17	1	9	17	0.719	2.832
−0.5	17	1	5	17	0.719	1.574
−0.5	17	1	0.25	17	0.719	0.080

Table 4. Estimates of Parameters (known μ and unknown but fixed ϕ)

true values				estimates		
ϕ	r	σ^2	τ^2	$\hat{r}$	$\hat{\sigma}^2$	$\hat{\tau}^2$
0.5	6	1	9	7	1.09	7.62
0.5	6	1	5	7	0.91	4.22
0.5	6	1	0.25	6	0.73	0.19
0.2	17	1	9	14	0.71	4.32
0.2	17	1	5	14	0.67	2.89
0.2	17	1	0.25	12	0.73	0.53
−0.5	10	1	9	10	0.71	9.11
−0.5	10	1	5	10	0.68	5.06
−0.5	10	1	0.25	8	0.70	0.25

In this paper, we consider the model which has only one change point. As was mentioned in Ryu and Cho(1987), we can extend our procedure to the model having more than one change points. For details, see Ryu and Cho(1987).

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