Journal of the KSQC
Vol. 17, No. 2, 1989

Estimation of Normal Variance Considered Prior Information

Lee, Sang-do*
Lee, Dong-choon*
Park, Ki-joo**

ABSTRACT

In this paper we present the shrunken testing estimator for the variance of normal population and we find the condition that can be used in seeking the situations in which the proposed estimator is superior to the minimum variance unbiased estimator.

1. Introduction

In parametric model, the estimating unknown parameter is one of the important problems. Usually we use minimum variance unbiased estimator for the unknown parameter. But we may consider one kind of biased estimators which are considerd for the substantial reduction of mean square error(MSE). Recently these problems are considered widly. (for well-known example, Bayesian approach, Ridge estimators, etc). In this paper we shall propose a new type biased estimator of so-called shrinkage estimator. And we shall obtain the testing condition that the proposed estimator may be used and the relative efficiency of proposed estimator with respect to minimum variance unbiased estimator.

*Dept. of Industrial Engineering, Dong-A University
**Dept. of Industrial Engineering, Kyungbuk Sanup University

2. Proposed Estimators for Normal Variance

A random variable x is distributed with the density function

$$f_x(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\quad -\infty<x<\infty \tag{1}$$

To astimate σ^2 in this distribution, s^2(sample variance) which is MVUE of σ^2 is often used. The shrunken estimator by Thompson for σ^2 is

$$\hat{\sigma}^2=hs^2+(1-h)\sigma_0^2 \tag{2}$$

Where, σ_0^2 is a prior value of σ^2, s^2 is minimum variance unbiased estimator for σ^2, and h is a constant between zero and one.
Ordinary the shrinkage estimators have more relative efficiency than s^2 if σ_0^2 is near to σ^2. So has this estimator. Then we can calculate the mean square error of $\hat{\sigma}^2$ easily as follow :

$$\text{MSE}(\hat{\sigma}^2)=E(\hat{\sigma}^2-\sigma^2)^2=\frac{2h^2\sigma^4}{n-1}+(1-h)^2(\sigma_0^2-\sigma^2)^2 \tag{3}$$

Since the mean square error of s^2 which denoted by Rohattgi is

$$\text{MSE}(s^2)=\text{Var}(s^2)=\frac{2\sigma^4}{n-1} \tag{4}$$

The relative efficiency of $\hat{\sigma}^2$ with respect to s^2 is

$$\text{REF}(\hat{\sigma}^2,\ s^2)=\frac{\text{MSE}(s^2)}{\text{MSE}(\hat{\sigma}^2)}=\frac{1}{h^2+(1-h)^2\left(\frac{n-1}{2}\right)\left(1-\frac{\sigma_0^2}{\sigma^2}\right)^2} \tag{5}$$

In this $\text{REF}(\hat{\sigma}^2,\ s^2)$, the condition of $\text{REF}(\hat{\sigma}^2,\ s^2)\geqq 1$ is

$$\frac{\left(\frac{n-1}{2}\right)\left(1-\frac{\sigma_0^2}{\sigma^2}\right)^2-1}{\left(\frac{n-1}{2}\right)\left(1-\frac{\sigma_0^2}{\sigma^2}\right)^2+1}\leqq h\leqq 1 \tag{6}$$

And the closer the value of σ_0^2/σ^2 to 1, the greater will be REF $(\hat{\sigma}^2,\ s^2)$. So we propose secondary shrinkage test estimator

$$\hat{\sigma}_2^2 = \begin{cases} hs^2 + (1-h)\sigma_0^2 & , \quad \text{for } s^2 \in R \\ s^2 & , \quad \text{for } s^2 \notin R \end{cases} \tag{7}$$

i.e. in estimation of σ^2, if s^2 satisfies the region R, we use

$$hs^2 + (1-h)\sigma_0^2 \tag{8}$$

and otherwise we use s. According to the methodology of Katti, the region R is determined by the testing hypothesis ;

$$H_0 : \sigma^2 = \sigma_0^2 \qquad \text{against } H_1 : \sigma^2 \neq \sigma_0^2$$

If the null hypothesis is accepted, then we use $hs^2 + (1-h)\sigma^2$ for estimating σ^2. So we obtained the condition that null hypothesis is accepted.

The condition by Katti is

$$A_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq A_2 \tag{9}$$

where, $X \sim \chi^2(n-1)$

$$P_r\{X > A_2\} = 1 - \alpha/2, \quad P\{X < A_1\} = \alpha/2$$

Therefore the complete estimator is

$$\hat{\sigma}_2^2 = \begin{cases} hs^2 + (1-h)\sigma_0^2 & , \quad \frac{A_1\sigma_0^2}{n-1} \leq s^2 \leq \frac{A_2\sigma_0^2}{n-1} \\ s^2 & , \quad \text{otherwise} \end{cases} \tag{10}$$

Then for the calculation of REF$(\hat{\sigma}_2^2, s^2)$, we obtained expected value of $\hat{\sigma}_2^2$ and the mean square error of $\hat{\sigma}_2^2$. These are ;

$$E(\hat{\sigma}_2^2) = \int_R \{hs^2 + (1-h)\sigma_0^2\} f(s^2)\, ds^2 + \int_{RC} s^2 f(s^2)\, ds^2 \tag{11}$$

where, $$f(s^2) = \frac{1}{\left(\frac{2\sigma^2}{n-1}\right)^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{(n-1)S^2}{2\sigma^2}} (S^2)^{\frac{n-1}{2}-1}$$

And $$E(\hat{\sigma}_2^2) = \sigma^2 - (1-h)\sigma^2 \cdot K + (1-h)\sigma_0^2 \cdot L \tag{12}$$

where , $F(x, r)=\frac{1}{\Gamma(r)}\int_0^x e^{-t}t^{r-1}dt$

$$K=F\left(\frac{A_2\sigma_0^2}{2\sigma^2}, \frac{n+1}{2}\right)-F\left(\frac{A_1\sigma_0^2}{2\sigma^2}, \frac{n+1}{2}\right)$$

$$L=F\left(\frac{A_2\sigma_0^2}{2\sigma^2}, \frac{n-1}{2}\right)-F\left(\frac{A_1\sigma_0^2}{2\sigma^2}, \frac{n-1}{2}\right)$$

So the estimator $\hat{\sigma}_2^2$ has bias $(1-h)\{\sigma_0^2 \cdot L-\sigma^2K\}$
Similarly,

$$\mathrm{MSE}(\hat{\sigma}_2^2)=E(\hat{\sigma}_2^2-\sigma^2)^2$$

$$=\int_{\frac{A_1\sigma_0^2}{n-1}}^{\frac{A_2\sigma_0^2}{n-1}}[hs^2+(1-h)\sigma_0^2-\sigma^2]^2f(s^2)ds^2$$

$$+\left[\int_0^{\frac{A_1\sigma_0^2}{n-1}}(s^2-\sigma^2)^2f(s^2)ds^2+\int_{\frac{A_2\sigma_0^2}{n-1}}^{\infty}(s^2-\sigma^2)^2f(s^2)ds^2\right]$$

$$=\frac{2\sigma^4}{n-1}[1-(1-h^2)\frac{n+1}{2}\cdot M+\left\{\frac{(1-h)\sigma_0^2}{2\sigma^2}-1\right\}(n-1)(1-h)\frac{\sigma_0^2}{\sigma^2}\cdot L$$

$$+\left(1+h\cdot\frac{\sigma_0^2}{\sigma^2}\right)(n-1)(1-h)\cdot K] \tag{13}$$

where, $M=F\left(\frac{A_2\sigma_0^2}{2\sigma^2}, \frac{n+3}{2}\right)-F\left(\frac{A_1\sigma_0^2}{2\sigma^2}, \frac{n+3}{2}\right)$

$$=\frac{1}{\Gamma\left(\frac{n+3}{2}\right)}\left\{\int_0^{\frac{A_2\sigma_0^2}{2\sigma^2}} e^{-t}, t^{\frac{n+1}{2}}dt-\int_0^{\frac{A_1\sigma_0^2}{2\sigma^2}} e^{-t}\cdot t^{\frac{n+1}{2}}dt\right\}$$

Therefore $\mathrm{REF}(\hat{\sigma}_2^2, s^2)=\mathrm{MSE}(s^2)/\mathrm{MSE}(\hat{\sigma}^2)$

$$=[1-(1-h^2)\frac{n+1}{2}M+\left\{\frac{(1-h)\sigma_0^2}{2\sigma^2}-1\right\}(n-1)(1-h)\frac{\sigma_0^2}{\sigma^2}\cdot L$$

$$+(1+h)\frac{\sigma_0^2}{\sigma^2}(n-1)(1-h)\cdot K]^{-1} \tag{14}$$

And the condition of $\mathrm{REF}(\hat{\sigma}_2^2, s^2)\geq 1$ can be easily obtained ;

$$
\left.
\begin{aligned}
&Q/P \le h \le 1 \\
&Q=\left(\frac{n-1}{2}\right)\left(\frac{\sigma_0^2}{\sigma^2}\right)^2 \cdot L+(n-1)K-\frac{n+1}{2}M-(n-1)\frac{\sigma_0^2}{\sigma^2}L \\
&P=\frac{n+1}{2}\cdot M+\left(\frac{n-1}{2}\right)\left(\frac{\sigma_0^2}{\sigma^2}\right)^2 \cdot L-(n-1)\left(\frac{\sigma_0^2}{\sigma^2}\right)\cdot K
\end{aligned}
\right\} \tag{15}
$$

Now, we consider another shrinkage estimator of s^2.

$$
\hat{\sigma}_2^2=\begin{cases} hs^2+(1-h)\sigma_0^2 & , \quad \dfrac{A_1\sigma_0^2}{n-1}\le s^2 \le \dfrac{A_2\sigma_0^2}{n+1} \\ s^2 & , \quad \text{otherwise} \end{cases} \tag{16}
$$

If $h=1$, then σ^2 always reduces to s^2, and if $h=0$, then σ^2 reduce to

$$
\hat{\sigma}_3^2=\begin{cases} \sigma_0^2 & , \quad \dfrac{A_1\sigma_0^2}{n-1}\le s^2 \le \dfrac{A_2\sigma_0^2}{n+1} \\ s^2 & , \quad \text{otherwise} \end{cases} \tag{17}
$$

We regards $\hat{\sigma}_3^2$ as a preliminary estimator.

Then in eq. (12), $E(\hat{\sigma}_2^2)=\sigma^2-\sigma^2K-\sigma_0^2L$

So Bias $(\hat{\sigma}_3^2)=\sigma_0^2L-\sigma^2K$

and eq. (13),

$$
\begin{aligned}
\text{MSE}(\hat{\sigma}_3^2) &= \frac{2\sigma^4}{n-1}\left[1-\frac{n+1}{2}M+\left\{\frac{\sigma_0^2}{2\sigma^2}-1\right\}(n-1)\frac{\sigma_0^2}{\sigma^2}\cdot L+(n-1)K\right] \\
&=2\sigma^4\left[\frac{1}{n-1}-\frac{n+1}{2(n-1)}M+\left\{\frac{\sigma_0^2}{2\sigma^2}-1\right\}\frac{\sigma_0^2}{\sigma^2}\cdot L+K\right]
\end{aligned} \tag{18}
$$

In eq. (13), to obtain the value of h which minimize MSE, we differentiate MSE$(\hat{\sigma}_2^2)$ with respect to h, putting it equal to zero.

Then we get ;

$$
h=\frac{[\sigma_0^4-\sigma^2\sigma_0^2]\cdot L+[\sigma^4-\sigma^2\sigma_0^2]\cdot K}{\frac{n+1}{n-1}\sigma^4\cdot M+\sigma_0^4\cdot L-2\sigma^2\cdot\sigma_0^2\cdot K} \tag{19}
$$

Again differentiating MSE$(\hat{\sigma}_2^2)$ with respect to h, we get

$$2\{(n+1)/(n-1)\}M\sigma^4+2\sigma_0^4-4\sigma^2\sigma_0^2K \tag{20}$$

This form will be always positive. (It is proved by Pandey(1980)). Hence the value of h in eq. (19) will give the minimum MSE. And in eq. (19) if $K=L=M=1$, which means we always use $\hat{\sigma}^2=hs^2+(1-h)\sigma_0^2$ for estimating σ^2, the value of h which minimizes the MSE($\hat{\sigma}^2$) is given by

$$h=\frac{(\sigma^2-\sigma_0^2)^2}{(\sigma^2-\sigma_0^2)+\dfrac{2\sigma^4}{n-1}} \tag{21}$$

In this form, h contains unknown parameter σ^2, so the value of h may not be calculated. If we replace σ^2 by its consistent estimator s^2, then it becomes

$$h'=\frac{(s^2-\sigma_0^2)^2}{(s^2-\sigma_0^2)+\dfrac{2s^4}{n-1}} \tag{22}$$

Using the h' we define a modified estimator in place of σ^2 such as

$$\sigma_N^2=h's^2+(1-h')\sigma_0^2$$

$$=h'(s^2-\sigma_0^2)+\sigma_0^2$$

$$=\frac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}+\sigma_0^2 \tag{23}$$

Actually this shrunken estimator is defined in case of $K=L=M=1$. So we propose the completed shrunken estimator.

$$\hat{\sigma}_{2N}^2=\begin{cases} h's^2+(1-h')\sigma_0^2 & , \quad \dfrac{A_1\sigma_0^2}{n-1}\leq s^2\leq\dfrac{A_2\sigma_0^2}{n-1} \\ s^2 & , \quad \text{otherwise}\end{cases} \tag{24}$$

where , $h'=\dfrac{(\sigma_0^4-s^2\sigma_0^2)\cdot L'+(s^4-s^2\sigma_0^2)\cdot K'}{\dfrac{n+1}{n-1}s^4M'+\sigma_0^4\cdot L'-2s^2\sigma_0^2\cdot K'}$

K', L', M' are obtained by putting s^2 in place of σ^2 in K, L, M. But this proposed estimator will be quite complicated one. So for simplicity we consider a modified estimator $\hat{\sigma}_{2N}^2$ such as

$$\hat{\sigma}_{3N}=\begin{cases}\dfrac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}+\sigma_0^2 & , \quad \text{for } \dfrac{A_1\sigma_0^2}{n-1}\le s^2\le\dfrac{A_2\sigma_0^2}{n-1} \\ s^2 & , \quad \text{otherwise}\end{cases} \tag{25}$$

Now we have shown that this estimator has higher relative efficienties than s^2

$$E(\hat{\sigma}_{3N}^2)=\int_{\frac{A_1\sigma_0^2}{n-1}}^{\frac{A_2\sigma_0^2}{n-1}}\left\{\frac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}+\sigma_0^2\right\}+f(s^2)\,ds^2$$

$$+\left[\int_0^{\frac{A_1\sigma_0^2}{n-1}}s^2f(s^2)\,ds^2+\int_{\frac{A_1\sigma_0^2}{n-1}}^{\infty}s^2f(s^2)\,ds^2\right]$$

$$=\sigma^2-\int_{\frac{A_1\sigma_0^2}{n-1}}^{\frac{A_2\sigma_0^2}{n-1}}\left\{\frac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}-(s^2-\sigma_0^2)\right\}f(s^2)\,ds^2 \tag{26}$$

$$\text{Bias}(\hat{\sigma}_{3N}^2)=\int_{\frac{A_1\sigma_0^2}{n-1}}^{\frac{A_2\sigma_0^2}{n-1}}\left\{\frac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}-(s^2-\sigma_0^2)\right\}f(s^2)\,ds^2 \tag{27}$$

$$\text{MSE}(\hat{\sigma}_{3N}^2)=\int_{\frac{A_1\sigma_0^2}{n-1}}^{\frac{A_2\sigma_0^2}{n-1}}\left\{\frac{(s^2-\sigma_0^2)^3}{(s^2-\sigma_0^2)^2+\dfrac{2s^4}{n-1}}+\sigma_0^2-\sigma^2\right\}^2f(s^2)\,ds^2$$

$$+\int_{R^c}(s^2-\sigma^2)^2f(s^2)\,ds^2 \tag{28}$$

$$\text{REP}(\hat{\sigma}_{3N}^2,\ s^2)=\frac{\text{MSE}(s^2)}{\text{MSE}(\hat{\sigma}_{3N}^2)} \tag{29}$$

These integral in REF$(\hat{\sigma}_{3N}^2,\ s^2)$ can be evaluated by using quadrature formular. (c. f. : Numerical Analysis)

So we have calculated the REF$(\hat{\sigma}_{3N}^2,\ s^2)$ for different values of n, σ_0^2/σ^2 and significant level α, using sixteen points Gauss-Legendre formular for integral. The results are shown in Table (1), (2).

3. Conclusions

Recently shrinkage method which reduces mean square error are widly used in estimation of

Table 1. The relative efficiencies of $\hat{\sigma}_{3N}^2$ w.r.to s^2 $(\alpha=0.01)$

σ_0^2/σ^2 \ n	3	5	7	9	15	25
0.25	0.9174	0.8703	0.8660	0.8794	0.9387	0.9887
0.50	1.1854	1.0572	0.9712	0.9115	0.8171	0.7829
0.75	1.7772	1.6652	1.5767	1.4027	1.3331	1.1485
1.00	2.4299	2.2949	2.2319	2.1950	2.1435	2.1125
1.25	2.6462	2.2479	2.0467	1.1900	1.6505	1.3990
1.50	2.6868	1.7963	1.5237	1.3532	1.0730	0.8688
2.00	1.6371	1.1130	0.9170	0.8153	0.6915	0.6431
3.00	0.9058	0.6668	0.6146	0.6094	0.6554	0.7795

Table 2. The relative efficiencies of $\hat{\sigma}_{3N}^2$ w.r.to s^2 $(\alpha=0.05)$

σ_0^2/σ^2 \ n	3	5	7	9	15	25
0.25	0.9355	0.9148	0.9191	0.9325	0.9723	0.9962
0.50	1.0303	0.9674	0.9204	0.8873	0.8379	0.9414
0.75	1.1289	1.2862	1.2560	1.2222	1.1292	1.0132
1.00	1.6639	1.7315	1.7601	1.7756	1.7993	1.8137
1.25	1.9509	1.9047	1.8282	1.7523	1.4581	1.3278
1.50	1.9919	1.6821	1.4681	1.3155	1.0411	0.8337
2.00	1.5742	1.1039	0.9085	0.8031	0.6724	0.6477
3.00	0.9053	0.6644	0.6080	0.5988	0.6688	0.8753

parameter. If we have prior information for unknown parameter, this method is useful tool. Thompson proposed a shrinkage estimator and said the estimator is useful if σ_0^2 nears to σ^2. But his estimator has arbitrary constant h between zero and one. So we considered the value of h which minimize MSE and proposed modified shrunken estimator. And we obtained the condition that the proposed estimator can be used. Next, we calculated the relative efficiencies of this modified testing estimator w.r.to s^2.

The result is that the proposed estimator has more relative efficiencies than s^2 when sample size is small and $0.5\leq\sigma_0^2/\sigma^2\leq1.5$. So in estimation of normal variance, if sample size is small and $0.5\leq\sigma_0^2/\sigma^2\leq1.5$, then $\hat{\sigma}_{3N}^2$ can be useful estimator in view of MSE. Finally, the region of $(A_1\sigma_0{}^2)/(n-1)\leq s^2\leq(A_2\sigma_0^2)/(n-1)$ says the condition that $\hat{\sigma}_{3N}^2$ can be used.

REFERANCES

1. Farebrother, R.W. (1978), *"A Class of shrinkage estimators,"* J. Royal Stat. Soc. B(40), pp 47-49.
2. Gunst, F.G. and Mason, R.L. (1977), *"Biased estimation in regression : An evaluation using mean square error"*, J. Amer. Stat. Assoc., pp 616-628.
3. Katti, S.K. (1962), *"Use of some apriori knowledge in the estimation of means from double samples"*, Biometrics 18, pp 139-147.
4. Lee, B.M.S. and Trivedi, P.K. (1982), *"Error misspecification and properties of the simple ridge estimator"*, Commn. Stat -Theo. 11(14), pp 1615-1624.
5. Ryzhik, I.M. and Gradshteyn, I.S. (1965), *"Tables of integrals, series and products"*, 4th edition, Academic Press.