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THE RELATION BETWEEN THE EXTERIOR RADIUS AND THE MEAN CURVATURE OF BOUNDED IMMERSION

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In this paper, we study an estimate for the exterior radius of a complete manifold immersed in some ambient spaces. In [Na], Nash showed that any noncompact n-dimensional Riemannian manifold can be isometrically imbedded in a ball of preassigned radius $\varepsilon > 0$ in $\mathbf{R}^{n+k}$ if the codimension k is large enough. First Calabi asked whether there is any complete minimal surface of $\mathbf{R}^3$ which is a subset of the unit ball ([Ya] problem section ♯91). The results in this direction without codimension assumption are of Aminov [Am], Jorge and Koutroufiotis [JK] and Jorge and Xavier [JX]. As a generalization of Jorge and Xavier [JX] we can prove the following.

Theorem 1. *Let M be a complete Riemannian manifold with scalar curvature S that satisfies $S(x) \geq -C(1+r^2(x))$ for some constant $C > 0$ where $r(x)$ is the distance from a fixed point $x_0 \in M$ to x in M and let N be a complete Riemannian manifold with sectional curvature bounded from above by δ^2, $\delta \geq 0$. For $y_0 \in N$, let $B_R(y_0)$ be a closed geodesic ball of radius R centered at y_0 in N which does not intersect the cut locus of y_0. Suppose $u : M \to B_R(y_0) \subset N$ is an isometric immersion with bounded mean curvature H (say $\|H\| \leq H_0$). Then the following holds*

(a) *if $\delta > 0$ and $R < \frac{\pi}{2\delta}$, $R \geq \frac{1}{\delta}\ arctan(\frac{\delta}{H_0})$*

(b) *if $\delta = 0$ and N is simply connected, $R \geq \frac{1}{H_0}$.*

In order to prove the theorem, we need some lemmas.

Lemma 2 [**Ka**]. *Suppose M^n is isometrically immersed in N^m. If $f : N \to \mathbf{R}$, then*

$$\triangle_M f = tr_M(\overline{\nabla}^2 f) + n\langle H, \quad grad_N f\rangle_N$$

where $\overline{\nabla}$ is the Riemannian connection on N and H is the mean curvature vector of the immersion.

Lemma 3 [**Ka**]. *Suppose M is a complete Riemannian manifold with $Ric(x) \geq -C(1 + r^2(x))$ for some constant $C > 0$ where $r(x)$ denotes the distance from a fixed point $x_0 \in M$ to x in M. If $u : M \to \mathbf{R}$ and* $\sup u < +\infty$, *then* $\inf_M \triangle u \leq 0$.

Lemma 4. *Suppose the sectional curvature on a closed geodesic ball $B_R(y_0)$ of radius R centered at y_0 in N which does not intersect the cut locus of y_0 is bounded above by 1 and $f(y) = 1 - \cos\rho(y)$ on $B_R(y_0)$ where $\rho(y)$ denotes the distance from a fixed point $y_0 \in N$ to y in N. Then $\bar{\nabla}^2 f \geq \cos\rho\, ds_N^2$ on $B_R(y_0)$.*

Proof. Let $\bar{\rho}(y)$ be a distance from a fixed point $\bar{y_0} \in S^m$ to y in the sphere S^m of dimension m with constant sectional curvature 1. Then

$$\nabla^2_{S^m}\bar{\rho} = \frac{\cos\bar{\rho}}{\sin\bar{\rho}}[ds^2_{S^m} - d\bar{\rho}\otimes d\bar{\rho}]([GW], p.30)$$

and so

$$\begin{aligned} \nabla^2_{S^m} g(\bar{\rho}) &= g''(\bar{\rho})d\bar{\rho}\otimes d\bar{\rho} + g'(\bar{\rho})\,\nabla^2_{S^m}\bar{\rho} \\ &= (\cos\bar{\rho})d\bar{\rho}\otimes d\bar{\rho} + \cos\bar{\rho}[ds^2_{S^m} - d\bar{\rho}\otimes d\bar{\rho}] \\ &= (\cos\bar{\rho})ds^2_{S^m} \end{aligned}$$

where $g : \mathbf{R} \to \mathbf{R}$ is a function defined by $g(x) = 1 - \cos x$. The Lemma follows by applying the Hessian comparison theorem ([GW], p. 19).

Proof of theorem 1. (a) Without loss of generality we may assume $\delta = 1$. First we show that $Ric(x) \geq -\bar{C}(1 + r^2(x))$ for some constant $\bar{C} > 0$. If $\{E_i\}_{i=1}^n$ is a local orthonormal frame for M, then we have the following identity, obtained from the Gauss equation by contraction:

$$S(x) = \sum_{i\neq j}\langle\bar{R}(E_i, E_j)E_j, E_i\rangle + n\|H\|^2 - \|B\|^2,$$

where $\|B\|^2$ is the square of the length of the second fundamental form of M. It follows that

$$\begin{aligned}\|B\|^2 &= \sum_{i\neq j}\langle \bar{R}(E_i,E_j)E_j,E_i\rangle + n\|H\|^2 - S(x) \\ &\leq n(n-1) + nH_0^2 + C(1+r^2(x)).\end{aligned}$$

Now for arbitrary plane $X \wedge Y \subset T_pM \subset T_pN$, the Gauss equation implies that

$$Sec_M(X\wedge Y) = Sec_N(X\wedge Y) + \langle B(X,X),B(Y,Y)\rangle - \|B(X,Y)\|^2.$$

Thus

$$\begin{aligned}|Sec_M(X\wedge Y)| &\leq |Sec_N(X\wedge Y)| + |\langle B(X,X),B(Y,Y)\rangle - \|B(X,Y)\|^2| \\ &\leq 1 + 2\|B\|^2 \leq C_1(1+r^2(x))\end{aligned}$$

for some constant $C_1 > 0$. And so $Ric(x) \geq -\bar{C}(1+r^2(x))$ for some $\bar{C} > 0$. Let $f(x) = 1 - \cos\rho(x)$ where $\rho(x)$ is the distance from $y_0 \in N$ to x in N. Then f is C^∞ on $B_R(y_0)$ by hypothesis. By lemmas 2 and 4,

$$\begin{aligned}\triangle_M f &= tr_M(\bar{\nabla}^2 f) + n\langle H, grad_N f\rangle_N \\ &\geq n\cos\rho(x) + n\sin\rho(x)\langle H,\bar{\nabla}\rho\rangle_N \\ &\geq n\cos R - nH_0\sin R\end{aligned}$$

for all $x \in M$. Since f is bounded on M, by lemma 3, $0 \geq n\cos R - nH_0\sin R$. The proof of (a) is complete.
(b) [Ka] Theorem 3.1.

Corollary 5. *If (M,ds^2) is a complete Riemannian manifold with scalar curvature S that satisfies $S(x) \geq -C(1+r^2(x))$ for some constant $C > 0$ and N is a complete Riemannian manifold with sectional curvature bounded above by a constant δ^2, $\delta > 0$, then for any $y_0 \in N$, (M^n, ds^2) cannot be isometrically minimally immersed in a closed geodesic ball $B_R(y_0)$ of radius $R < \frac{\pi}{2\delta}$ in N which does not intersect the cut locus of y_0.*

Proof. For a minimal immersion, $H = 0$ (i.e., $H_0 = 0$) and so the theorem 1 implies the corollary.

Remark 6. If the volume growth restriction is removed, such immersion exists. For example, Jones [Jo] constructed complete minimal surfaces entirely contained in balls of $\mathbf{R}^4$.

Remark 7. Our result is sharp as the following example shows. Let S^n be the sphere of dimension n with constant sectional curvature K. Then the inclusion map $i : S^{n-1} \to S^n$ as the equator is minimal, since S^{n-1} is the totally geodesic submanifold of S^n. But S^{n-1} lies in the closed ball of radius $\frac{\pi}{2\sqrt{K}}$ centered at the north pole.

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