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NEARLY SASAKIAN MANIFOLDS WITH VANISHING CONTACT CONFORMAL CURVATURE TENSOR FIELD

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1. Introduction

The notion of a nearly Sasakian structure was introduced by D.E. Blair, D.K. Showers and K. Yano in their paper [1]. They also showed ([1]) that S^5 properly imbedded in S^6 inherits a nearly Sasakian structure which is not Sasakian.

Z. Olszak ([4]) studied nearly Sasakian manifolds whose curvature tensor satisfies Cartan's condition, conformally flat nearly Sasakian manifolds and those of constant ϕ–sectional curvature, and also proved that if they are not Sasakian, they are 5–dimensional and of constant curvature.

In this paper, we study nearly Sasakian manifolds with vanishing contact conformal curvature tensor field and prove the following theorem

Theorem. *Any $m(\neq 5)$-dimensional nearly Sasakian manifold with vanishing contact conformal curvature tensor field is always Sasakian.*

Throughout this paper, manifolds are assumed to be connected and of class C^∞, and all tensor fields are of class C^∞.

2. Nearly Sasakian manifolds

A $(2n+1)$–dimensional manifold M^{2n+1} is said to have an *almost contact structure with an associated Riemannian metric tensor* g_{ji} if there exist on M^{2n+1} a tensor field $\phi_j{}^i$ of type (1.1), a unit vector field ξ^i and its dual 1–form η_i with respect to g_{ji} which satisfy ([6])

$$(2.1)\qquad \phi_j{}^h\phi_h{}^i = -\delta_j^i + \eta_j\xi^i, \phi_h{}^i\xi^h = 0, \phi_j{}^k\phi_i{}^h g_{kh} = g_{ji} - \eta_j\eta_i,$$

where here and in the sequel the indices $a, b, c, \cdots, h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and the Einstein summation convention will be used. It is clear that the tensor field $\phi_{ji} = \phi_j{}^h g_{hi}$ is skew–symmetric.

Such a manifold M^{2n+1} is said to be *nearly Sasakian* if it satisfies ([1])

$$\nabla_k \phi_{ji} + \nabla_j \phi_{ki} = -2g_{kj}\eta_i + g_{ki}\eta_j + g_{ji}\eta_k, \tag{2.2}$$

where ∇ denotes the Riemannian connection with respect to g_{ji}. Every Sasakian manifold is nearly Sasakian, but the converse statement fails in general ([1], [4]). For a nearly Sasakian manifold, the vector field ξ^i is Killing ([1]), that is,

$$\nabla_j \eta_i + \nabla_i \eta_j = 0. \tag{2.3}$$

Here we define a tensor field H_{ji} by setting

$$\nabla_j \eta_i = \phi_{ji} + H_{ji}. \tag{2.4}$$

Then, from the skew–symmetry of ϕ_{ji} and (2.3), it follows that H_{ji} is skew–symmetric. Here and in the sequel, we set

$$H_j{}^i = H_{jh} g^{hi}, H^{ji} = H_h{}^i g^{hj}, H_{ji}^{ba} = H_j{}^b H_i{}^a, H_{kji}^{cba} = H_k{}^c H_{ji}^{ba},$$

$$\phi^{ji} = \phi_h{}^i g^{hj}, \phi_{ji}^{ba} = \phi_j{}^b \phi_i{}^a, \phi_{kji}^{cba} = \phi_k{}^c \phi_{ji}^{ba}, \phi_{kjih}^{dcba} = \phi_k{}^d \phi_{jih}^{cba},$$

where $(g^{ji}) = (g_{ji})^{-1}$.

3. Fundamental properties of nearly Sasakian structure

We first of all consider the second equation of (2.1) as in the form

$$\phi_{ja}\xi^a = 0. \tag{3.1}$$

Differentiating (3.1) covariantly and using (2.1) and (2.4), we have

$$(\nabla_j \phi_{ia})\xi^a = -g_{ji} + \eta_j \eta_i - H_{ja}\phi_i a, \tag{3.2}$$

from which, taking the symmetric part and substituting (2.2),

$$H_{ja}\phi_i{}^a + H_{ia}\phi_j{}^a = 0. \tag{3.3}$$

Transvecting (3.3) with $\phi_b{}^j$ and using (2.1), we obtain

$$H_{ja}\phi_{bi}^{ja} = H_{ib} - H_{ia}\xi^a\eta_b,$$

which together with

$$H_{ia}\xi^a = 0 \tag{3.4}$$

implies

$$H_{ba}\phi_{ji}^{ba} = -H_{ji}. \tag{3.5}$$

We now apply the operator ∇_l to the both side of (2.2). Denoting by R_{kjih} the components of curvature tensor and using Ricci identity, we have

$$\begin{aligned}\nabla_k\nabla_l\phi_{ji} - R_{lkia}\phi_j{}^a + R_{lkja}\phi_i{}^a + \nabla_l\nabla_j\phi_k \\ = -2g_{kj}\nabla_l\eta_i + g_{ki}\nabla_l\eta_j + g_{ji}\nabla_l\eta_k,\end{aligned}$$

from which, taking account of (2.2) and (2.3),

$$\begin{aligned}-R_{lkia}\phi_j{}^a + R_{lkja}\phi_i{}^a + \nabla_j\nabla_l\phi_{ki} - R_{ljia}\phi_k{}^a + R_{ljka}\phi_i{}^a - \nabla_k\nabla_j\phi_{li} \\ = -2g_{kj}\nabla_l\eta_i + 2g_{lj}\nabla_k\eta_i + g_{ki}\nabla_l\eta_j - g_{li}\nabla_k\eta_j + 2g_{ji}\nabla_l\eta_k\end{aligned}$$

and consequently

$$\begin{aligned}-R_{lkia}\phi_j{}^a + R_{lkja}\phi_i{}^a - 2\nabla_k\nabla_j\phi_{li} + R_{jkia}\phi_l{}^a - R_{jkla}\phi_i{}^a \\ -2g_{kl}\nabla_j\eta_i + g_{li}\nabla_j\eta_k - R_{ljia}\phi_k{}^a + R_{ljka}\phi_i{}^a \\ = -2g_{kj}\nabla_l\eta_i + 2g_{lj}\nabla_k\eta_i + g_{ki}\nabla_l\eta_j - g_{li}\nabla_k\eta_j + 2g_{ji}\nabla_l\eta_k.\end{aligned}$$

Thus, applying Bianchi identity to the above equation, we have

$$\begin{aligned}2R_{ljka}\phi_i{}^a - R_{lkia}\phi_j{}^a - R_{kjia}\phi_l{}^a - R_{ljia}\phi_k{}^a + 2g_{kj}\nabla_l\eta_i \\ -2g_{lj}\nabla_k\eta_i - 2g_{ji}\nabla_l\eta_k + 2g_{ki}\nabla_j\eta_l - 2g_{kl}\nabla_j\eta_i = 2\nabla_k\nabla_j\phi_{li},\end{aligned} \tag{3.6}$$

from which, using (2.2),

$$\begin{aligned}2R_{ljka}\phi_i{}^a - R_{lkia}\phi_j{}^a - R_{kjia}\phi_l{}^a - R_{ljia}\phi_k{}^a + 2g_{kj}\nabla_l\eta_i \\ -2g_{ji}\nabla_k\eta_l + 2g_{ki}\nabla_j\eta_l - 2g_{kl}\nabla_j\eta_i + 2g_{il}\nabla_k\eta_j \\ -2\nabla_k\nabla_i\phi_{jl} = 0.\end{aligned}$$

Here, taking the skew–symmetric part with respect to k and i, and using Ricci and Bianchi identities, we can find

$$R_{lkja}\phi_i{}^a + R_{lkai}\phi_j{}^a + R_{laji}\phi_k{}^a + R_{akji}\phi_l{}^a = 0 \tag{3.7}$$

with the help of (2.3).

Next, we transvect (3.7) with $\phi_h{}^l$. Then it follows from (2.1) that

$$R_{baji}\phi^{ba}_{hk} + R_{bkai}\phi^{ba}_{hj} + R_{bkja}\phi^{ba}_{hi} - R_{hkji} + R_{akji}\xi^a\eta_h = 0,$$

from which, alternating with respect to h and k, and taking account of (2.1) and (3.7), we can see that

$$\begin{aligned}(3.8)\qquad 2R_{baji}\phi^{ba}_{hk} - 2R_{hkba}\phi^{ba}_{ji} + R_{akji}\xi^a\eta_h - R_{ahji}\xi^a\eta_k \\ -R_{hkai}\xi^a\eta_j + R_{ajhk}\xi^a\eta_i = 0.\end{aligned}$$

Replacing h, k in (3.8) by d, c respectively, and transvecting the resulting equation with ϕ^{dc}_{hk}, we obtain

$$\begin{aligned}(3.9)\qquad 2R_{hkji} - 2R_{dcba}\phi^{dcba}_{hkji} - 2R_{akji}\xi^a\eta_h + 2R_{ahji}\xi^a\eta_k \\ -R_{dcai}\phi^{dc}_{hk}\xi^a\eta_j + R_{dcaj}\phi^{dc}_{hk}\xi^a\eta_i = 0\end{aligned}$$

with the help of (2.1). Transvecting (3.9) with ξ^i yields

$$(3.10)\quad R_{dcaj}\xi^a\phi^{dc}_{hk} + 2R_{hkja}\xi^a - 2R_{akjb}\xi^a\xi^b\eta_h + 2R_{ahjb}\xi^a\xi^b\eta_k = 0,$$

from which, transvecting with ϕ^{hk}_{lm} and using (2.1),

$$2R_{dcja}\xi^a\phi^{dc}_{lm} + R_{lmaj}\xi^a - R_{lcaj}\xi^c\xi^a\eta_m - R_{dmaj}\xi^d\xi^a\eta_l = 0,$$

and consequently

$$(3.11)\quad 2R_{dcja}\xi^a\phi^{dc}_{hk} + R_{hkaj}\xi^a - R_{akbj}\xi^a\xi^b\eta_h - R_{habj}\xi^a\xi^b\eta_k = 0.$$

Multipling (3.11) by 2 and adding the resulting equaiton to (3.10), we can easily obtain

$$(3.12)\qquad R_{ajdc}\xi^a\phi^{dc}_{hk} = 0,$$

which and (3.9) imply

$$(3.13)\qquad R_{dcba}\phi^{dcba}_{hkji} = R_{hkji} - R_{akji}\xi^a\eta_h + R_{ahji}\xi^a\eta_k.$$

Transvecting (3.13) with ϕ^{hk}_{lm} and using (2.1), we have

$$(3.14)\qquad R_{lmba}\phi^{ba}_{ji} = R_{baji}\phi^{ba}_{lm}.$$

From now on we prepare the following lemma.

Lemma 3.1 (cf. [4], [5]). *On a $(2n+1)$-dimensional nearly Sasakian manifold M^{2n+1}*

$$R_{ji}\xi^i = (2n + H_{ba}H^{ba})\eta_j \tag{3.15}$$

and

$$H_{ba}H^{ba}_{\cdot} = const., \tag{3.16}$$

where R_{ji} denote the components of Ricci tensor of M^{2n+1}.

Proof. Differentiating (3.4) covariantly, we have

$$(\nabla_j H_{ia})\xi^a = -H_{ia}\nabla_j\xi^a,$$

which and (2.4) imply

$$(\nabla_j H_{ia})\xi^a = -H_{ia}(\phi_j{}^a + H_j{}^a). \tag{3.17}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}
\nabla_k\nabla_j\eta_i + \nabla_k\nabla_i\eta_j = 0,\\
\nabla_j\nabla_i\eta_k + \nabla_j\nabla_k\eta_i = 0,\\
\nabla_i\nabla_k\eta_j + \nabla_i\nabla_j\eta_k = 0,
\end{aligned}$$

from which together with Ricci identity, we have

$$\begin{aligned}
\nabla_k\nabla_j\eta_i + \nabla_k\nabla_i\eta_j = 0,\\
\nabla_j\nabla_i\eta_k + \nabla_k\nabla_j\eta_i + R_{kjia}\xi^a = 0,\\
\nabla_k\nabla_i\eta_j + R_{kija}\xi^a + \nabla_j\nabla_i\eta_k + R_{jika}\xi^a = 0,
\end{aligned}$$

which and Bianchi identity give

$$\nabla_k\nabla_j\eta_i + \nabla_j\nabla_i\eta_k + \nabla_k\nabla_i\eta_j + R_{kija}\xi^a = 0. \tag{3.18}$$

Since

$$\nabla_k\nabla_j\eta_i = \nabla_k\phi_{ji} + \nabla_k H_{ji},$$

(3.18) implies

$$R_{akji}\xi^a = -\nabla_k\phi_{ji} - \nabla_k H_{ji}. \tag{3.19}$$

Now we transvect (3.19) with ϕ^{ji}_{cb}. Then, from (2.1) and (3.12), we find

$$(\nabla_k\phi_{ji} + \nabla_k H_{ji})\phi^{ji}_{cb} = 0,$$

from which, transvecting with ϕ^{cb}_{rs} and making use of (2.1),

$$\nabla_k\phi_{rs} + \nabla_k H_{rs} = [(\nabla_k\phi_{rj})\xi^j + (\nabla_k H_{rj})\xi^j]\eta_s - [(\nabla_k\phi_{sj})\xi^j + (\nabla_k H_{sj})\xi^j]\eta_r,$$

and consequently

$$(3.20)\quad \nabla_k\phi_{ji} + \nabla_k H_{ji} = (g_{ki} + H_{ka}H_i{}^a)\eta_j - (g_{kj} + H_{ka}H_j{}^a)\eta_i.$$

Hence it follows from (3.19) and (3.20) that

$$(3.21)\quad \begin{aligned} R_{akji}\xi^a &= -\nabla_k\phi_{ji} - \nabla_k H_{ji} \\ &= (g_{kj} + H_{ka}H_j{}^a)\eta_i - (g_{ki} + H_{ka}H_i{}^a)\eta_j, \end{aligned}$$

from which, transvecting with g^{kj}, we obtain

$$R_{ia}\xi^a = (2n + H_{ba}H^{ba})\eta_i,$$

which is the first assertion of the lemma.

On the other side, transvecting (3.21) with ϕ^{ji} and H^{ji}, respectively and using (3.1) and (3.4), we can see that

$$(3.22)\quad (\nabla_k\phi_{ba} + \nabla_k H_{ba})\phi^{ba} = 0, \quad (\nabla_k\phi_{ba} + \nabla_k H_{ba})H^{ba} = 0,$$

which together with $\phi_{ji}\phi^{ji} = 2n$ gives

$$(3.23)\quad (\nabla_k H_{ba})\phi^{ba} = 0.$$

Furthermore, applying the operator ∇_k to (3.3) and transvecting the resulting equation with g^{ji}, we have

$$(\nabla_k H_{ba})\phi^{ba} + (\nabla_k\phi_{ba})H^{ba} = 0,$$

which and (3.23) yield

$$(\nabla_k\phi_{ba})H^{ba} = 0,$$

and consequently

$$(\nabla_k H_{ba})H^{ba} = 0$$

with the aid of (3.22). Hence $H_{ba}H^{ba}$ = const., which is the second assertion of the lemma.

We next prove the following lemma.

Lemma 3.2. (cf. [4], [5]). *On a nearly Sasakian manifold*

$$(3.24)\qquad (\nabla_k\phi_{js})\phi_a{}^s H_i{}^a = -H_{ka}H_i{}^a\eta_j + H_{ja}H_i{}^a\eta_k + H_{ka}\phi_i{}^a\eta_j.$$

Proof. Differentiating the first equation of (2.1) covariantly, we have

$$(\nabla_k\phi_{ai})\phi_j{}^a + \phi_{ai}\nabla_k\phi_j{}^a = (\nabla_k\eta_i)\eta_j + \eta_i\nabla_k\eta_j,$$

which together with (2.4) leads to

$$(3.25)\quad (\nabla_k\phi_{ai})\phi_j{}^a = (\nabla_k\phi_{ja})\phi_i{}^a + (\phi_{kj} + H_{kj})\eta_i + (\phi_{ki} + H_{ki})\eta_j.$$

Therefore, (3.25) together with (2.2) implies

$$\begin{aligned}(\nabla_a\phi_{ki})\phi_j{}^a &= -(\nabla_k\phi_{ja})\phi_i{}^a - 2\phi_{jk}\eta_i + \phi_{ji}\eta_k \\ &\quad -(\phi_{kj} + H_{kj})\eta_i - (\phi_{ki} + H_{ki})\eta_j,\end{aligned}$$

from which, interchanging pairwise the indices k, j and then using (2.2), we obtain

$$(3.26)\ (\nabla_a\phi_{ji})\phi_k{}^a = (\nabla_k\phi_{ja})\phi_i{}^a - (\phi_{kj} - H_{kj})\eta_i + 2\phi_{ki}\eta_j - H_{ji}\eta_k.$$

Transvecting (3.26) with $\phi_l{}^j$ and changing the indices a, j, l to b, a, j, respectively, we find

$$(\nabla_b\phi_{ai})\phi_{kj}^{ba} = (\nabla_k\phi_{ab})\phi_j{}^a\phi_i{}^b - (\phi_{ka}\phi_j{}^a - H_{ka}\phi_j{}^a)\eta_i - H_{ai}\phi_j{}^a\eta_k,$$

from which, using (2.1), (3.2) and (3.25),

$$(3.27)\qquad \begin{aligned}(\nabla_b\phi_{ai})\phi_{kj}^{ba} &= -\nabla_k\phi_{ji} - 2g_{kj}\eta_i + g_{ki}\eta_j \\ &\quad +\eta_k\eta_j\eta_i - \phi_{ka}H_i{}^a\eta_j + \phi_{ja}H_i{}^a\eta_k.\end{aligned}$$

From now on, we differentiate (3.25) covariantly and use (2.1), (3.3), (3.6), (3.14) and (3.21). Then we can easily verify that

$$(3.28)\qquad \begin{aligned}&(\nabla_k\phi_{ia})(\nabla_l\phi_j{}^a) + (\nabla_k\phi_{ja})(\nabla_l\phi_i{}^a) + R_{kilj} + R_{kjli} \\ &-R_{liba}\phi_{kj}^{ba} - R_{kiba}\phi_{lj}^{ba} + 2g_{lk}g_{ji} - 2g_{ki}g_{lj} - 2g_{li}g_{kj} \\ &+\phi_{ki}\phi_{lj} - \phi_{li}\phi_{kj} + H_{ki}H_{lj} + H_{li}H_{kj} - 2g_{lk}\eta_j\eta_i + g_{ki}\eta_l\eta_j \\ &+g_{li}\eta_k\eta_j + g_{lj}\eta_k\eta_i + g_{kj}\eta_l\eta_i - g_{ki}H_{la}\phi_j{}^a - g_{li}H_{ka}\phi_j{}^a \\ &-g_{lj}H_{ka}\phi_i{}^a - g_{kj}H_{la}\phi_i{}^a = 0.\end{aligned}$$

Interchanging pairwise the indices i, l to r, s in (3.28) and transvecting it with ϕ_{il}^{rs}, we can find

$$
\begin{aligned}
&(\nabla_k\phi_{ia})(\nabla_l\phi_j{}^a) - \nabla_k\phi_{ja}\nabla_l\phi_i{}^a + 2(\nabla_l\phi_{kj})\eta_i + 2(\nabla_k\phi_{li})\eta_j \\
&\quad -(\nabla_k\phi_{is})\phi_a{}^s H_j{}^a\eta_l + (\nabla_k\phi_{js})\phi_a{}^s H_l{}^a\eta_i + (\nabla_l\phi_{js})\phi_a{}^s H_k{}^a\eta_i \\
(3.29)\qquad &\quad -(\nabla_k\phi_{js})\phi_a{}^s H_i{}^a\eta_l + R_{kilj} - R_{kiba}\phi_{lj}^{ba} + R_{liba}\phi_{kj}^{ba} - R_{likj} \\
&\quad +4g_{kl}\eta_j\eta_i - g_{ki}\eta_l\eta_j - g_{li}\eta_k\eta_j - g_{kj}\eta_l\eta_i - g_{lj}\eta_k\eta_i \\
&\quad -\phi_{ki}\phi_{lj} - \phi_{li}\phi_{kj} - 2\phi_{lk}\phi_{ji} + H_{ki}H_{lj} - H_{li}H_{kj} \\
&\quad -g_{ki}H_{la}\phi_j{}^a + g_{li}H_{ka}\phi_j{}^a + g_{kj}H_{la}\phi_i{}^a - g_{lj}H_{ka}\phi_i{}^a \\
&\quad +H_{ka}H_i{}^a\eta_l\eta_j - 2H_{ka}H_j{}^a\eta_l\eta_i + H_{la}H_j{}^a\eta_k\eta_i + 2H_{ka}\phi_l{}^a\eta_j\eta_i \\
&\quad -H_{ka}\phi_i{}^a\eta_l\eta_j + H_{ka}\phi_j{}^a\eta_l\eta_i = 0.
\end{aligned}
$$

Equations (3.28) and (3.29) give

$$
\begin{aligned}
&2(\nabla_k\phi_{ia})(\nabla_l\phi_j{}^a) + 2(\nabla_k\phi_{li})\eta_j + 2(\nabla_l\phi_{kj})\eta_i - (\nabla_k\phi_{is})\phi_a{}^s H_j{}^a\eta_l \\
&\quad +(\nabla_k\phi_{js})\phi_a{}^s H_i{}^a\eta_l + (\nabla_l\phi_{js})\phi_a{}^s H_k{}^a\eta_i - (\nabla_k\phi_{js})\phi_a{}^s H_l{}^a\eta_i + 2R_{kilj} \\
&\quad -2R_{kiba}\phi_{lj}^{ba} + 2g_{lk}g_{ji} - 2g_{ki}g_{lj} - 2g_{li}g_{kj} - 2\phi_{lk}\phi_{ji} - 2\phi_{li}\phi_{kj} \\
&\quad +2H_{ki}H_{lj} + 2g_{lk}\eta_j\eta_i - 2g_{ki}H_{la}\phi_j{}^a - 2g_{lj}H_{ka}\phi_i{}^a + H_{ka}H_i{}^a\eta_1\eta_j \\
&\quad -2H_{ka}H_j{}^a\eta_l\eta_i + H_{la}H_j{}^a\eta_k\eta_i + 2H_{ka}\phi_l{}^a\eta_j\eta_i \\
&\quad -H_{ka}\phi_i{}^a\eta_l\eta_j + H_{ka}\phi_j{}^a\eta_l\eta_i = 0,
\end{aligned}
$$

from which, interchanging pairwise the indices k, i and l, j and subtracting, we have

$$
\begin{aligned}
&(\nabla_l\phi_{is})\phi_a{}^s H_k{}^a\eta_j - (\nabla_l\phi_{is})\phi_a{}^s H_j{}^a\eta_k + (\nabla_l\phi_{js})\phi_a{}^s H_k{}^a\eta_i \\
&\quad +(\nabla_l\phi_{js})\phi_a{}^s H_i{}^a\eta_k - (\nabla_k\phi_{is})\phi_a{}^s H_l{}^a\eta_j - (\nabla_k\phi_{is})\phi_a{}^s H_j{}^a\eta_l \\
&\quad -(\nabla_k\phi_{js})\phi_a{}^s H_l{}^a\eta_i + (\nabla_k\phi_{js})\phi_a{}^s H_i{}^a\eta_l + H_{la}\phi_j{}^a\eta_k\eta_i \\
&\quad +2H_{la}H_i{}^a\eta_k\eta_j - 2H_{ka}H_j{}^a\eta_l\eta_i - H_{ka}\phi_i{}^a\eta_l\eta_j - 4H_{la}\phi_k{}^a\eta_j\eta_i \\
&\quad -H_{la}\phi_i{}^a\eta_k\eta_j + H_{ka}\phi_j{}^a\eta_l\eta_i = 0
\end{aligned}
$$

with the help of (3.3) and (3.14). Transvecting the above equation with ξ^l and using (2.1), (3.4) and $(\nabla_a\phi_{ji})\xi^a = -H_{ja}\phi_i{}^a$, which is a direct consequence of (2.2) and (3.3), we can obtain

$$
\begin{aligned}
(3.30)\quad (\nabla_k\phi_{js})\phi_a{}^s H_i{}^a - (\nabla_k\phi_{is})\phi_a{}^s H_j{}^a &= H_{ka}H_j{}^a\eta_i - H_{ka}H_i{}^a\eta_j \\
&\quad -H_{ka}\phi_j{}^a\eta_i + H_{ka}\phi_i{}^a\eta_j.
\end{aligned}
$$

Taking the symmetric part of (3.30) with respect to k and j, and using (2.1), (2.2) and (3.3), we find

$$(\nabla_i \phi_{ks})\phi_a{}^s H_j{}^a + (\nabla_i \phi_{js})\phi_a{}^s H_k{}^a = -H_{ia}H_k{}^a\eta_j - H_{ia}H_j{}^a\eta_k$$
$$+2H_{ka}H_j{}^a\eta_i + H_{ia}\phi_k{}^a\eta_j + H_{ia}\phi_j{}^a\eta_k,$$

which together with (3.30) leads to our assertion (3.24).

Finally we prepare the following lemma.

Lemma 3.3. *On a nearly Sasakian manifold*

$$\begin{aligned} H^{as}_{ja}R_{sikl} + H^{as}_{ia}R_{sjkl} &= -(H_{la}H_i{}^a - H^{abs}_{lab}H_{is})\eta_k\eta_j \\ &+(H_{ka}H_i{}^a - H^{abs}_{kab}H_{is})\eta_l\eta_j - (H_{la}H_j{}^a - H^{abs}_{lab}H_{js})\eta_k\eta_i \\ &+(H_{ka}H_j{}^a - H^{abs}_{kab}H_{js})\eta_l\eta_i - H^{as}_{ka}(\phi_{is} + H_{is})(\phi_{lj} + H_{lj}) \\ &+H^{as}_{la}(\phi_{is} + H_{is})(\phi_{kj} + H_{kj}) - H^{as}_{ka}(\phi_{js} + H_{js})(\phi_{li} + H_{li}) \\ &+H^{as}_{la}(\phi_{js} + H_{js})(\phi_{ki} + H_{ki}). \end{aligned} \tag{3.31}$$

Proof. At first we transvect (3.24) with $\phi_l{}^i$ and make use of (2.1), (3.4) and (3.5). Then we get

$$(\nabla_k \phi_{ja})H_i{}^a = H^{as}_{ka}\phi_{is}\eta_j - H^{as}_{ja}\phi_{is}\eta_k - H_{ki}\eta_j. \tag{3.32}$$

On the other hand, transvecting (3.20) with $H_h{}^i$ and using (3.4), we have

$$(\nabla_k \phi_{ja})H_h{}^a + (\nabla_k H_{ja})H_h{}^a = (H_{hk} + H_{ka}H_s{}^a H_h{}^s)\eta_j,$$

from which, substituting (3.32),

$$(\nabla_k H_{ja})H_h{}^a = -H^{as}_{ka}(\phi_{hs} + H_{hs})\eta_j + H^{as}_{ja}\phi_{hs}\eta_k$$

and consequently

$$\nabla_k(H_{ja}H_h{}^a) = -H^{as}_{ka}(\phi_{hs} + H_{hs})\eta_j - H^{as}_{ka}(\phi_{js} + H_{js})\eta_h.$$

Hence, applying the opertor ∇_l to the above equation and using (2.1), (2.4), (3.3), (3.4) and (3.21), we can easily verify that

$$\begin{aligned} \nabla_l\nabla_k(H_{ja}H_i{}^a) &= 2(H_{la}H_k{}^a - H^{abs}_{lab}H_{ks})\eta_j\eta_i - (H_{la}H_i{}^a - H^{abs}_{lab}H_{is})\eta_k\eta_j \\ &-(H_{la}H_j{}^a - H^{abs}_{lab}H_{js})\eta_k\eta_i) - H^{as}_{ka}(\phi_{is} + H_{is})(\phi_{lj} + H_{lj}) \\ &-H^{as}_{ka}(\phi_{js} + H_{js})(\phi_{li} + H_{li}), \end{aligned}$$

which and Ricci identity imply our assertion (3.31).

4. Proof of main theorem

In a $(2n+1)$–dimensional nearly Sasakian manifold M^{2n+1}, the contact conformal curvature tensor field $C_{0,kji}{}^h$ is defined by

$$\begin{aligned} C_{0,kji}{}^h = {} & R_{kji}{}^h + \frac{1}{2n}(\delta_k^h R_{ji} - \delta_j^h R_{ki} + R_k{}^h g_{ji} - R_j{}^h g_{ki} \\ & - R_k{}^h \eta_j \eta_i + R_j{}^h \eta_k \eta_i - \eta_k \xi^h R_{ji} + \eta_j \xi^h R_{ki} - \phi_k{}^h S_{ji} \\ & + \phi_j{}^h S_{ki} - S_k{}^h \phi_{ji} + S_j{}^h \phi_{ki} + 2\phi_{kj} S_i{}^h + 2S_{kj}\phi_i{}^h) \\ & + \frac{1}{2n(n+1)}[2n^2 - n - 2 + \frac{(n+2)s}{2n}](\phi_k{}^h \phi_{ji} - \phi_j{}^h \phi_{ki} \\ & - 2\phi_{kj}\phi_i{}^h) + \frac{1}{2n(n+1)}[n + 2 - \frac{(3n+2)s}{2n}](\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\ & + \frac{1}{2n(n+1)}[-(4n^2 + 5n + 2) + \frac{(3n+2)s}{2n}] \\ & (\delta_k^h \eta_j \eta_i - \delta_j^h \eta_k \eta_i + \eta_k \xi^h g_{ji} - \eta_j \xi^h g_{ki}), \end{aligned} \tag{4.1}$$

where s denotes the scalar curvature of M^{2n+1}, $S_{ji} = \phi_j{}^h R_{hi}$ and $S_j^h = S_{ji} g^{ih}$.

From now on we assume that the contact conformal curvature tensor field of M^{2n+1} vanishes identically. Then, from (4.1) with $C_{0,kji}{}^h = 0$, we have

$$\begin{aligned} H_j{}^a H_a{}^s R_{sikl} = {} & -\frac{1}{2n}[H_j{}^a H_{al} R_{ik} - H_j{}^a H_a{}^s R_{sk} g_{il} + H_j{}^a H_a{}^s R_{sl} g_{ik} \\ & - H_j{}^a H_{ak} R_{il} - H_j{}^a H_a{}^s R_{sl} \eta_i \eta_k + H_j{}^a H_a{}^s R_{sk} \eta_i \eta_l \\ & - H_j{}^a H_a{}^s \phi_{sl} S_{ik} + H_j{}^a H_a{}^s S_{sk} \phi_{il} - H_j{}^a H_a{}^s S_{sl} \phi_{ik} \\ & + H_j{}^a H_a{}^s \phi_{sk} S_{il} + 2H_j{}^a H_a{}^s \phi_{si} S_{kl} + 2H_j{}^a H_a{}^s S_{si} \phi_{kl}] \\ & - \frac{1}{2n(n+1)}[2n^2 - n - 2 + \frac{(n+2)s}{2n}](H_j{}^a H_a{}^s \phi_{sl} \phi_{ik} \\ & - H_j{}^a H_a{}^s \phi_{sk} \phi_{il} - 2H_j{}^a H_a{}^s \phi_{si} \phi_{kl}) \\ & - \frac{1}{2n(n+1)}[n + 2 - \frac{(3n+2)s}{2n}](H_j{}^a H_{al} g_{ik} - H_j{}^a H_{ak} g_{il}) \\ & - \frac{1}{2n(n+1)}[-(4n^2 + 5n + 2) + \frac{(3n+2)s}{2n}] \\ & (H_j{}^a H_{al} \eta_i \eta_k - H_j{}^a H_{ak} \eta_i \eta_l), \end{aligned}$$

which together with (3.31) gives

$$\begin{aligned}
&\frac{1}{2n}[H_i{}^a H_{al} R_{jk} - H_i{}^a H_a{}^s R_{sk} g_{jl} + H_i{}^a H_a{}^s R_{sl} g_{jk} - H_i{}^a H_{ak} R_{jl} \\
&\quad - H_i{}^a H_a{}^s R_{sl}\eta_j\eta_k + H_i{}^a H_a{}^s R_{sk}\eta_j\eta_l - H_i{}^a H_a{}^s \phi_{sl} S_{jk} + H_i{}^a H_a{}^s S_{sk}\phi_{jl} \\
&\quad - H_i{}^a H_a{}^s S_{sl}\phi_{jk} + H_i{}^a H_a{}^s \phi_{sk} S_{jl} + 2H_i{}^a H_a{}^s \phi_{sj} S_{kl} + 2H_i{}^a H_a{}^s S_{sj}\phi_{kl}] \\
&\quad + \frac{1}{2n(n+1)}[2n^2 - n - 2 + \frac{(n+2)s}{2n}](H_i{}^a H_a{}^s \phi_{sl}\phi_{jk} - H_i{}^a H_a{}^s \phi_{sk}\phi_{jl} \\
&\quad - 2H_i{}^a H_a{}^s \phi_{sj}\phi_{kl}) + \frac{1}{2n(n+1)}[n + 2 - \frac{(3n+2)s}{2n}](H_i{}^a H_{al} g_{jk} - H_i{}^a H_{ak} g_{jl}) \\
&\quad + \frac{1}{2n(n+1)}[-(4n^2 + 5n + 2) + \frac{(3n+2)s}{2n}](H_i{}^a H_{al}\eta_j\eta_k - H_i{}^a H_{ak}\eta_j\eta_l) \\
&\quad - H_j{}^a H_a{}^s R_{sikl} = (H_{la} H_i{}^a - H_{lab}^{abs} H_{is})\eta_k\eta_i \\
&\quad - (H_{ka} H_i{}^a - H_{kab}^{abs} H_{is})\eta_l\eta_j + (H_{la} H_j{}^a - H_{lab}^{abs} H_{js})\eta_k\eta_i \\
&\quad - (H_{ka} H_j{}^a - H_{kab}^{abs} H_{js})\eta_l\eta_i + H_{ka}^{as}(\phi_{is} + H_{is})(\phi_{lj} + H_{lj}) \\
&\quad - H_{la}^{as}(\phi_{is} + H_{is})(\phi_{kj} + H_{kj}) + H_{ka}^{as}(\phi_{js} + H_{js})(\phi_{li} + H_{li}) \\
&\quad - H_{la}^{as}(\phi_{js} + H_{js})(\phi_{ki} + H_{ki}).
\end{aligned}$$

Transvecting the above equation with $\xi^k\xi^j$ and using (3.4) and (3.15), we can easily obtain

$$(H_{ba}H^{ba})H_i{}^a H_{al} = -2nH_{lab}^{abs}H_{is},$$

which implies

$$-(H_{ba}H^{ba})^2 = 2n\|H_l{}^a H_{ab}\|^2$$

and consequently $H_{ji} = 0$. Thus we complete the proof of the main theorem stated in the first section.

Remark. The contact conformal curvature tensor field of S^5 (properly imbedded in S^6) never vanishes identically.

Remark. (cf. [1]). Any 3–dimensional nearly Sasakian manifold is Sasakian.

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