Kyungpook Mathematical Journal
Volume 29, Number 2, December, 1989

SYMMETRIC CANTOR SETS AND DIRICHLET SETS

Hung Hwan Lee

The notion of a Dirichlet set has been studied for several decades. Such sets are named in honour of Dirichlet's Theorem [3, p235] which, in modern terminology, simply says that every finite set in R is a Dirichlet set.

In this paper, we present a Criterion for proving that a symmetric Cantor set is a Dirichlet set.

Definition 1. A bounded subset A of R is called a *Dirichlet set* (in short, D-set) if there exists a sequence $(\alpha_k)_{k=1}^{\infty}$ in R such that

$$\lim_{k\to\infty} \alpha_k = \infty \quad \text{and} \quad \lim_{k\to\infty} (\sup_{x\in A} |\sin \alpha_k x|) = 0.$$

[Define $\sup \emptyset = 0$ for the empty set $\emptyset$, so $\emptyset$ is a D-set.]

Notation 2. Let $C = (c_n)_{n=1}^{\infty}$ be a fixed sequence of real numbers such that $0 < 2c_n < c_{n-1}$ for $n \geq 1$ and put $r_n = c_{n-1} - c_n$ for $n \geq 1$. Let

$$F_n = \{\sum_{j=1}^{n} \varepsilon_j r_j \,|\, \varepsilon_j = 0 \quad \text{or} \quad 1 \quad \text{for all} \quad j\}.$$

Then it is clear that $|s - t| > c_n$ for $s \neq t \in F_n$. In particular, F_n has exactly 2^n points. Next, put $E_n = \cup_{t\in F_n}[t, t + c_n]$ which, by the above, is a disjoint union of 2^n closed intervals of length c_n each. Note that for $t \in F_n$, we have $t \in F_{n+1}$, $t + r_{n+1} \in F_{n+1}$, and $t < t + c_{n+1} < t + r_{n+1} < t + c_n$. This shows that $E_{n+1} \subset E_n$ for all $n \geq 1$. The set $E = E_C = \cap_{n=1}^{\infty} E_n$ will be called the *symmetric Cantor set* on $[0, c_0]$ determined by C. It is easy to show that $E = \{\sum_{i=1}^{\infty} \varepsilon_i r_i | \varepsilon_i = 0 \text{ or } 1 \text{ for all } i\}$

This work was partially supported by KOSEF research grant, 881-0102-005-2

In what follows we write the following classical result of Dirichlet taken from [3].

Lemma 3. *Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be any k real numbers and let Q be any positive integer. Then we can find an integer q with $1 \leq q \leq Q^k$ and integers $p_1, p_2, \cdots, p_k$ such that*

$$|\alpha_j - \frac{p_j}{q}| < \frac{1}{Qq} \leq \frac{1}{q^{1+\frac{1}{k}}} (j = 1, 2, \cdots, k).$$

In particular, $|\sin \pi q \alpha_j| < \frac{\pi}{Q} (j = 1, 2, \cdots, k)$.

Proposition 4. *Adopt the Notation (2). If* $\underline{\lim}_n \sum_{k=1}^{\infty} |\sin n r_k| = 0$ *then E is a D-set.*

Proof. Let $\{n_p\}(\uparrow \infty)$ be a sequence in N such that

$$\sum_{k=1}^{\infty} |\sin n_p r_k| < \eta_p \quad \text{with} \quad \{\eta_p\} \downarrow 0.$$

For $x \in E$, we have

$$|\sin n_p x| \leq \sum_{k=1}^{\infty} |\sin n_p r_k| < \eta_p.$$

Thus

$$\sup_{x \in E} |\sin n_p x| < \eta_p.$$

It follows that

$$\lim_{p \to \infty} \sup_{x \in E} |\sin n_p x| = 0.$$

Now we are ready for the main theorem.

Theorem 5. *Adopt the Notation as before. If* $\underline{\lim}_p p c_p^{\frac{1}{p}} = 0$ *then E_C is a D-set.*

Proof. Let

$$c_k^{\frac{1}{k}} = \frac{1}{k\psi(k)} \quad \text{with} \quad \overline{\lim}_k \psi(k) = \infty.$$

It follows from Lemma (3) that for given p, $A \in Z^+$ and $t \in R (t \geq 1)$, there exist $n = n(p)$ such that

$$A \leq n \leq At^p \quad \text{and} \quad |\sin n r_k| < \frac{\pi}{[t]} \quad \text{for} \quad 1 \leq k \leq p.$$

Let $A = p$, and $t = p\sqrt{\psi(p)} \geq 2$. Then there exists $n = n(p)$ such that

$$p \leq n < p(p\sqrt{\psi(p)})^p \quad \text{and} \quad |\sin nr_k| < \frac{\pi}{|p\sqrt{\psi(p)}|}(k = 1, 2, \cdots, p).$$

Thus $\sum_{k=1}^{p} |\sin nr_k| < \frac{2\pi}{\sqrt{\psi(p)}}$ since $p\sqrt{\psi(p)} \geq 2$ and

$$\sum_{k=p+1}^{\infty} |\sin nr_k| \leq n \sum_{k=p+1}^{\infty} r_k = nc_p < p(p\sqrt{\psi(p)})^p \cdot (\frac{1}{p\psi(p)})^p = \frac{p}{(\sqrt{\psi(p)})^p}$$

Therefore we have

$$\sum_{k=1}^{\infty} |\sin nr_k| \leq \frac{2\pi}{\sqrt{\psi(p)}} + \frac{p}{(\sqrt{\psi(p)})^p} \to 0 \quad \text{as} \quad p \to \infty.$$

Let us choose a increasing sequence $\{n = n(p)\}$ by letting A and $p \to \infty$. It follows from proposition (4) that E_C is a D-set.

References

[1] Lee, Hung Hwan, *Structure and Dimension of Dirichlet Sets*, Ph.D. Dissertation (Kansas State University), 1986.

[2] Lindahl, L.A. and Poulsen, F., *Thin sets in Harmonic Analysis*, Marcel Dekker, Inc., New York, 1971.

[3] Stromberg, Karl, *Introduction to Classical Real Analysis*, Wadsworth, Inc., Belmont, California, 1981.

[4] Zygmund, A., *Trigonometric Series*, Cambridge University Press, New York, 1979.

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, KOREA