Kyungpook Mathematical Journal
Volume 29, Number 2, December, 1989

ON SOLUTIONS OF VOLTERRA–FREDHOLM INTEGRAL EQUATIONS

Thabet A.A. and A.Hadi Alim

Abstract

The existence and uniqueness of solutions of nonlinear Volterra–Fredholm integral equations of the more general type are investigated. The main tool employed in our analysis is the method of successive approximation based on the general idea of T.Wazewski.

1. Introduction

In this paper we wish to study the existence and uniqueness of solutions of nonlinear Volterra–Fredholm integral equations of the more general type of the form

$$x(t) = F(t, \int_0^t f(t,s,x(s))ds, \int_0^T g(t,s,x(s))ds), \quad 0 \le t \le T, \quad (1.1)$$

where $x(t)$ is an unknown function. In our analysis we shall apply the method of Wazewski [5]. In recent years, there have been several results which investigate the existence and uniqueness of solutions of various special forms of equation (1.1) by using different techniques (see, [1], [2], [3], [4], [5]).

Our main hypotheses which will be used in the subsequent analysis are:

Hypothesis $\mathbf{H}_1$: Suppose that

$\mathbf{H}_{11}$: E be a Banach space with norm $\|.\|$, $I = [0,T]$, $S = \{(t,s) : 0 \le s \le t \le T\}$, $f, g \in C[S \times E, E]$, $F \in C[I \times E^2, E]$, and if $x \in C[I,E]$ and

$$Z(t) = F(t, \int_0^t f(t,s,x(s))ds, \int_0^T g(t,s,x(s))ds),$$

then, $Z \in C[I, E]$.

H$_{12}$: there exist functions $W_1(t,s,r), W_2(t,s,r)$ such that $W_1, W_2 \in C[S \times R^+, R^+], R^+ = (0, \infty)$ which are nondecreasing in r and fulfill the conditions:

$$W_1(t,s,0) \equiv 0, \quad W_2(t,s,0) \equiv 0 \quad \text{and}$$

$$\|f(t,s,x) - f(t,s,\bar{x})\| \leq W_1(t,s,\|x-\bar{x}\|),$$
$$\|g(t,s,x) - g(t,s,\bar{x})\| \leq W_2(t,s,\|x-\bar{x}\|),$$

for $x, \bar{x} \in C[I, E]$.

H$_{13}$: there exists a function $H(t, r_1, r_2, r_3)$ defined for $t \in I$ and $0 \leq r_1, r_2, r_3 < \infty$ such that $H(t,0,0,0) \equiv 0$ and

(a) if $u \in C[I, T]$ and

$$V(t) = H(t, \int_0^t W_1(t,s,u(s))ds, \int_0^T W_2(t,s,u(s))ds),$$

then $V \in C[I, I]$;

(b) if $u, \bar{u} \in C[I, I]$ and $u(t) \leq \bar{u}(t)$ for $t \in I$, then

$$\begin{aligned} &H(t, \int_0^t W_1(t,s,u(s))ds, \int_0^T W_2(t,s,u(s))ds), \\ &\leq H(t, \int_0^t W_1(t,s,\bar{u}(s))ds, \int_0^T W_2(t,s,\bar{u}(s))ds), \text{ for } t \in I; \end{aligned}$$

(c) if $u_n \in C[I, I], u_{n+1} \leq u_n, n = 0, 1, 2, \cdots$, and $\lim_{n\to\infty} u_n(t) = u(t)$, then

$$\begin{aligned} &\lim_{n\to\infty} H(t, \int_0^t W_1(t,s,u_n(s))ds, \int_0^T W_2(t,s,u_n(s))ds) \\ &= H(t, \int_0^t W_1(t,s,u(s))ds, \int_0^T W_2(t,s,u(s))ds), \text{ for } t \in I, \end{aligned}$$

H$_{14}$: the inequality

$$\begin{aligned} &\|F(t, x_1, x_2, x_3) - F(t, \bar{x}_1, \bar{x}_2, \bar{x}_3\| \\ &\leq H(t, \|x_1 - \bar{x}_1\|, \|x_2 - \bar{x}_2\|, \|x_3 - \bar{x}_3\|), \end{aligned}$$

holds for $x_i, \bar{x}_i \in C[I, E], i = 1, 2, 3$ and $t \in I$.

Hypothesis $\mathbf{H}_2$: Suppose that
$\mathbf{H}_{21}$: there exists a nonnegative continuous function $\bar{u} : I \to R^+$ being the solution of the inequality

$$H(t, \int_0^t W_1(t, s, u(s))ds, \int_0^T W_2(t, s, u(s))ds) + h(t) \leq u(t) \qquad (1.2)$$

where

$$h(t) = \sup_{t \in I} \|F(t, \int_0^t f(t, s, o)ds, \int_0^T g(t, s, 0)ds)\|$$

$\mathbf{H}_{22}$: in the class of functions satisfying the condition $0 \leq u(t) \leq \bar{u}(t)$, $t \in I$, the function $u(t) \equiv 0$, $t \in I$, is the only solution of the equation

$$u(t) = H(t, \int_0^t W_1(t, s, u(s))ds, \int_0^T W_2(t, s, u(s))ds), t \in I. \qquad (1.3)$$

In order to prove the existence of a solution of equation (1.1), we define the sequence $x_0(t) \equiv 0$,

$$x_{n+1}(t) = F(t, \int_0^t f(t, s, x_n(s))ds, \int_0^T g(t, s, x_n(s))ds) \qquad (1.4)$$

for $n = 0, 1, 2, \cdots$
To prove the convergence of sequence $\{x_n\}$ to the solution $\bar{x}$ of the equation (1.1), we define the sequence $\{u_n\}$ by the relations:

$$u_0(t) = \bar{u}(t),$$

$$u_{n+1}(t) = H(t, \int_0^t f(t, s, u_n(s))ds, \int_0^T g(t, s, u_n(s))ds) \qquad (1.5)$$

for $n = 0, 1, 2, \cdots$, where the funciton $\bar{u}(t)$ is from H_2.

Now, we prove the following basic lemma which will be used in our subsequent discussion.

Lemma 1.1. *If condition* H_{13} *and hypothesis* H_2 *are satisfied, then*

$$0 \leq u_{n+1}(t) \leq u_n(t) \leq \bar{u}(t), t \in I, n = 0, 1, 2, \cdots \qquad (1.6)$$

$$\lim_{n \to \infty} u_n(t) = 0, \quad t \in I$$

and the convergence is uniform in each bounded set.

Proof. Using (1.5) and (1.2) we obtain

$$\begin{aligned} u_1(t) &= H(t, \int_0^t f(t,s,u_0(s))ds, \int_0^T g(t,s,u_0(s))ds) \\ &\leq H(t, \int_0^t f(t,s,\bar{u}(s))ds, \int_0^T g(t,s,\bar{u}(s))ds) + h(t). \end{aligned}$$

$$\bar{u}(t) = u_0(t), \quad t \in I.$$

Further, we obtain (1.6) by induction. But (1.6) implies the convergence of the sequence $\{u_n(t)\}$ to some nonnegative function $\phi(t)$ for $t \in I$. By Lebesgue's theorem and continuity of H it follows that the function $\phi(t)$ satisfies equation (1.3). Now, from H_2 we have $\phi \equiv 0, t \in I$. The uniform convergence of the sequence $\{u_n\}$ in I follows from Dini's theorem. Thus, the proof of the lemma is complete.

2. Main Result

In this section we shall establish our main results on the existence and uniqueness of the solutions of equation (1.1).

Theorem 2.1. *If Hypotheses* H_1 *and* H_2 *are satisfied, then there exists a continuous solution* $\bar{x}$ *of equation (1.1). The sequence* $\{x_n\}$ *defined by (1.4) converges uniformly on* I *to* $\bar{x}$*, and the following estimates*

$$\|\bar{x}(t) - x_n(t)\| \leq u_n(t), \quad t \in I, \quad n = 0, 1, 2, \cdots \tag{2.1}$$

and

$$\|\bar{x}(t)\| \leq \bar{u}(t), \quad t \in I, \tag{2.2}$$

hold. The solution $\bar{x}$ *of equation (1.1) is unique in the class of functions satisfying the condition (2.2).*

Proof. We first prove that the sequence $\{x_n(t)\}$, $t \in I$, fulfills the condition

$$\|x_n(t)\| \leq \bar{u}(t), t \in I, n = 0, 1, 2, \cdots \tag{2.3}$$

Evidently, we see that

$$\|x_0(t)\| \equiv 0 \leq \bar{u}(t), \quad t \in I.$$

Further, if we suppose that the inequality (2.3) is true for $n \geq 0$, then,

$$\begin{aligned} \|x_{n+1}(t)\| &= \|F(t, \int_0^t f(t,s,x_n(s))ds, \int_0^T g(t,s,x_n(s))ds) \\ &\quad -F(t, \int_0^t f(t,s,0)ds, \int_0^T g(t,s,0)ds) \\ &\quad +F(t, \int_0^t f(t,s,0)ds, \int_0^T g(t,s,0)ds)\| \\ &\leq H(t, \int_0^t W_1(t,s,\|x_n(s)\|)ds, \int_0^T W_2(t,s,\|x_n(s)\|)ds + h(t)) \\ &\leq H(t, \int_0^t W_1(t,s,\bar{u}(s))ds, \int_0^T W_2(t,s,\bar{u}(s))ds) + h(t)) \\ &\leq \bar{u}(t) \end{aligned}$$

For $t \in I$. Now, we obtain (2.3) by induction.

Next, we prove that:

$$\|x_{n+r}(t) - x_n(t)\| \leq u_n(t), t \in I, \quad r = 0, 1, 2, \cdots \tag{2.4}$$

By (2.3) we have:

$$\|x_r(t) - x_0(t)\| = \|x_r(t)\| \leq \bar{u}(t) = u_0(t), \quad t \in I, \quad r = 0, 1, 2, \cdots$$

Suppose that (2.4) is true for $n, r \geq 0$, then

$$\begin{aligned} \|x_{n+r+1}(t) - x_{n+1}(t)\| &= \|F(t, \int_0^t f(t,s,x_{n+r}(s))ds, \int_0^T g(t,s,x_{n+r}(s))ds) \\ &\quad -F(t, \int_0^t f(t,s,x_n(s))ds, \int_0^T g(t,s,x_n(s))ds)\| \\ &\leq H(t, \int_0^t W_1(t,s,\|x_{n+r}(s) - x_n(s)\|)ds, \\ &\quad \int_0^t W_2(t,s,\|x_{n+r}(s) - x_n(s)\|)ds) \\ &\leq H(t, \int_0^t W_1(t,s,u_n(s))ds, \int_0^T W_2(t,s,u_n(s))ds) \\ &= u_{n+1}(t) \quad \text{for} \quad t \in I. \end{aligned}$$

Now, we obtain (2.4) by induction.

Because of lemma, $\lim_{n\to\infty} u_n(t) = 0$ in I, we get from (2.4) $x_n \to \bar{x}$ in I. The continuity of $\bar{x}$ follows from the uniform convergence of the sequence $\{x_n\}$ and the continuity of all functions x_n. If $r \to \infty$, then

(2.4) gives estimation (2.1). Estimation (2.2) is implied by (2.3). It is obvious that $\bar{x}$ is a solution of equation (1.1).

To prove that the solution $\bar{x}$ is a unique solution of equation (1.1) in the class of functions satisfying the condition (2.2), let us suppose that there exists another solution $\hat{x}$ defined in I and such that $\bar{x}(t) \not\equiv \hat{x}(t)$ for $t \in I$ and $\|\hat{x}(t)\| \leq \bar{u}(t)$ for $t \in I$.

From (2.1) we have: $\|\hat{x}(t) - x_n(t)\| \leq u_n(t), t \in I, n = 0, 1, 2, \cdots$, and it follows that $\bar{x}(t) = \hat{x}(t)$ for $t \in I$. This contradiction proves the uniqueness of $\bar{x}$ in the class of functions satisfying relation (2.2). This completes the proof of the theorem.

We next establish a theorm which gives conditions under which equation (1.1) has at most one solution, those conditions do not guarantee existence.

Theorem 2.2. *If* H_1 *is satisfied and the function* $m(t) \equiv 0, t \in I$ *is the only nonnegative continuous solution of the inequality*

$$m(t) \leq H(t, \int_0^t W_1(t, s, m(s))ds, \int_0^T W_2(t, s, m(s))ds), 0 \leq t \leq T, \quad (2.5)$$

Then, equation (1.1) has at most one solution in I.

Proof. Let us suppose that there exist two solutions $\bar{x}$ and $\hat{x}$ of equation (1.1) such that

$$\bar{x}(t) \not\equiv \hat{x}(t), \quad t \in I.$$

Put

$$m(t) = \|\bar{x}(t) - \hat{x}(t)\|, \quad t \in I,$$

then

$$\begin{aligned} m(t) &= \|F(t, \int_0^t f(t, s, \bar{x}(s))ds, \int_0^T g(t, s, \bar{x}(s))ds) \\ &\quad -F(t, \int_0^t f(t, s, \hat{x}(s))ds, \int_0^T g(t, s, \hat{x}(s))ds)\| \\ &\leq H(t, \int_0^t W_1(t, s, \|\bar{x}(s) - \hat{x}(s)\|)ds, \\ &\quad \int_0^T W_2(t, s, \|\bar{x}(s) - \hat{x}(s)\|)ds) \\ &= H(t, \int_0^t W_1(t, s, m(s))ds, \int_0^T W_2(t, s, m(s))ds) \end{aligned}$$

and by (2.5) we conclude that $m(t) \equiv 0$ for $t \in I$, i.e. $\bar{x}(t) = \hat{x}(t)$, $t \in I$. This contradiction proves the theorem.

References

[1] Asirov, S. and Mamedov, J., *Investigation of solutions of nonlinear Volterra-Fredholm operator equations*, Dokl. Akad. Nauk, SSSR 229(1976), 982–986.

[2] Bihari, I., *Notes on a nonlinear integral equation*, Studia Sci. Math. Hungar. 2(1967), 1–6.

[3] Grossman, S., *Existence and stability of a class of nonlinear Volterra integral equations*, Trans. Amer. Math. Soc. 150(1970), 541–556.

[4] Mamedov, M. and Musaev, V., *On the theory of solutions of nonlinear operator equations*, Dokl. Akad. Nauk, SSSR. 195(1970), 1420–1423.

[5] Wazewski, T., *Sur une procédé de prouver la convergence des approximations successive sans utilisation des séries de comparison*. Bull. Acad. Polon. Sei. Sér. Sci. Math. Astr. et phy. 8(1960), 45–52.

Mathematics Department, Faculty of Science, Sana's University, Sana's, Yemen Arab Republic