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GENERALIZATION CLASS OF CERTAIN MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract

A generalization class $\sum_p(\alpha, \beta, \gamma)$ of certain meromorphic univalent functions with positive coefficients is introduced. The class $\sum_p(\alpha, \beta, \gamma)$ is a generalization of the class which was stuied by N.E. Cho, S.H. Lee and S. Owa [1]. The object of the present paper is to prove some properties of functions in the class $\sum_p(\alpha, \beta, \gamma)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$(1.1)\qquad f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0; \quad p \in N = \{1, 2, 3, \cdots\})$$

which are analytic and univalent in the domain $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue one at $z = 0$.

A function $f(z)$ in $\sum_p$ is said to be a member of the class $\sum_p(\alpha, \beta, \gamma)$ if it satisfies

$$(1.2)\qquad |z^2 f'(z) + 1| < \beta |(2\gamma - 1) z^2 f'(z) + (2\alpha\gamma - 1)|$$

for some $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$, $\gamma(\frac{1}{2} \leq \gamma \leq 1)$ and for all $z \in D$.

The class $\sum_1(\alpha, \beta, \gamma)$ when $p = 1$ was introduced and was studied by Cho, Lee and Owa ([1]). Therefore, the class $\sum_p(\alpha, \beta, \gamma)$ is a generalization of $\sum_1(\alpha, \beta, \gamma)$.

2. Distortion inequalities

We begin with the statement of the following lemma due to Cho, Lee and Owa ([1]).

Lemma 1. *Let a function $f(z)$ be in the class $\sum_1$. Then $f(z)$ belongs to the class $\sum_1(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)a_n \leq 2\beta\gamma(1-\alpha) \tag{2.1}$$

for some $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$, and $\gamma(\frac{1}{2} \leq \gamma \leq 1)$.

By virtue of the above Lemma 1, it is easy to see that

Lemma 2. *Let a function $f(z)$ be in the class $\sum_p$. Then $f(z)$ belongs to the class $\sum_p(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=p} n(1+2\beta\gamma-\beta)a_n \leq 2\beta\gamma(1-\alpha) \tag{2.2}$$

for some $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$, and $\gamma(\frac{1}{2} \leq \gamma \leq 1)$.

Now, we prove

Theorem 1. *If $f(z) \in \sum_p(\alpha,\beta,\gamma)$, then*

$$\frac{1}{|z|} - \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}|z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}|z|^p \tag{2.3}$$

and

$$\frac{1}{|z|^2} - \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}|z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}|z|^{p-1} \tag{2.4}$$

for $z \in D$. Equalities in (2.3) and (2.4) are attained for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}z^p. \tag{2.5}$$

Proof. Since

$$\sum_{n=p}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)} \tag{2.6}$$

and

$$\sum_{n=p}^{\infty} na_n \leq \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta} \tag{2.7}$$

for $f(z) \in \sum_p(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - |z|^p \sum_{n=p}^{\infty} a_n \\ &\geq \frac{1}{|z|} - \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}|z|^p, \end{aligned} \tag{2.8}$$

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + |z|^p \sum_{n=p}^{\infty} a_n \\ &\leq \frac{1}{|z|} + \frac{2\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}|z|^p, \end{aligned} \tag{2.9}$$

$$\begin{aligned} |f'(z)| &\geq \frac{1}{|z|^2} - |z|^{p-1} \sum_{n=p}^{\infty} n a_n \\ &\geq \frac{1}{|z|^2} - \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}|z|^{p-1}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} |f'(z)| &\leq \frac{1}{|z|^2} + |z|^{p-1} \sum_{n=p}^{\infty} n a_n \\ &\leq \frac{1}{|z|^2} - \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}|z|^{p-1}, \end{aligned} \tag{2.11}$$

which completes the proof of Theorem 1.

Remark 1. Taking $p = 1$ in Theorem 1, we have the corresponding results by Cho, Lee and Owa ([1]).

By the same way as in the proof by Cho, Lee and Owa ([1]), we have

Theorem 2. *If $f(z) \in \sum_p(\alpha, \beta, \gamma)$, then $f(z)$ is meromorphically starlike of order $\delta (0 \leq \delta < 1)$ in $0 < |z| < \gamma(\alpha, \beta, \gamma, \delta, p)$, where*

$$\gamma(\alpha, \beta, \gamma, \delta, p) = \inf_{n \geq p} \left\{ \frac{n(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{\frac{1}{n+1}}. \tag{2.12}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} z^n \quad (n \geq p). \tag{2.13}$$

Theorem 3. *If $f(z) \in \sum_p(\alpha, \beta, \gamma)$, then $f(z)$ is meromorphically convex of order $\delta(0 \leq \delta < 1)$ in $0 < |z| < \gamma(\alpha, \beta, \gamma, \delta, p)$, where*

$$\gamma(\alpha, \beta, \gamma, \delta, p) = \inf_{n \geq p} \left\{ \frac{(1 + 2\beta\gamma - \beta)(1 - \delta)}{2\beta\gamma(1 - \alpha)(n + 2 - \delta)} \right\}^{\frac{1}{n+1}}. \tag{2.14}$$

The result is sharp for the function $f(z)$ given by (2.13).

Remark 2. *A function $f(z) \in \sum_p$ is said to be meromorphically starlike of order $\delta(0 \leq \delta < 1)$ if*

$$Re\{-\frac{zf'(z)}{f(z)}\} > \delta \quad (z \in D). \tag{2.15}$$

Further, a function $f(z) \in \sum_p$ is said to be meromorphically convex of order $\delta(0 \leq \delta < 1)$ if

$$Re\{-(1 + \frac{zf''(z)}{f'(z)})\} > \delta \quad (z \in D). \tag{2.16}$$

3. Convolution properties

For the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{j,n} z^n \quad (a_{j,n} \geq 0; j = 1, 2) \tag{3.1}$$

belonging to $\sum_p$, we denote by $f_1 * f_2(z)$ the convolution of $f_1(z)$ and $f_2(z)$, or

$$f_1 * f_2(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_{1,n} a_{2,n} z^n. \tag{3.2}$$

Theorem 4. *If $f_j(z)(j = 1, 2)$ are in the class $\sum_p(\alpha, \beta, \gamma)$, then $f_1 * f_2(z)$ in $\sum_p(\delta, \beta, \gamma)$, where*

$$\delta = 1 - \frac{2\beta\gamma(1 - \alpha)^2}{p(1 + 2\beta\gamma - \beta)}. \tag{3.3}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1 - \alpha)}{p(1 + 2\beta\gamma - \beta)} z^p \quad (j = 1, 2). \tag{3.4}$$

Proof. We shall find the largest δ such that

$$\sum_{n=p}^{\infty} n(1+2\beta\gamma-\beta)a_{1,n}a_{2,n} \leq 2\beta\gamma(1-\delta) \tag{3.5}$$

for $f_j(z) \in \sum_p(\alpha,\beta,\gamma)$. Note that $f_j(z) \in \sum_p(\alpha,\beta,\gamma)$ imply

$$\sum_{n=p}^{\infty} n(1+2\beta\gamma-\beta)a_{j,n} \leq 2\beta\gamma(1-\alpha) \quad (j=1,2). \tag{3.6}$$

By using the Cauchy-Schwarz inequality, we have

$$\sum_{n=p}^{\infty} n(1+2\beta\gamma-\beta)\sqrt{a_{1,n}a_{2,n}} \leq 2\beta\gamma(1-\alpha). \tag{3.7}$$

Therefore, we only need to prove that

$$\frac{a_{1,n}a_{2,n}}{1-\delta} \leq \frac{1}{1-\alpha}\sqrt{a_{1,n}a_{2,n}} \quad (n \geq p), \tag{3.8}$$

or

$$\sqrt{a_{1,n}a_{2,n}} \leq \frac{1-\delta}{1-\alpha} \quad (n \geq p). \tag{3.9}$$

Using (3.7), we have to show that

$$\frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} \leq \frac{1-\delta}{1-\alpha} \quad (n \geq p), \tag{3.10}$$

that is, that

$$\delta \leq 1 - \frac{2\beta\gamma(1-\alpha)^2}{n(1+2\beta\gamma-\beta)} \quad (n \geq p). \tag{3.11}$$

Noting that

$$\phi(n) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{n(1+2\beta\gamma-\beta)} \quad (n \geq p) \tag{3.12}$$

is an increasing function of n, we have

$$\delta \leq \phi(p) = 1 - \frac{2\beta\gamma(1-\alpha)^2}{p(1+2\beta\gamma-\beta)} \tag{3.13}$$

which completes the proof of Theorem 4.

Taking $p=1$ in Theorem 4, we have

Corollary 1. *If $f_j(z)$ $(j=1,2)$ are in the class $\sum_1(\alpha,\beta,\gamma)$, then $f_1 * f_2(z) \in \sum_p(\delta,\beta,\gamma)$, where*

$$\delta = 1 - \frac{2\beta\gamma(1-\alpha)^2}{1+2\beta\gamma-\beta}. \tag{3.14}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}z \quad (j=1,2). \tag{3.15}$$

Finally, we prove

Theorem 5. *If $f_j(z)$ $(j=1,2)$ are in the class $\sum_p(\alpha,\beta,\gamma)$, then*

$$h(z) = \frac{1}{z} + \sum_{n=p}^{\infty}(a_{1,n}^2 + a_{2,n}^2)z^n \tag{3.16}$$

belongs to the class $\sum_p(\delta,\beta,\gamma)$, where

$$\delta = 1 - \frac{4\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}. \tag{3.17}$$

The result is sharp for the function $f(z)$ given by (3.4).

Proof. It follows from $f_j(z) \in \sum_p(\alpha,\beta,\gamma)$ that

$$\begin{aligned} &\sum_{n=p}^{\infty} \frac{n^2(1+2\beta\gamma-\beta)^2}{4\beta^2\gamma^2(1-\alpha)^2} a_{j,n}^2 \\ &\le \Big(\sum_{n=p}^{\infty} \frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_{j,n}\Big)^2 \\ &\le 1. \end{aligned} \tag{3.18}$$

Therefore, we have

$$\sum_{n=p}^{\infty} \frac{n^2(1+2\beta\gamma-\beta)^2}{4\beta^2\gamma^2(1-\alpha)^2}(a_{1,n}^2 + a_{2,n}^2) \le 2. \tag{3.19}$$

Thus, we need to find the largest δ such that

$$\frac{1}{1-\delta} \le \frac{n(1+2\beta\gamma-\beta)}{4\beta\gamma(1-\alpha)} \quad (n \ge p). \tag{3.20}$$

or

$$\delta \leq 1 - \frac{4\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} \quad (n \geq p). \tag{3.21}$$

Since the function

$$\psi(n) = 1 - \frac{4\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} \quad (n \geq p) \tag{3.22}$$

is increasing on n, we see that

$$\delta \leq \psi(p) = 1 - \frac{4\beta\gamma(1-\alpha)}{p(1+2\beta\gamma-\beta)}. \tag{3.23}$$

This completes the proof of Theorem 5.

Making $p = 1$, Theorem 5 leads to

Corollary 2. *If $f_j(z)$ $(j = 1, 2)$ are in the class $\sum_1(\alpha, \beta, \gamma)$, then $h(z) \in \sum_1(\delta, \beta, \gamma)$, where*

$$\delta = 1 - \frac{4\beta\gamma(1-\alpha)}{1+2\beta\gamma-\beta}. \tag{3.24}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by (3.15).

References

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