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ON NEAT-INJECTIVE GROUPS

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It has been proved that every group can be embedded as a neat subgroup in a reduced neat–injective group provided its Frattini subgroup vanishes. Connecting Ext to Next and Next to Hom in the form of exact sequences the existence of two subgroups of Next has been shown. Splitting of Next and the quotient groups obtained are discussed.

Introduction

An exact sequence $0 \to A \to G \to C \to 0$ is called neat exact if A is a neat subgroup of G. The elements of the group $Next(C, A)$ are the neat exact sequences. $Next(C, A)$ is a cotorsion group for all groups A and C. $Next(C, A)$ is the Frattini subgroup of $Ext(C, A)$.

A group E is called neat–injective if every neat exact sequence $0 \to E \to G \to H \to O$ splits or equivalently if $Ext(Q, E) = 0 = Next(Q/Z, E)$. A group E is neat–injective if and only if $E = D \oplus \Pi_p T_p$, where D is divisible, $pT_p = 0$ and p ranges over all primes see [2].

Lemma 1. *If* $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ *is a neat exact sequence, then for any group* G *the induced homomorphisms*

$$v^* : Ext(C, G) \to Ext(B, G) \quad and \quad u_* : Ext(G, A) \to Ext(G, B)$$

map upon neat subgroups.

Lemma 2. *If the sequence* $N : 0 \to A \to B \to C \to 0$ *is neat exact, then the images of the connecting homomorphisms*

$$N^* : Hom(A, G) \to Ext(C, G), \quad N_* : Hom(G, C) \to Ext(G, A)$$

are contained in $Next(C, G)$ *and* $Next(G, A)$ *respectively.*

The proof of the above lemmas is on similar lines as that of lemmas 53.5 and 53.6 of [1].

In general we adopt the notations used in [1]. The Frattini subgroup of a group A will be dented by $\phi(A)$ and the Frattini factor $A/\phi(A)$ by A_ϕ.

Main Results

Every group is a neat subgroup of a neat–injective group. See [2]. In case of reduced neat–injective groups we prove the following.

Lemma 3. *A group G can be embedded as a neat subgroup in a reduced neat-injective group $\overline{G}$ if and only if $\phi(G) = 0$. Moreover, the quotient group $\overline{G}/G$ is divisible.*

Proof. By lemma 4 of [2] $\overline{G} = \Pi_{p\in P}(G/pG)$. Define a homomorphism $f : G \to \overline{G}$ such that

$$f(g) = (\cdots, g + pG, \cdots) \quad \text{for} \quad g \in G \quad \text{and} \quad p \in P.$$

Now $f(g) = 0 \Rightarrow g + pG = 0 \Rightarrow g \in pG$, for every p implies $g \in \cap_p pG = \phi(G) = 0$ and f is monomorphism.

We proceed to prove that $f(G)$ is neat in $\overline{G}$. Let the equation $p\overline{g} = f(g)$ has a solution in $\overline{G}$ for $g \in G$ and $p \in P$, then for $\overline{g} = (\cdots, g_p + pG, \cdots) \in \overline{G}$ we have

$$p(\cdots, g_p + pG, \cdots) = (\cdots, g + pG, \cdots)$$

which implies $pg_p - g \in pG \Rightarrow g \in pG \Rightarrow f(g) \in pf(G) \Rightarrow \overline{g} \in f(G)$.

Converse follows from the fact that G neat in $\overline{G}$ implies $\phi(G) = G \cap \phi(\overline{G})$ whereas $\phi(\overline{G}) = 0$.

Since G/pG is bounded it is complete in its Z–adic topology. Corollary 13.4 of [1] implies $\overline{G}$ is complete in the Z–adic topology. Theorems 13.6 and 39.5 of [1] implies that the induced topology of a complete group is Z–adic and hence G is dense in $\overline{G}$ and exercise 10(b) page 34 of [1] implies $\overline{G}/G$ is divisible.

We establish the isomorphism between Hom and $Next$ which is contained in

Lemma 4. *Let A be a group such that $\phi(A) = 0$. Then*

$$Hom(D, G/A) \cong Next(D, A)$$

for any divisible group D and a reduced neat-injective group G containing A as a neat subgroup. Furthermore, $Next(D,A)$ is torsion-free.

Proof. By lemma 3 we have a neat exact sequence $0 \to A \to G \to G/A \to 0$ with G/A divisible. It induces the exact sequence

$$Hom(D,G) \to Hom(D,G/A) \to Next(D,A) \to Next(D,G)$$

The first group is zero, and the last group is zero because G is neat–injective. Divisibility of D implies that $Hom(D,G/A)$ and hence $Next(D,A)$ is torsion–free.

Next, we connect Hom to Ext to $Next$ in the form of an exact sequence.

Theorem 5. *For any group G, the sequence*

$$0 \to Hom(G,\phi(A)) \to Hom(G,A) \to Hom(G,A_\phi) \to Ext(G,\phi(A))$$

$$\xrightarrow{\alpha_*} Next(G,A) \xrightarrow{\beta_*} Next(G,A_\phi) \to 0$$

is exact.

Proof. The exact sequence $0 \to \phi(A) \xrightarrow{\alpha} A \xrightarrow{\beta} A_\phi \to 0$ and the free resolution $0 \to H \to F \to G \to 0$ of G give the following commutative diagram with exact rows

$$\begin{array}{ccccc} Hom(H,\phi(A)) & \xrightarrow{\eta} & Ext(G,\phi(A)) & \longrightarrow & Ext(F,\phi(A)) = 0 \\ {\scriptstyle \alpha'_*} \downarrow & & {\scriptstyle \alpha_*} \downarrow & & \\ Hom(H,A) & \xrightarrow{\psi} & Ext(G,A) & \longrightarrow & Ext(F,A) = 0 \end{array}$$

where η and ψ stands for the connecting homomorphism.

Now, $Im\alpha_* = Im\alpha_*\eta = Im\psi\alpha'_*$.

Since H is free $Im\alpha'_*$ must be contained in the Frattini subgroup of $Hom(H,A) \cong \Pi A$. The epimorphism of ψ implies that

$$Im\psi\alpha'_* \subseteq \phi(Ext(G,A)) = Next(G,A).$$

Since the sequence

$$Ext(G,\phi(A)) \xrightarrow{\alpha_*} Ext(G,A) \xrightarrow{\beta_*} Ext(G,A_\phi) \to 0$$

is exact, the homomorphism β_* maps Frattini subgroup into Frattini subgroup and hence the sequence

$$Ext(G,\phi(A)) \xrightarrow{\alpha_*} Next(G,A) \xrightarrow{\beta_*} Next(G,A_\phi)$$

is exact. We are required to prove that every N in $Next(G,A_\phi)$ is the image of some N' in $Next(G,A)$. But in $Next(G,A)$ we do have an element N' such that $\beta_* N' = N$.

Since $Im\alpha_* = Ker\beta_* \subseteq Next(G,A)$, by lemma 37.1 of [1] no element not in $Next(G,A)$ can be mapped into the Frattini subgroup of $Im\beta_*$ and hence $N' \in Next(G,A)$.

Corollary 6. *$Next(D,G) \cong Ext(D,\phi(G)) \oplus Next(D,G_\phi)$ for any divisible group D.*

Proof. Follows from theorem 5 and lemma 4.

Now we connect $Next$ to Hom in the form of an exact sequence.

Theorem 7. *If A satisfies $\phi(A) = 0$, then the sequence*

$$0 \to Next(C_\phi, A) \to Next(C,A) \to Hom(\phi(C), G/A) \to 0$$

is exact, where G is a reduced neat-injective group containing A as a neat subgroup.

Proof. The neat exact sequence $0 \to A \to G \to G/A \to 0$ with G/A divisible, together with the exact sequence $0 \to \phi(C) \to C \to C_\phi \to 0$ gives the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & Hom(C_\phi,G) & \xrightarrow{k} & Hom(C,G) & & \\
 & & \downarrow u & & \downarrow \lambda & & \\
0 & \longrightarrow & Hom(C_\phi,G/A) & \xrightarrow{v} & Hom(C,G/A) & \longrightarrow & Hom(\phi(C),G/A) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & Next(C_\phi,A) & \longrightarrow & Next(C,A) & & \\
 & & \downarrow & & \downarrow & & \\
 & & Next(C_\phi,G)=0 & & Next(C,G)=0 & &
\end{array}$$

with exact rows and columns, see Theorem 44.4 in [1]. Since G is reduced neat–injective $\phi(G) = 0$ and every homomorphism $C \to G$ is induced by some $C_\phi \to G$. Hence k is isomorphism and the top row stays exact if continue it with $\to 0$. The bottom row is zero since G is neat–injective.

Now we can complete the third row with $Hom(\phi(C), G/A)$, this homomorphism exists in view of both $Next(C, A)$ and the group $Hom(\phi(C), G/A)$ are epimorphic images of $Hom(C, G/A)$ with kernels $Im\lambda$ and Imv, where $Im\lambda = Im\lambda k = Imvu \subseteq Imv$. Hence the third row can be completed with $Hom(\phi(C), G/A) \to 0$.

Corollary 8. *If A and C satisfy $\phi(A) = 0$ and $\phi(C)$ is divisible, then the following hold.*

(a) $Next(C, A) \cong Next(C_\phi, A) \oplus Hom(\phi(C), G/A)$

(b) $Next(C, A) \cong Next(C_\phi, A) \oplus Next(\phi(C), A)$

Proof. The proof follows from the fact that in the exact sequence.

$$0 \to Next(C_\phi, A) \to Next(C, A) \to Hom(\phi(C), G/A) \to 0$$

the first group is cotorsion and the last group is torsion–free.

The proof of (b) follows from Lemma 4.

Now we construct two subgroups of Next one contained in another and discuss the decomposition of the quotient groups.

Theorem 9. *For arbitrary groups A and C $Next(C, A)$ has two subgroups $K(C, A) \subseteq L(C, A)$ such that*

$$\frac{Next(C, A)}{L(C, A)} \cong Hom(\phi(C), \frac{G}{A_\phi})$$

where G is a reduced neat–injective group that contains A_ϕ as a neat subgroup and

$$\frac{L(C, A)}{K(C, A)} \cong Next(C_\phi, A_\phi) \oplus Ext(\phi(C), \phi(A))$$

If the Frattini factor C_ϕ of C is an elementary p–group then $K(C, A) = 0$ and $L(C, A) \cong Ext(\phi(C), \phi(A))$.

Proof. The exact sequences $0 \to \phi(C) \to C \to C_\phi \to 0$ and $0 \to \phi(A) \to A \to A_\phi \to 0$, by Theorem 51.3 of [1] and Theorem 5 and Theorem 7 give the following commutative diagram.

$$\begin{array}{ccccccc}
Ext(C_\phi,\phi(A)) & \xrightarrow{\alpha} & Ext(C,\phi(A)) & \longrightarrow & Ext(\phi(C),\phi(A)) & \longrightarrow & 0\\
\gamma\downarrow & & \downarrow\beta & & & & \\
Next(C_\phi,\lambda) & \xrightarrow{\delta} & Next(C,A) & & & & \\
\kappa\downarrow & & \downarrow\lambda & & & & \\
Next(C_\phi,A_\phi) & \xrightarrow{u} & Next(C,A_\phi) & \xrightarrow{v} & Hom(\phi(C),G/A_\phi) & \longrightarrow & 0\\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

with exact rows and columns.

Now let $K(C,A)=Im\beta\alpha$ and $L(C,A)=Ker\,v\lambda$. It is clear that $v\lambda: Next(C,A)\to Hom(\phi(C),G/A_\phi)$ is epimorphism, and so

$$\frac{Next(C,A)}{L(C,A)}\cong Hom(\phi(C),G/A_\phi).$$

Theorem 8.3 of [1] implies $L(C,A)=Im\beta+Im\delta$. Since $Im\beta\alpha\subseteq Im\beta$ and $Im\delta\gamma\subseteq Im\delta$ it follows that $K(C,A)\subseteq Im\beta\cap Im\delta$.

Let $x\in Im\beta\cap Im\delta$, then $x\in Im\beta=Ker\lambda\Rightarrow \lambda x=0$ and for some $y\in Next(C_\phi,A)$ we have $x=\delta y\Rightarrow \lambda x=0=\lambda\delta y=uky\Rightarrow ky=0\Rightarrow$ $y\in Ker\,k=Imv$ showing that $x\in Im\delta\gamma=Im\beta\alpha=K(C,A)$. Thus $K(C,A)=Im\beta\cap Im\delta$. Hence

$$\frac{L(C,A)}{K(C,A)}=\frac{(Im\beta+Im\delta)}{Im\beta\alpha}=\frac{Im\beta}{Im\beta\alpha}+\frac{Im\delta}{Im\delta\gamma}$$

It is clear that $Im\beta/Im\beta\alpha\cong Ext(\phi(C),\phi(A))$ and $Im\delta/Im\delta\gamma\cong Next(C_\phi,A_\phi)$. The required isomorphism follows.

Now, by [3] $Next(C_\phi,A)=0=Next(C_\phi,A_\phi)$ if C_ϕ is an elementary p-group. It follows that $Im\delta=0$ and so $K(C,A)=Im\beta\cap Im\delta=0$.

Corollary 10. *If $\phi(C)$ is divisible, then the following hold.*

(a) $\frac{Next(C,A)}{K(C,A)}\cong Next(C_\phi,A_\phi)\oplus Ext(\phi(C),\phi(A))\oplus Hom(\phi(C),G/A_\phi)$

(b) $\frac{Next(C,A)}{K(C,A)} \cong Next(C_\phi, A_\phi) \oplus Ext(\phi(C), \phi(A)) \oplus Next(\phi(C), A_\phi)$

(c) $\frac{Next(C,A)}{K(C,A)}$ *is cotorsion.*

Proof. $K(C,A) \subseteq L(C,A) \subseteq Next(C,A)$ implies the isomorphism

$$\frac{Next(C,A)}{K(C,A)} \Big/ \frac{L(C,A)}{K(C,A)} \cong \frac{Next(C,A)}{L(C,A)}$$

and hence the sequence

$$0 \to \frac{L(C,A)}{K(C,A)} \to \frac{Next(C,A)}{K(C,A)} \to \frac{Next(C,A)}{L(C,A)} \to 0$$

is exact. The first group in the sequence being direct sum of two cotorsion groups is cotorsion. Divisibility of $\phi(C)$ implies $Hom(\phi(C), G/A_\phi)$ and therefore $\frac{Next(C,A)}{L(C,A)}$ is torsion–free and hence the sequence splits.

The proof of (b) follows from lemma 4. Since all direct summands in (b) are cotorsion (c) follows.

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