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MAXIMAL OPERATORS CONCERNING THE DIFFERENTIABILITY OF FUNCTIONS

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1. Introduction

In this paper we will introduce certain maximal operators in terms of which we will characterize the first order Sobolev functions. The first order Sobolev space $L_1^p(\boldsymbol{R}^n)$ is defined to be the set of all functions f belonging to $L^p(\boldsymbol{R}^n)$ whose distributional derivatives $\frac{\partial f}{\partial x_j}$, $j=1, \cdots, n$, also belong to $L^p(\boldsymbol{R}^n)$. It is well known that if a function f and its distributional derivative $\frac{\partial f}{\partial x_j}$ are locally integrable then f (possibly modified on a set of measure zero) is in fact partially differentiable with respect to x_j almost everywhere. For this and other properties of Sobolev functions we refer the readers to [1] and [4]. The differentiability of a function at almost every point in a given set has been studied by many persons. We refer the readers to Stein [3], which shows a systematic approach to the problem, and also to Neugebauer [2] for a succinct condition for the differentiability property. In their studies the even part of a function played an important role. We are, however, concerned with the odd part. The even and odd parts of a function f on $\boldsymbol{R}^1$ at x are defined to be the functions φ and ψ, respectively, given by $\varphi(t)=\frac{1}{2}(f(x+t)+f(x-t))$ and $\psi(t)=\frac{1}{2}(f(x+t)-f(x-t))$.

2. Definitions

For a function $f \in C^1(\boldsymbol{R}^n)$ and for $j=1, \cdots, n$ and $h>0$ we define the *mean difference quotient* $\delta_{j,h}f(x)$ of f at $x \in \boldsymbol{R}^n$ by the equation

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(1) $$\delta_{j,h}f(x)=\frac{1}{h}\int_0^h \frac{f(x+te_j)-f(x-te_j)}{2t}dt,$$

where e_j is the j-th standard basis element of $\boldsymbol{R}^n$. The *maximal derivate* $D_j f(x)$ is then defined to be the associated maximal function given by the equation

(2) $$D_j f(x)=\sup_{0<h\leq 1}|\delta_{j,h}f(x)|.$$

We interpret the singular integral in (1) as the limit of the following integrals when $\epsilon\to 0$:

$$\frac{1}{h}\int_\epsilon^h \frac{f(x+te_j)-f(x-te_j)}{2t}dt.$$

LEMMA 1. *Let* $f\in L^1_{loc}(\boldsymbol{R}^n)$. Then for every $j=1,\cdots,n$ and every $h>0$, *the integral defining* $\delta_{j,h}f(x)$ *converges and is finite for a. e.* $x\in\boldsymbol{R}^n$, *and so* $D_j f(x)$ *is a well-defined measurable function.*

Proof. Suppose first $n=1$ and fix $h>0$. The function φ_h defined to be $1/s$ if $|s|\leq h$ and zero otherwise is a Calderón-Zygmund kernel, and so for every $g\in L^1(\boldsymbol{R}^1)$ the singular integral $g*\varphi_h(x)$ exists for a. e. $x\in\boldsymbol{R}^1$. Now for each positive integer N let $f_N(x)$ to be $f(x)$ if $|x|\leq N$ and zero otherwise. Then since $f_N\in L^1(\boldsymbol{R}^1)$, $f_N*\varphi_h(x)$ exist for a. e. $x\in\boldsymbol{R}^1$. If $|x|\leq N-h$, then $f*\varphi_h(x)=f_N*\varphi_h(x)$, and so $f*\varphi_h(x)$ exists for a. e. x with $|x|\leq N-h$. Letting $N\to\infty$ we now see that $f*\varphi_h(x)$ exists for a. e. $x\in\boldsymbol{R}^1$. But, $\delta_{1,h}f(x)=-\frac{1}{2h}f*\varphi_h(x)$. The assertion for $\delta_{j,h}f$ is thus proved for the case $n=1$.

Suppose now $n>1$, and fix $j=1,\cdots,n$. Let V_j be the hyperplane of $\boldsymbol{R}_n$ perpendicular to e_j, and for each $x'\in V_j$ let $f_{x'}(t)=f(x'+te_j)$, $t\in\boldsymbol{R}^1$. By the Fubini's theorem it follows that $f_{x'}\in L^1_{loc}(\boldsymbol{R}^1)$ for a. e. $x'\in V_j$. The previous case then implies that $f_{x'}*\varphi_h(t)$ exists for a. e. $x'\in V_j$ and for a. e. $t\in\boldsymbol{R}^1$. But, $\delta_{j,h}f(x)=\delta_{j,h}f(x'+te_j)=-\frac{1}{2h}f_{x'}*\varphi_h(t)$. Thus $\delta_{j,h}f(x)$ exists for a. e. $x\in\boldsymbol{R}_n$.

The measurability of $D_j f$ follows from the equation $\sup\{\delta_{j,h}f(x) : 0<h\leq 1\}=\sup\{\delta_{j,r}f(x) : 0<r\leq 1,\ r \text{ rational}\}$, which in turn follows from the fact that for each fixed x, $\delta_{j,h}f(x)$ is continuous in h.

Note that $D_j f$ is well-defined in particular for every $f\in L^p(\boldsymbol{R}^n)$,

$1\leq p\leq\infty$. The readers are referred to [4] for the Calderón-Zygmund kernel.

3. Main results

THEOREM 1. *Let* $f\in{}_1^p(\boldsymbol{R}^n)$, $1\leq p\leq\infty$. *Then for* $j=1,\cdots,n$

(a) *if* $1<p\leq\infty$, *then* $D_jf\in L^p(\boldsymbol{R}^n)$ *and*

$$\|D_jf\|_p\leq c_p\left\|\frac{\partial f}{\partial x^j}\right\|_p;$$

and

(b) *if* $p=1$, *then for every* $r>0$

$$|\{x: D_jf(x)>r\}|\leq\frac{c_1}{r}\left\|\frac{\partial f}{\partial x_j}\right\|_1,$$

where $|\cdot|$ *denotes the Lebesgue measure on* $\boldsymbol{R}^n$. *The constants* c_p *depend only on the parameters* p *and* n.

We need the following lemma for the proof of the above theorem.

LEMMA 2. *Let* $f\in L_1^p(\boldsymbol{R}^n)$, $1\leq p\leq\infty$. Then for $j=1,\cdots,n$ and $h>0$

$$\delta_{j,h}f(x)=\frac{1}{h}\int_0^h\left(\frac{1}{2t}\int_{-t}^t\frac{\partial f}{\partial x_j}(x+se_j)ds\right)dt$$

for a.e. $x\in\boldsymbol{R}^n$.

Proof. By the Fubini's theorem we may assume $n=1$. Fix $h>0$ and let

$$I(x)=\delta_{1,h}f(x)-\frac{1}{h}\int_0^h\left(\frac{1}{2t}\int_{-t}^t f'(x+s)ds\right)dt.$$

It suffices to show that for every $\varphi\in C_c^\infty(\boldsymbol{R}^1)$

$$\int I(x)\varphi(x)dx=0. \tag{3}$$

Setting

$$J(t)=\int(f(x+t)-f(x-t)-\int_{-t}^t f'(x+s)ds)\varphi(x)dx,$$

we get

$$\int I(x)\varphi(x)dx=\frac{1}{h}\int_0^h\frac{1}{2t}J(t)dt.$$

But

$$\begin{aligned}J(t)&=\int f(x)(\varphi(x-t)-\varphi(x+t)+\int_{-t}^t\varphi'(x-s)ds)dx\\&=0.\end{aligned}$$

Now (3) follows from this.

Proof of Theorem 1. Let M_j be the Hardy-Littlewood maximal operator acting in the direction of e_j, defined by the equation $M_j g(x)=\sup_{t>0}\frac{1}{2t}\int_{-t}^{t}|g(x+se_j)|ds$. Then by Lemma 2

$$|\delta_{j,h}f(x)|\leq\frac{1}{h}\int_0^h\frac{1}{2t}\int_{-t}^{t}\left|\frac{\partial f}{\partial x_j}(x+se_j)\right|ds\,dt$$
$$\leq\frac{1}{h}\int_0^h M_j\Big(\frac{\partial f}{\partial x_j}\Big)(x)\,ds=M_j\Big(\frac{\partial f}{\partial x_j}\Big)(x).$$

Hence $D_jf(x)\leq M_j\Big(\frac{\partial f}{\partial x_j}\Big)(x)$ for a. e. $x\in\boldsymbol{R}^n$. Now the inequalities for D_j follows from the corresponding inequalities for M_j.

We refer readers to [4] for the properties of the Hardy-Littlewood maximal operators.

Remark. The weak-type boundedness of the maximal operators D_j on $L_1^1(\boldsymbol{R}^n)$ is the best we can expect. There are indeed functions in $L_1^1(\boldsymbol{R}^n)$ whose maximal derivates do not belong to $L^1(\boldsymbol{R}^n)$. An example can be constructed as follows. For each positive integer m define a function g_m on $\boldsymbol{R}^1$ by setting $g_m(x)=m^{-2}$ for $2^{-m-2}\leq x\leq 1+2^{-m-2}$, $g_m(x)=0$ for $x\leq 0$ or $x\geq 1+2^{-m-1}$, and linear otherwise. Then, $\|g_m\|_1\leq 2m^{-2}$ and $\left\|\frac{d}{dx}g_m\right\|_1\leq 2m^{-2}$, where $\frac{d}{dx}g_m$ is the distributional derivative of g_m. Furthermore, if $-\frac{1}{4}\leq x\leq -2^{-m-2}$, then

$$D_1g_m(x)\geq\delta_{1,4|x|}g_m(x)\geq\frac{1}{4|x|}\int_{2|x|}^{4|x|}\frac{g_m(x+t)}{2t}dt$$
$$=\frac{1}{4|x|}\int_{2|x|}^{4|x|}\frac{m^{-2}}{2t}dt\geq\frac{1}{16m^2|x|}$$

Hence,

$$\int_{-1}^{0}D_1g_m(x)\,dx\geq\frac{1}{16m^2}\int_{-1/4}^{-2^{-m-2}}\frac{1}{|x|}dx\geq\frac{c}{m},$$

where $c=\log 2/16$. Now letting $g(x)=\sum_{m=1}^{\infty}g_m(x-4(m-1))$ we obtain a desired function. It follows that

$$\|g\|_1 \leq \sum_{m=1}^{\infty} \|g_m\|_1 \leq 2\sum_{m=1}^{\infty} m^{-2} < \infty,$$

and

$$\left\| \frac{dg}{dx} \right\|_1 \leq \sum_{m=1}^{\infty} \left\| \frac{d}{dx} g_m \right\|_1 \leq 2\sum_{m=1}^{\infty} m^{-2} < \infty,$$

and so $g \in L_1^1(\boldsymbol{R}^1)$. But, since $D_1 g(x) = D_1 g_m(x - 4(m-1))$ for $4(m-1) - 1 \leq x \leq 4(m-1)$,

$$\|D_1 g\|_1 \geq \sum_{m=1}^{\infty} \int_{4(m-1)-1}^{4(m-1)} D_1 g(x)\, dx$$
$$= \sum_{m=1}^{\infty} \int_{-1}^{0} D_1 g_m(t)\, dt \geq c \sum_{m=1}^{\infty} \frac{1}{m}.$$

Thus $D_1 g \notin L^1(\boldsymbol{R}^1)$ since the last series diverges to ∞.

An immediate consequence of Theorem 1 is

COROLLARY. *Let* $f \in L_1^p(\boldsymbol{R}^n)$, *and* $j = 1, \cdots, n$. *Then* $\delta_{j,h} f(x) \to \frac{\partial f}{\partial x_j}(x)$ *as* $h \to 0$ *for a.e.* $x \in \boldsymbol{R}^n$. *Futhermore, the convergence is also in the* L^p*-norm provided* $1 < p \leq \infty$.

As a converse of Theorem 1 we have

THEOREM 2. *Let* $f \in L^p(\boldsymbol{R}^n)$, $1 \leq p \leq \infty$. *If* $D_j f \in L^p(\boldsymbol{R}^n)$ *for some* $j = 1, \cdots, n$, *then* $\frac{\partial f}{\partial x_j} \in L^p(\boldsymbol{R}^n)$ and $\left\| \frac{\partial f}{\partial x_j} \right\|_p \leq \|D_j f\|_p$. Hence, if $D_j f \in L^p(\boldsymbol{R}^n)$ *for every* $j = 1, \cdots, n$, *then* $f \in L_1^p(\boldsymbol{R}^n)$.

The following lemma will be used to prove the above theorem in the case $p = 1$.

LEMMA 3. *Let* $\{f_k\}$ *be a sequence of functions* $L^1(\boldsymbol{R}^n)$, $g \in L^1(\boldsymbol{R}^n)$, *and* μ *a finite Borel measure on* $\boldsymbol{R}^n$. *Suppose* $|f_k| \leq g$ *for every* k *and* f_k *converges weakly to* μ, *i.e.*, $\int f_k \varphi \to \int \varphi\, d\mu$ *for every* $\varphi \in C_0(\boldsymbol{R}^n)$. *Then* μ *is absolutely continuous.*

Proof. We may assume each of f_k and μ is real-valued (by splitting them into the real and imaginary parts, and by apylying the following arguments to each part.) It suffices to show each of μ^+ and μ^- is absolutely continuous. The absolute continuity of μ^+ (or μ^-) is obtained once we show that $\mu^+(E) > 0$ (or $\mu^-(E) > 0$) implies $|E| > 0$ for every Borel set E.

By the Hahn's decomposition theorem there exist Borel sets P and N such that $P\cap N=\phi$, $P\cap N=\boldsymbol{R}^n$, and $\mu^+(E)=\mu(E\cap P)$ and $\mu^-(E)=-\mu(E\cap N)$ for every Borel set E.

To prove the absolute continuity of μ^+ suppose E is a Borel set and $\epsilon=\mu^+(E)>0$. We may assume $E\subset P$ (otherwise we can consider $E\cap P$). Choose a compact set $K\subset E$ (and so $K\subset P$) with $\mu^+(E\sim K)<\epsilon/4$, and an open set $V\supset E$ with $|\mu|(V\sim E)<\epsilon/4$. Such sets can be chosen by the regularity of μ^+ and $|\mu|$. Note that $\mu^+(K)>3\epsilon/4$. Let G be an arbitary open set such that $E\subset G\subset V$, and choose $\varphi\in C_0(\boldsymbol{R}^n)$ such that $0\leq\varphi\leq 1$, $\varphi(x)=1$ for $x\in K$, and supp $\varphi\subset G$. Then for $k=1,2,\cdots$,

$$\int_G g\geq\int_G\varphi g\geq\int\varphi|f_k|\geq\left|\int\varphi f_k\right|.$$

Since $\int\varphi f_k\to\int\varphi d\mu$, it now follows that $\int_G g\geq\left|\int\varphi d\mu\right|$. On the other hand,

$$\left|\int\varphi d\mu\right|=\left|\int_K d\mu+\int_{G\sim K}\varphi d\mu\right|\geq|\mu(K)|-|\mu|(G\sim K)$$
$$\geq\mu^+(K)-|\mu|(V\sim K)>\frac{3}{4}\epsilon-\frac{1}{4}\epsilon=\frac{\epsilon}{2}>0.$$

Thus we get $\int_G g>\epsilon/2$. Now

$$\int_E g=\inf\left\{\int_G g : E\subset G,\ G \text{ open}\right\}$$
$$=\inf\left\{\int_G g : E\subset G\subset V,\ G \text{ open}\right\}\geq\frac{\epsilon}{2}>0,$$

and this implies $|E|>0$. We thus obtain the absolute continuity of μ^+. Similary we see that μ^- is also absolutely continuous.

Proof of THEOREM 2. Suppose first $1<p\leq\infty$, and choose q such that $1/p+1/q=1$. From the hypothesis we see that each $\delta_{j,h}f$ belongs to the ball of radius $\|D_jf\|_p$ in the space $L^p(\boldsymbol{R}^n)$, which is the dual space of $L^q(\boldsymbol{R}^n)$. By the weak-compactness of balls in dual spaces it then follows that there exists a function $g\in L^p(\boldsymbol{R}^n)$ with $\|g\|_p\leq\|D_jf\|_p$ and a sequence $\{h_k\}$ with $h_k\to 0$ as $k\to\infty$ such that

$$(4)\qquad \int\delta_{j,h_k}f(x)\varphi(x)dx\to\int g(x)\varphi(x)dx$$

as $k\to\infty$ for every $\varphi\in L^q(\boldsymbol{R}^n)$. This is a fortiori true for $\varphi\in C_c^\infty(\boldsymbol{R}^n)$. Then, for $\varphi\in C_c^\infty(\boldsymbol{R}^n)$ it is easy to see that

$$\int\delta_{j,h_k}f(x)\varphi(x)dx=-\int f(x)\delta_{j,h_k}\varphi(x)dx. \tag{5}$$

Since $\varphi\in L^\infty(\boldsymbol{R}^n)$, Theorem 1 implies $D_j\varphi\in L^\infty(\boldsymbol{R}^n)$. Observe that $D_j\varphi$ has compact support. Hence $D_j\varphi\in L^r(\boldsymbol{R}^n)$ for every r with $1\leq r\leq\infty$. In particular, $D_j\varphi\in L^q(\boldsymbol{R}^n)$. Thus we get

$$|f(x)\delta_{j,h_k}\varphi(x)|\leq|f(x)D_j\varphi(x)|\in L^1(\boldsymbol{R}^n)$$

for every k, and

$$f(x)\delta_{j,h_k}\varphi(x)\to f(x)\frac{\partial\varphi}{\partial x_j}(x)$$

as $k\to\infty$. It now follows from the Lebesgue's dominated convergence theorem that

$$\int f(x)\delta_{j,h_k}\varphi(x)dx\to\int f(x)\frac{\partial\varphi}{\partial x_j}(x)dx \tag{6}$$

as $k\to\infty$. from (4), (5), and (6) we now get

$$\int g(x)\varphi(x)dx=-\int f(x)\frac{\partial\varphi}{\partial x_j}(x)dx$$

for every $\varphi\in C_c^\infty(\boldsymbol{R}^n)$, which indicates $\frac{\partial f}{\partial x_j}=g\in L^p(\boldsymbol{R}^n)$ and completes the proof for the case $1<p\leq\infty$.

Suppose next $p=1$. Considering $L^1(\boldsymbol{R}^n)$ as a subspace of the space of all finite Borel measures on $\boldsymbol{R}^n$, which is the dual space of $C_0(\boldsymbol{R}^n)$ consisting of all continuous functions vanishing at infinity, and applying the weak-compactness argument as above, we get a sequence $\{h_k\}$ with $h_k\to 0$ as $k\to\infty$ and a finite Borel measure μ on $\boldsymbol{R}^n$ with $\|\mu\|\leq\|D_jf\|_1$ such that for every $\varphi\in C_0(\boldsymbol{R}^n)$

$$\int\delta_{j,h_k}f(x)\varphi(x)dx\to\int\varphi(x)d\mu(x)$$

as $k\to\infty$. But, $\delta_{j,h_k}f$, $D_jf\in L^1(\boldsymbol{R}^n)$ and $|\delta_{j,h_k}f|\leq D_jf$ Hence it follows from Lemma 3 that μ is absolutely continuous, that is, there exists a function $g\in L^1(\boldsymbol{R}^n)$ such that $d\mu=g\,dx$ We thus obtain, by the same argument as above, $\frac{\partial f}{\partial x_j}=g\in L^1(\boldsymbol{R}^n)$ and $\left\|\frac{\partial f}{\partial x_j}\right\|_1=\|\mu\|\leq\|D_jf\|_1$, and finish the proof.

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