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GAUSSIAN MEASURES ON REARRANGEMENT INVARIANT FUNCTION SPACES ON [0, 1] AND ON SEPARABLE σ-COMPLETE BANACH LATTICES

Hi Ja Song

1. Introduction

In 1977 S. Chobanjan and V. Tarieladze[3] showed that the necessary and sufficient condition for a nonnegative symmetric bounded linear operator R from X^* into X, where X is a Banach space which has cotype p for some $p<\infty$ and an unconditional basis $\{x_k\}_{k=1}^{\infty}$, to be a covariance operator of a Gaussian measure on X is that the series $\sum_{k=1}^{\infty} \langle Rx_k^*, x_k^*\rangle^{\frac{1}{2}} x_k$ is convergent in X, where $\{x_k^*\}$ is the sequence of biorthogonal functionals associated with the basis $\{x_k\}$.

As we know, if $\{x_k\}$ is an unconditional basis of a Banach space X then we can always define on X an equivalent norm so that the unconditional constant becomes one. Further every Banach space with an unconditional basis $\{x_k\}$, whose unconditional constant is equal to one, is a Banach lattice when the order is defined by $\sum_{n=1}^{\infty} a_n x_n \geqq 0$ if and only if $a_n \geqq 0$ for all n.

The natural question is how to extend the above theorem (on the characterization of Gaussian measures) to a general Banach lattice, which is of cotype p for some $p<\infty$. Here, we describe the Gaussian measures on the following Banach lattices: (a) Rearrangement invariant function spaces on [0, 1] of cotype p for some $p<\infty$ and, (b) separable σ-complete Banach lattices of cotype p for some $p<\infty$.

Let $([0,1], \Sigma, \mu)$ be the Lebesgue measure space. Suppose that f is an integrable function on [0, 1] and that $\mathscr{B}$ is a σ-algebra of measurable sets in [0, 1]. There exists a unique, up to equality almost everywhere, $\mathscr{B}$-measurable integrable function $E_{\mathscr{B}} f$ so that

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$\int_\sigma E_{\mathscr{B}} f(\omega)\, d\mu = \int_\sigma f(\omega)\, d\mu$ for every $\mathscr{B}$-measurable set σ. $E_{\mathscr{B}} f$ is called the conditional expectation of f with respect to $\mathscr{B}$.

Let œ be the σ-algebra generated by a sequence $\{A_k\}_{k=1}^{\infty}$ of disjoint measurable sets in [0, 1]. Let X be a rearrangement invariant function space on [0, 1]. It follows from theorem 2. a. 4 of [10] that the conditional expectation $E_{œ}$ is a projection of norm one from X onto the subspace of X consisting of all the œ-measurable functions. It follows easily that if a nonnegative symmetric bounded linear operator R from X^* into X is a Gaussian covariance, then the nonnegative symmetric bounded linear operator $E_{œ} R E_{œ}^*$ from $(E_{œ}(X))^*$ into $E_{œ}(X)$ is a Gaussian covariance. Let $f_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_X}$ and $g_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_{X^*}}$ for all i. Since $\{f_i\}$ is a 1-unconditional basis of $E_{œ}(X)$ and $\{g_i\}$ is the sequence of biorthogonal functionals associated with the basis $\{f_i\}$, by the theorem of [3] the series $\sum_{i=1}^{\infty} \langle E_{œ} R E_{œ}^* g_i, g_i \rangle^{\frac{1}{2}} f_i$ converges in $E_{œ}(X)$. Moreover, we show that there exists a constant K such that $\|\sum_{i=1}^{\infty} \langle E_{œ} R E_{œ}^* g_i, g_i \rangle^{\frac{1}{2}} f_i\| \leq K$ for every σ-algebra œ generated by a sequence $\{A_k\}$ of disjoint measurable sets in [0, 1].

Next by using the definition of a rearrangement invariant function space X on [0, 1] we find σ-algebras œ generated by finite sequences $\{A_k\}$ of disjoint measurable sets in [0, 1] such that $E_{œ} A^*$ is not uniformly γ-summing when $A^* : H \to X$ is not γ-summing. By using the above result we find a sufficient condition for R to be a Gaussian covariance. Hence we prove the following result (Theorem 1) : Let X be a rearrangement invariant function space on [0, 1] which has cotype p for some $p < \infty$. A nonnegative symmetric bounded linear operator R from X^* into X is a Gaussian covariance if and only if there exists a constant K such that for every σ-algebra œ generated by a sequence $\{A_k\}_{k=1}^{\infty}$ of disjoint measurable sets in [0, 1], $\|\sum_{i=1}^{\infty} \langle E_{œ} R E_{œ}^* g_i, g_i \rangle^{\frac{1}{2}} f_i\| \leq K$, where $f_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_X}$, $g_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_{X^*}}$, $E_{œ}$ is the conditional expectation and $E_{œ} R$

E_∞^* is the induced map from $(E_\infty(X))^*$ to $E_\infty(X)$.

In the case of a separable σ-complete Banach lattice E, which is of cotype p for some $p<\infty$, we show the following result (Theorem 2) : A nonnegative symmetric bounded linear operator R from E^* into E is a Gaussian covariance if and only if there is a constant K such that for all finite disjoint sequences $\{g_i\}$ in E and $\{g_i^*\}$ in E^* with $\langle g_i^*, g_j\rangle=\delta_{ij}$, $\|\sum\langle QRQ^* g_i^*, g_i^*\rangle^{\frac{1}{2}}Q(g_i)\|\leq K$, where Q is the canonical map of E onto $E/[g_i^*]_\perp$ and $[g_i^*]_\perp=\{f\in E : \langle f, f^*\rangle=0$ for all $f^*\in[g_i^*]\}$.

In fact, the idea of the proof of Theorem 2 is essentially the same as that of Theorem 1. In proving Theorem 2 we use the following : Let $\{g_i\}\subset E$ and $\{g_i^*\}\subset E^*$ be finite disjoint sequences with $\langle g_i^*, g_j\rangle=\delta_{ij}$. Then $\{Q(g_i)\}$ is an unconditional basis for $E/[g_i^*]_\perp$ and $\{g_i^*\}$ is the sequence of biorthogonal functionals associated to the basis $\{Q(g_i)\}$.

2, Definitions and notation

Here are some of the definitions and notation we use.

The canonical Gaussian cylindrical measure γ_H on a Hilbert space H is the cylindrical measure with characteristic functional $\hat{\gamma}_H(h)=\exp\left\{-\frac{\|h\|^2}{2}\right\}$, $h\in H$.

A cylindrical measure μ on a Banach space X is called a Gaussian cylindrical measure if there exists a Hilbert space H and a continuous linear map T from H into X such that $\mu=\gamma_H\circ T^{-1}$.

Let X be a Banach space, X^* its dual. For any nonnegative symmetric bounded linear operator R from X^* into X there exists a Hilbert space H and a bounded linear operator A from X^* into H such that $R=A^*\circ A$. A is uniquely defined up to isometry (cf. [12]). Thus every nonnegative symmetric bounded linear operator R from X^* into X determines a cylindrical Gaussian measure $\gamma_H\circ(A^*)^{-1}$ with characteristic functional $\hat{\gamma}_H\circ(A^*)^{-1}(x^*)=\exp\{-\frac{1}{2}\langle Rx^*, x^*\rangle\}$, $x^*\in X^*$. If a cylindrical Gaussian measure $\gamma_H\circ(A^*)^{-1}$ admits extension to a tight Borel measure then R is called a Gaussian covariance.

A bounded linear operator T from a Hilbert space H into a Banach space X is called γ-Radonifying if $\gamma_H \circ T^{-1}$ admits extension to a tight Borel measure on the Borel field.

A bounded linear operator T from a Hilbert space H into a Banach space X is called γ-summing if there exists $C \geqq 0$ such that for any finite subset $\{h_i\}_{i=1}^{n} \subset H$, $\left(E \| \sum_{k=1}^{n} Th_k \gamma_k \|^2\right)^{\frac{1}{2}} \leqq C \sup_{\|h\|=1} \left\{\left(\sum_{k=1}^{n} |\langle h, h_k \rangle|^2\right)^{\frac{1}{2}}\right\}$, where $\{\gamma_k\}$ is a sequence of identically distributed independent standard Gaussian random variables. The infimum of such C is denoted by $\Pi_\gamma(T)$.

The sequence of Rademacher functions $\{\varepsilon_n(t)\}_{n=1}^{\infty}$ on $[0, 1]$ is defined by $\varepsilon_n(t) = \text{sign} \sin 2^n \pi t$ and is a sequence of independent identically distributed random variables taking the values ± 1 with probability $\frac{1}{2}$.

A Banach space X is of cotype p for some $p \geqq 2$ if there exists a constant C such that for any finite subset $\{x_i\}_{i=1}^{n} \subset X$, $\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}} \leqq C \left(E \| \sum_{k=1}^{n} x_k \varepsilon_k \|^p\right)^{\frac{1}{p}}$.

Let (Ω, Σ, μ) be a σ-finite measure space. Let X be a Banach space whose elements are (equivalence classes modulo equality almost everywhere) measurable functions on Ω. X is called a Köthe function space if the following conditions hold.

(1) For every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$, the characteristic function χ_σ of σ belongs to X.

(2) If $|f(\omega)| \leqq |g(\omega)|$ almost everywhere on Ω with f measurable and $g \in X$ then $f \in X$ and $\|f\| \leqq \|g\|$.

(3) If $f \in X$ then $f\chi_\sigma \in X$ and $\int |f(\omega)\chi_\sigma(\omega)| d\mu < \infty$ for every $\sigma \in \Sigma$ with $\mu(\sigma) < \infty$.

Let (Ω, Σ, μ) be a σ-finite measure space. Let X be a Banach space whose elements are measurable functions on Ω. X' is the space of all measurable functions g such that $\int | f(\omega) g(\omega) | d\mu < \infty$ for each $f \in X$. X' is a norming subspace of X^* if for every $f \in X$, $\|f\| = \sup \left\{ |\int f(\omega)\, g(\omega)\, d\mu| : g \in X', \|g\|_{X^*} = 1 \right\}$.

A map τ from a measure space (Ω, Σ, μ) into (Ω, Σ, μ) is called an automorphism of Ω if τ is one-to-one, τ and τ^{-1} are measurable and $\mu(\sigma)=\mu(\tau(\sigma))$ for every measurable subset σ of Ω.

Let $([0,1], \Sigma, \mu)$ be a Lebesgue measure space. A rearrangement invariant(r. i.) function space X on $[0,1]$ is a Kōthe function space X on $[0,1]$ such that

(1) If τ is an automorphism of $[0,1]$ into $[0,1]$ and f is a measurable function on $[0,1]$ then $f \in X$ if and only if $f \circ \tau^{-1} \in X$ and if this is the case then $\|f\| = \|f \circ \tau^{-1}\|$.

(2) X' is a norming subspace of X^*.

(3) $L_\infty([0,1]) \subset X \subset L_1([0,1])$ with norm-one inclusions.

Notation. Let X be a Banach space, M be a subspace of X and N be a subspace of X*.

(1) $N_\perp = \{x \in X : \langle x, x^* \rangle = 0 \text{ for all } x^* \in N\}$

(2) $M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \text{ for all } x \in M\}$

A Banach lattice X is said to be σ-complete if every order bounded sequence in X has a least upper bound. A Banach lattice X is said to be σ-order continuous if for every downward directed sequence $\{x_n\}$ in X with $\wedge x_n = 0$, $\lim_n \|x_n\| = 0$.

3. Results

The proof of Theorem 1 is obtained by means of a few lemmas and the known results from [3] and [4]. We begin with a few lemmas.

Lemma 1. *Let X be a separable Banach space. If a bounded linear operator T from a Hilbert space H into X is γ-Radonifying then* $\left(\int_X \|x\|^2 \, d\mu(x)\right)^{\frac{1}{2}} = \Pi_\gamma(T)$, *where $\mu = \gamma_H \circ T^{-1}$ is the (extension) Gaussian measure on X.*

Proof. Let $\{h_k\}_{k=1}^n$ be a finite sequence in H. Since for every $\varepsilon \geqq 0$ there exists a finite dimensional subspace F of X^* such that $\|Q_{F_\perp} u\|_{X/F_\perp} \leqq \|u\|_X \leqq (1+\varepsilon) \|Q_{F_\perp} u\|_{X/F_\perp}$ for all $u \in [Th_k]_{k=1}^n$, where $F_\perp = \{x \in X : \langle x^*, x \rangle = 0 \text{ for all } x^* \in F\}$ and $Q_{F_\perp}$ is the canonical map of

X onto $X/F_\perp$, we have that $\left(E\|\sum_{k=1}^{n} Q_{F\perp}Th_k\gamma_k\|^2\right)^{\frac{1}{2}}$
$\leqq\left(E\|\sum_{k=1}^{n} Th_k\gamma_k\|^2\right)^{\frac{1}{2}}\leqq(1+\varepsilon)\left(E\|\sum_{k=1}^{n} Q_{F\perp}Th_k\gamma_k\|^2\right)^{\frac{1}{2}}$. (1)

By (1) and the definition of a γ-summing operator $Q_{F\perp}T$, we get that $\left(E\|\sum_{k=1}^{n} Th_k\gamma_k\|^2\right)^{\frac{1}{2}}=\sup_{\substack{F\subset X^* \\ \dim F<\infty}}\left\{\left(E\|\sum_{k=1}^{n} Q_{F\perp}Th_k\gamma_k\|^2\right)^{\frac{1}{2}}\right\}$

$\leqq\sup_{\substack{F\subset X^* \\ \dim F<\infty}}\left\{\Pi_\gamma(Q_{F\perp}T)\sup_{\substack{\|h\|=1 \\ h\in H}}\left\{\left(\sum_{k=1}^{n}|\langle h,h_k\rangle|^2\right)^{\frac{1}{2}}\right\}\right\}$.

Hence we have $\Pi_\gamma(T)\leqq\sup_{\substack{F\subset X^* \\ \dim F<\infty}}\{\Pi_\gamma(Q_{F\perp}T)\}$ (2) from the definition of $\Pi_\gamma(T)$.

Since $Q_{F\perp}T: H\to X/F_\perp$, where $F\subset X^*$ and $\dim F<\infty$, is a finite rank operator, it follows from lemma 3 of [4] that $\Pi_\gamma(Q_{F\perp}T)$
$=\left(\int_H\|Q_{F\perp}Th\|^2 d\gamma_H(h)\right)^{\frac{1}{2}}$

$$\leqq\left(\int_H\|Th\|^2 d\gamma_H(h)\right)^{\frac{1}{2}}=\left(\int_X\|x\|^2 d_{\gamma_H\circ T^{-1}}(x)\right)^{\frac{1}{2}}.$$

The last equality is due to the fact that $\gamma_H\circ T^{-1}$ is a Gaussian measure. Together with (2), we have that

$$\Pi_\gamma(T)\leqq\left(\int_X\|x\|^2 d_{\gamma_H\circ T^{-1}}(x)\right)^{\frac{1}{2}}. \quad (3)$$

By the separability of X, there exists a countable subset $\{x_i\}_{i=1}^{\infty}$ of X such that $X=[x_i]_{i=1}^{\infty}$. For $1\leqq k\leqq n$, suppose that we have chosen a finite dimensional subspace F_k of X^* such that
$\|Q_{F_{k\perp}}u\|_{X/F_{k\perp}}\leqq\|u\|_X\leqq\left(1+\frac{1}{k}\right)\|Q_{F_{k\perp}}u\|_{X/F_{k\perp}}$ for all $u\in[x_i]_{i=1}^{k}$ and $F_1\subset F_2\subset\cdots\subset F_n$. Now we choose a finite dimensional subspace $\tilde{F}_{n+1}$ of X^* such that $\|u\|_X\leq\left(1+\frac{1}{n+1}\right)\|Q_{\tilde{F}_{n+1\perp}}u\|_{X/\tilde{F}_{n+1\perp}}$ for all $u\in[x_i]_{i=1}^{n+1}$.
If we let $F_{n+1}=F_n+\tilde{F}_{n+1}$ then $F_n\subset F_{n+1}$ and
$\|u\|_X\leqq\left(1+\frac{1}{n+1}\right)\|Q_{F_{n+1\perp}}u\|_{X/F_{n+1\perp}}$ for all $u\in[x_i]_{i=1}^{n+1}$

Therefore there exists an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of finite dimensional subspaces of X^* such that for each n, $\|Q_{F_{n\perp}}u\|_{X/F_{n\perp}}\leqq\|u\|_X\leqq\left(1+\frac{1}{n}\right)\|Q_{F_{n\perp}}u\|_{X/F_{n\perp}}$ for all $u\in[x_i]_{i=1}^{n}$.

Now by Fautou's lemma and the normed operator ideal property of the class of all γ-summing operators, we get that

$$\left(\int\|x\|^2\, d\gamma_H\circ T^{-1}(x)\right)^{\frac{1}{2}}=\left(\int_H \|Th\|^2\, d\gamma_H(h)\right)^{\frac{1}{2}}$$

$$=\left(\int\liminf_n \|Q_{F_{n\perp}}\, Th\|^2\, d\gamma_H(h)\right)^{\frac{1}{2}}\leqq\liminf_n\left(\int\|Q_{F_{n\perp}}\, Th\|^2\, d\gamma_H(h)\right)^{\frac{1}{2}}$$

$$=\liminf_n \{\Pi_\gamma(Q_{F_{n\perp}}\, T)\}\leqq\Pi_\gamma(T). \quad (4)$$

(3) and (4) conclude the proof.

It should be noted that Linde and Pietsch [8] have proved the analogous result for γ-summing operators. There, however, one gets that $\Pi_\gamma(T)=\left(\int_{X^{**}}\|\phi\|^2\, d\mu(\phi)\right)^{\frac{1}{2}}$, where $\mu=\gamma_H\circ(JT)^{-1}$ is the Gaussian measure on X^{**} and $J: X\to X^{**}$ is the canonical embedding of a space X into its bidual.

LEMMA 2. *Let X be a Banach space with a* 1*-unconditional basis $\{x_i\}_{i=1}^\infty$ and let $\{x_i^*\}_{i=1}^\infty$ be the sequence of biorthogonal functionals associated with the basis $\{x_i\}_{i=1}^\infty$, If $R: X^*\to X$ is a Gaussian covariance then $\|\sum_{k=1}^\infty\langle R x_k^*,\ x_k^*\rangle^{\frac{1}{2}}x_k\|_X\leq\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Pi_\gamma(A^*)$, where $A^*: H\to X$ is the operator in the representation $R=A^*\circ A$.*

Proof. Let x be a Gaussian random element in X with expectation zero, μ be the distribution of x and R be its covariance operator. Then for each k, $\langle x_k^*,\ x\rangle$ is a Gaussian random variable in $\boldsymbol{R}$ and so we have $\boldsymbol{E}\,|\langle x_k^*,\ x\rangle| =\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\langle R\, x_k^*,\ x_k^*\rangle^{\frac{1}{2}}$. By the 1-unconditionality of the basis $\{x_k\}$ and Hölder's inequality, we get that

$$\|\boldsymbol{E}\left(\sum_{k=1}^\infty|\langle x_k^*,\ x\rangle|x_k\right)\|\leqq \boldsymbol{E}\,\|\sum_{k=1}^\infty|\langle x_k^*,\ x\rangle|x_k\|=\boldsymbol{E}\,\|x\|\leq\left(\boldsymbol{E}\,\|x\|^2\right)^{\frac{1}{2}}.$$

Since X has a basis $\{x_i\}$, X is separable and since $R=A^*\circ A$ is a Gaussian, A^* is γ-Radonifying. Therefore, by Lemma 1 we have that

$$\|\sum_{k=1}^\infty\langle Rx_k^*,\ x_k^*\rangle^{\frac{1}{2}}x_k\|=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\ \|\boldsymbol{E}\left(\sum_{k=1}^\infty|\langle x_k^*,\ x\rangle|x_k\right)\|\leqq\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\boldsymbol{E}\,\|x\|^2\right)^{\frac{1}{2}}$$

$=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Pi_\gamma(A^*)$, which completes the proof.

LEMMA 3. *Let X be an r.i. function space on* [0, 1]. *If f is an element of X then for any $\varepsilon>0$ there exists an algebra* œ *whih is* *generated by a finite sequence of disjoint measurable sets in* [0, 1] *such that* $\|E_{œ} f-f\|\leqq\varepsilon$.

Proof. Let $A_k=[t\in[0,1]: k\varepsilon\leqq f(t)<(k+1)\varepsilon]$, $g_\varepsilon=\sum_{k=-\infty}^{\infty} k\varepsilon\chi_{A_k}$ and $g_\varepsilon{}^N=\sum_{k=-N}^{N} k\varepsilon\,\chi_{A_k}$. Then $\|g_\varepsilon-f\|_{L_\infty}\leq\varepsilon$ and $\|g_\varepsilon-g_\varepsilon{}^N-f\,\chi_{[t:|f(t)|\geq(N+1)\varepsilon]}\|_{L\infty}\leqq\varepsilon$. It follows from the definition of an r. i. function space X that $L_\infty([0,1])\subset X\subset L_1([0,1])$ with norm one inclusions and so we have that $\|g_\varepsilon-f\|_X\leqq\varepsilon$ and $\|g_\varepsilon-g_\varepsilon^N-f\,\chi_{[t:|f(t)|\geq(N+1)\varepsilon]}\|_X\leqq\varepsilon$. (1)

Let g be an element of X' such that $\|g\|_{X^*}=1$. Since $f\in L_1([0,1])$, we get that $|f|\chi_{[t:|f(t)|\geq\lambda]}|g|$ goes to 0 almost everywhere as $\lambda\to\infty$ and is dominated by $|f|\,|g|$. This fact and the fact that $|f|\,|g|\in L_1([0,1])$ allow us to use the dominated convergence theorem to conclude that $\lim_{\lambda\to\infty}\int_0^1|f(t)|\chi_{[t:|f(t)|\geq\lambda]}(t)\,|g(t)|\,dt=0$. Now X' is a norming subspace of X^* by the definition of an r. i. function space X. So

$$\|f\,\chi_{[t:|f(t)|\geq\lambda]}\|_X=\sup_{\substack{g\in X'\\ \|g\|_{X^*}=1}}\left\{\left|\int_0^1 f(t)\,\chi_{[t:|f(t)|\geq\lambda]}(t)g(t)\,dt\right|\right\}.$$

Hence for any $\varepsilon>0$, we have that $\|f\,\chi_{[t:|f(t)]\geq(N+1)\varepsilon]}\|_X\leqq\varepsilon$ for large N. (2)

From (1) and (2), for any $\varepsilon>0$ we get that $\|f-g_\varepsilon^N\|_X\leqq3\,\varepsilon$ for large N.

Now we take œ as the algebra generated by $\{A_k\}_{k=-N}^{N}$. Since the conditional expectation $E_{œ}$ is a projection of norm one by Theorem 2. a. 4 of[10], we have the following : $\|E_{œ} f-f\|_X=\|(E_{œ} f-g_\varepsilon^N)+(g_\varepsilon^N-f)\|_X\leqq\|E_{œ}(f-g_\varepsilon^N)\|_X+\|g_\varepsilon^N-f\|_X\leq2\|f-g_\varepsilon^N\|_X\leq6\varepsilon$.

This completes the proof of the lemma, because ε is an arbitrary positive number.

Lemma 4. *Let X be an r. i. function space on* [0, 1]. *If the σ-algebra* $œ_1$ *is contained in the σ-algebra* $œ_2$ *and if f is an element of X then* $\|f-E_{œ_2} f\|_X\leq2\,\|f-E_{œ_1} f\|_X$.

Proof. Let $g=f-E_{œ_1}f$. By the definition of the conditional expectation, we get that

$g-E_{œ_2}g=g-E_{œ_2}f+E_{œ_1}f=f-E_{œ_2}f$. Since $E_{œ_2}$ is a projection of norm one, it follows that

$$\| f-\boldsymbol{E}_{œ_2} f \| = \| g-\boldsymbol{E}_{œ_2} g \| \leqq 2\| g \| = 2\| f-\boldsymbol{E}_{œ_1} f \|.$$

We are now prepared to prove Theorem 1.

THEOREM 1. *Let X be a rearrangement invariant function space on [0, 1] which has cotype p for some $p<\infty$. A nonnegative symmetric bounded linear operator R from X^* into X is a Gaussian covariance if and only if there exists a constant K such that for every σ-algebra œ generated by a sequence $\{A_k\}_{k=1}^{\infty}$ of disjoint measurable sets in [0, 1],*

$$\| \sum_{i=1}^{\infty} \langle \boldsymbol{E}_{œ} R \boldsymbol{E}_{œ}^{*} g_i,\ g_i \rangle^{\frac{1}{2}} f_i \| \leqq K, \quad \textit{where } f_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_X},\quad g_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_{X^*}},$$

$\boldsymbol{E}_{œ}$ is the conditional expectation and $\boldsymbol{E}_{œ} R \boldsymbol{E}_{œ}^{*}$ is a map from $(\boldsymbol{E}_{œ}(X))^*$ to $\boldsymbol{E}_{œ}(X)$.

Proof. Necessity.

Let œ be a σ-algebra generated by the sequence $\{A_k\}_{k=1}^{\infty}$ of disjoint measurable sets in [0, 1]. Let $A^*: H \to X$ be the operator in the representation $R = A^* \circ A$. Suppose that R is a Gaussian covariance. Then by the definition of γ-Radonifying operators, $\gamma_H \circ (A^*)^{-1}$ extends to a tight Borel measure $\gamma_{H^\circ}(A^*)^{-1}$. That is, for every $\varepsilon > 0$ there exists a compact subset K of X so that $\gamma_{H^\circ}(A^*)^{-1}(K) > 1-\varepsilon$. It follows from Theorem 2. a. 4 of [10] that the conditional expectation $\boldsymbol{E}_{œ}$ is a projection of norm one from X onto the subspace of X consisting of all the œ-measurable functions. Hence $\boldsymbol{E}_{œ}(K)$ is a compact set and so the cylindrical Gaussian measure $\gamma_H \circ (\boldsymbol{E}_{œ} A^*)^{-1}$ has a tight extension to a Borel measure on $\boldsymbol{E}_{œ}(X)$. In other words, $\boldsymbol{E}_{œ} R \boldsymbol{E}_{œ}^{*}$ is a Gaussian covariance.

Now let $f_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_X}$ and $g_i = \frac{\chi_{A_i}}{\|\chi_{A_i}\|_{X^*}}$ for all i. Since $\{f_i\}$ is a sequence of mutually disjoint elements of a Banach lattice X (with respect to the pointwise order), we have $|\sum_{i=1}^{\infty} a_i f_i| = \sum_{i=1}^{\infty} |a_i| |f_i|$ for every sequence of scalars $\{a_i\}$ and hence

$\| \sum_{i=1}^{\infty} a_i f_i \| = \| |\sum_{i=1}^{\infty} a_i f_i| \| = \| \sum_{i=1}^{\infty} |a_i| |f_i| \| = \| \sum_{i=1}^{\infty} |a_i f_i| \|$ for every sequence of scalars $\{a_i\}$. Therefore $\{f_i\}$ is a 1-unconditional basis of $\boldsymbol{E}_{œ}(X)$

and $\{g_i\}$ is the sequence of the biorthogonal functionals associated with the basis $\{f_i\}$. By lemma 2 and the normed operator ideal property of the class of all γ-summing operators, we get that

$$\|\sum_{i=1}^{\infty}\langle E_{œ} R E_{œ}^{*} g_i,\ g_i\rangle^{\frac{1}{2}} f_i\| \leqq \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(E_{œ} A^*) \leqq \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(A^*).$$

If we write K for the constant $\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(A^*)$ then this proves the necessity.

Sufficiency.

Assume that R is not a Gaussian covariance. Then the operator $A^*: H \to X$ in the representation $R = A^* \circ A$ is not γ-Radonifying. Therefore it follows from Theorem 1 of [4] that A^* is not γ-summing, because X doesn't contain a subspace isomorphic to C_0. Since A^* is not γ-summing, by definition, for any constant $M>0$ there exists a finite orthonormal sequence $\{h_k\}_{k=1}^{n}$ in H so that $\left(E\|\sum_{i=1}^{n} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} > M$. By Lemma 3, for any $\varepsilon>0$ there exists an algebra $œ_i$ generated by a finite sequence $\{A_{i,j}\}_{j=1}^{N_i}$ of disjoint measurable sets in $[0,1]$ such that $\|E_{œ_i}(A^* h_i) - A^* h_i\|_X < \varepsilon$ for $i=1, 2, \cdots, n$. Let the sequence $\{A_i\}$ be the common refinement of all of the $A_{i,j}$'s, $i=1, 2, \cdots, n$, $j=1, \cdots, N_i$, and let $œ$ be the σ-algebra generated by $\{A_i\}$. Then $œ_i \subset œ$ for $i=1, 2, \cdots, n$ and hence by Lemma 4, we have $\|E_{œ}(A^* h_i) - A^* h_i\| \leqq 2\ \|E_{œ_i}(A^* h_i) - A^* h_i\|$ for $i=1, 2, \cdots, n$.

We get the following estimate by Minkowski's inequality.

$$\left(E\|\sum_{i=1}^{n} E_{œ} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} = \left(E\|\sum_{i=1}^{n} A^* h_i \gamma_i + \sum_{i=1}^{n}(E_{œ} A^* h_i - A^* h_i)\gamma_i\|^2\right)^{\frac{1}{2}}$$
$$\geqq \left(E\|\sum_{i=1}^{n} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} - \left(E\|\sum_{i=1}^{n}(E_{œ} A^* h_i - A^* h_i)\gamma_i\|^2\right)^{\frac{1}{2}}$$
$$\geqq \left(E\|\sum_{i=1}^{n} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} - \left(\sum_{i=1}^{n}\|E_{œ} A^* h_i - A^* h_i\|^2\right)^{\frac{1}{2}}\left(E\left(\sum_{i=1}^{n}|\gamma_i|^2\right)\right)^{\frac{1}{2}}$$
$$\geqq M - 2n\varepsilon.$$

Since ε is an arbitrary positive number, we can take ε as $\frac{M}{4n}$ and then $\left(E\|\sum_{i=1}^{n} E_{œ} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} \geqq \frac{M}{2}$ for any constant $M>0$. Hence $E_{œ} A^*$ is not γ-summing(*). Next we show that (*) is impossible and this contradiction proves the sufficiency. Since $E_{\alpha e}(X)$ is a subspace of X and X has cotype p for some $p<\infty$, $E_{\alpha e}(X)$ also has cotype p. By hypothesis, the series

$$\sum_{i=1}^{\infty} \langle E_{\alpha e} R E_{\alpha e}^{*} g_i,\ g_i\rangle^{\frac{1}{2}} f_i, \text{ where } f_i=\frac{\chi_{A_i}}{\|\chi_{A_i}\|_X} \text{ and } g_i=\frac{\chi_{A_i}}{\|\chi_{A_i}\|_{X^*}},$$

converges in $E_{\alpha e}(X)$. Therefore according to Theorem 2. 1. of [3], we have that $E_{\alpha e} R E_{\alpha e}^{*}$ is a Gaussian covariance. Thus $E_{\alpha e} A^{*}$ should be γ-summing. That is, (*) is impossible. This completes the proof of sufficiency.

we need the following lemma which is needed in proving the sufficiency of Theorem 2.

LEMMA 5. *Let E be a separable σ-complete Banach lattice, which is of cotype p for some $p<\infty$. If f is an element of E then for any $\varepsilon>0$ there exists a simple function $\tilde{f}$ such that $\|f-\tilde{f}\|\leqq\varepsilon$.*

Proof. Since E is a σ-complete Banach lattice with cotype p for some $p<\infty$, E is a σ-complete Banach lattice which doesn't contain a subspace isomorphic to C_0. Hence it follows from Proposition 1. a. 7 of [10] that E is σ-complete and σ-order continuous. Moreover, E has a weak unit because E is separable. Therefore, by Theorem 1. b. 14 of [10], there exist a probability space (Ω, Σ, μ), an ideal $\tilde{E}$ of $L_1(\Omega, \Sigma, \mu)$ and a lattice norm $\|\ \|_{\tilde{E}}$ on $\tilde{E}$ so that

(i) E is order isometric to $\tilde{E}$

(ii) $\tilde{E}$ is dense in $L_1(\Omega, \Sigma, \mu)$ and $L_\infty(\Omega, \Sigma, \mu)$ is dense in $\tilde{E}$

(iii) $\|f\|_1\leqq\|f\|_{\tilde{E}}\leqq 2\|f\|_\infty$ whenever $f\in L_\infty(\Omega, \Sigma, \mu)$.

Hence we can consider E as $L_\infty(\Omega, \Sigma, \mu)\subset E\subset L_1(\Omega, \Sigma, \mu)$ with dense inclusions.

Let $A_k=[\omega : k\varepsilon\leqq f(\omega)<(k+1)\varepsilon]$, $g_\varepsilon=\sum_{k=-\infty}^{\infty} k\varepsilon\,\chi_{A_k}$ and $g_\varepsilon^N=\sum_{k=-N}^{N} k\varepsilon\,\chi_{A_k}$. $E^{*}=E'$, since E is σ-order continuous. Just as above, then, for any $\varepsilon>0$ we get $\|f-g_\varepsilon^N\|\leq 5\,\varepsilon$ for large N. If we write $\tilde{f}$ for a simple function $g_\varepsilon^N=\sum_{k=-N}^{N} k\varepsilon\,\chi_{A_k}$ then this completes the proof of the lemma, because ε is an arbitrary positive number.

Now we prove Theorem 2. The proof of Theorem 2 is nearly identical to the proof of Theorem 1.

THEOREM 2. *Let E be a separable σ-complete Banach lattice, which is of cotype p for some $p<\infty$. A nonnegative symmetric bounded linear operator R from E^* into E is a Gaussian covariance if and only if there is a constant K such that for all finite disjoint sequences $\{g_i\}$ in E and $\{g_i^*\}$ in E^* with $\langle g_i^*, g_j\rangle=\delta_{ij}$, $\|\sum\langle QRQ^*g_i^*, g_i^*\rangle^{\frac{1}{2}} Q(g_i)\|\leq K$, where Q is the canonical map of E onto $E/[g_i^*]_\perp$.*

Proof. Necessity

Let $\{g_i\}\subset E$ and $\{g_i^*\}\subset E^*$ be finite disjoint sequences with $\langle g_i^*, g_j\rangle=\delta_{ij}$. Suppose that R is a Gaussian covariance. Let $A^*: H\to E$ be the operator in the representation $R=A^*\circ A$. Then, by definition, the cylindrical Gaussian measure $\gamma_H\circ(A^*)^{-1}$ extends to a tight Borel measure $\gamma_H\circ(A^*)^{-1}$ as above. Since Q is a continuous map of E onto $E/[g_i^*]_\perp$, $Q(K)$ is a compact set and hence the cylindrical Gaussian measure $\gamma_H\circ(QA^*)^{-1}$ has a tight extension to a Borel measure on $E/[g_i^*]_\perp$. In other words, QRQ^* is a Gaussian covariance.

Now $\langle Q(g_i), g_j^*\rangle=\delta_{ij}$ and since $[g_i^*]$ is a finite dimensional subspace of E^*, $(E/[g_i^*]_\perp)^*=[g_i^*]_\perp^\perp=\overline{[g_i^*]}^{w^*}=[g_i^*]$. Since $\{g_i^*\}$ is an unconditional basis for $[g_i^*]$ and has coefficient functionals $\{Q(g_i)\}\subset[g_i^*]^*$, $\{Q(g_i)\}$ is an unconditional basis for $[Q(g_i)]$. But dim $(E/[g_i^*]_\perp)=\dim[g_i^*]^*=\dim[Q(g_i)]$ and so $\{Q(g_i)\}$ is an unconditional basis for $E/[g_i^*]_\perp$. By Lemma 2 and the normed operator ideal property of the class of all γ-summing operators, we get that

$$\|\sum\langle QRQ^*g_i^*, g_i^*\rangle^{\frac{1}{2}} Q(g_i)\|\leq\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(QA^*)\leq\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(A^*).$$

If we write K for the constant $\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Pi_\gamma(A^*)$ then this proves the necessity.

Sufficiency

As we mentioned in the proof of Lemma 5, since E is a σ-complete and σ-order continuous Banach lattice which has a weak unit, by Theorem 1.b.14 of [10] we can consider E as $L_\infty(\Omega, \Sigma, \mu)\subset E\subset L_1(\Omega, \Sigma, \mu)$ with dense inclusions.

Assume that R is not a Gaussian covariance. Then the operator $A^*: H \to E$ in the representation $R = A^* \circ A$ is not γ-Radonifying. Therefore it follows from theorem 1 of [4] that A^* is not γ-summing, because E doesn't contain a subspace isomorphic to C_0. Since A^* is not γ-summing, by definition, for any constant $M > 0$ there exists a finite orthonormal sequence $\{h_k\}_{k=1}^n$ in H so that $\left(E \| \sum_{i=1}^n A^* h_i \gamma_i \|^2\right)^{\frac{1}{2}} > M$. By lemma 5, for any $\varepsilon > 0$ there exists a simple function $\sum_{l=1}^{N_i} \alpha_{i,l}\, \chi_{A_{i,l}}$ such that $\| A^* h_i - \sum_{l=1}^{N_i} \alpha_{i,l}\, \chi_{A_{i,l}} \| < \varepsilon$ for $i = 1, 2, \cdots, n$. Let the sequence $\{A_i\}$ be the common refinement of $\{A_{i,l}\}_{i=1, l=1}^{n, N_i}$. Then a simple function $\sum_{l=1}^{N_i} \alpha_{i,l}\, \chi_{A_{i,l}}$ can be written as $\sum_{j=1}^{M_i} \beta_{i,j}\, \chi_{A_j}$ and for any $\varepsilon > 0$, we also have $\| \sum_{j=1}^{M_i} \beta_{i,j} \chi_{A_j} - A^* h_i \| < \varepsilon$ for $i = 1, 2, \cdots n$. Now we define a map $\tilde{A}^*$ from $[h_i]_{i=1}^n$ into E by $\tilde{A}^* h_i = \sum_{j=1}^{M_i} \beta_{i,j} \chi_{A_j}$, $i = 1, 2, \cdots, n$. Again, as above by Minkowski's inequality, we get $\left(E \| \sum_{i=1}^n \tilde{A}^* h_i \gamma_i \|^2\right)^{\frac{1}{2}} \geqq M - n\varepsilon$. Since ε is an arbitrary positive number, we can take ε as $\dfrac{M}{4n}$ and then we get $\left(E \| \sum_{i=1}^n \tilde{A}^* h_i \gamma_i \|^2\right)^{\frac{1}{2}} \geqq \frac{3}{4} M$ for any constant $M > 0$.

Now we choose a finite dimensional subspace F of E^* such that for every $\varepsilon > 0$, $\| g \| \leqq (1+\varepsilon) \| Q_{F_\perp} g \|$ for all $g \in [\chi_{A_j}]$, where $Q_{F_\perp}$ is the cannoical map of E onto $E/F_\perp$. We can assume without loss of generality that F is a subspace of $[\chi_{C_k}]$, where $\{C_k\}$ is a finite sequence of disjoint measurable sets. Since the space $[\chi_{C_k}]_\perp$ is a subspace of the space $F_\perp$, we have $\| Q_{F_\perp} g \|_{E/F_\perp} \leqq \| Q_{\{\chi_{C_k}\}_\perp} g \|_{E/\{\chi_{C_k}\}_\perp}$ and hence for every $\varepsilon > 0$, $\| g \| \leqq (1+\varepsilon) \| Q_{\{\chi_{C_k}\}_\perp} g \|$ for all $g \in [\chi_{A_j}]$. Since $\sum_{i=1}^n \tilde{A}^* h_i \gamma_i$ is an element of $[\chi_{A_j}]$, for every $\varepsilon > 0$ we have

$$\| Q_{\{\chi_{C_k}\}_\perp} \left(\sum_{i=1}^n \tilde{A}^* h_i \gamma_i \right) \| \geqq \left(\frac{1}{1+\varepsilon} \right) \| \sum_{i=1}^n \tilde{A}^* h_i \gamma_i \| \text{ and so}$$

$$\left(E \| \sum_{i=1}^n Q_{\{\chi_{C_k}\}_\perp} \tilde{A}^* h_i \gamma_i \|^2 \right)^{\frac{1}{2}} \geqq \left(\frac{1}{1+\varepsilon} \right) \left(E \| \sum_{i=1}^n \tilde{A}^* h_i \gamma_i \|^2 \right)^{\frac{1}{2}} \geqq \left(\frac{1}{1+\varepsilon} \right) \cdot \frac{3}{4} M$$

for any constant $M>0$. Since $\|Q_{[\chi_{c_k}]_\perp}\tilde{A}^* h_i - Q_{[\chi_{c_k}]_\perp} A^* h_i\| < \frac{M}{4n}$ for $i=1,2,\cdots n$, by Minkowski's inequality we get that $\left(E\|\sum_{i=1}^{n} Q_{[\chi_{c_k}]_\perp} A^* h_i \gamma_i\|^2\right)^{\frac{1}{2}} \geqq \left(\frac{1}{1+\varepsilon}\right)\frac{3}{4}M - \frac{M}{4}$ for any constant $M>0$. Hence $Q_{[\chi_{c_k}]_\perp} A^*$ is not γ-summing(*). Next we show that(*) is impossible and this contradiction proves the sufficiency. Let $\{g_k\}$ be the sequence of elements of E such that $\langle g_k, \chi_{c_j}\rangle = \delta_{kj}$. Then $\langle Q_{[\chi_{c_k}]_\perp}(g_k), \chi_{c_j}\rangle = \delta_{kj}$ and since $[\chi_{c_j}]$ is a finite dimensional subspace of E^*, we have $(E/[\chi_{c_j}]_\perp)^* = [\chi_{c_j}]_\perp^\perp = \overline{[\chi_{c_j}]}^{w^*} = [\chi_{c_j}]$. Since $\{\chi_{c_k}\}$ is an unconditional basis for $[\chi_{c_k}]$ and has coefficient functionals $\{Q(g_k)\} \subset [\chi_{c_k}]^*$, $\{Q(g_k)\}$ is an unconditional basis for $[Q(g_k)]$. But $\dim(E/[\chi_{c_j}]_\perp) = \dim[\chi_{c_j}]^* = \dim[Q(g_j)]$ and so $\{Q(g_k)\}$ is an unconditional basis for $E/[\chi_{c_j}]_\perp$. By hypothesis, the series $\sum\langle QRQ^*\chi_{c_k}, \chi_{c_k}\rangle^{\frac{1}{2}} Q(g_k)$ converges in $E/[\chi_{c_j}]_\perp$. Therefore according to Theorem 2.1 of [3], we have that QRQ^* is a Gaussian covariance. Thus QA^* should be γ-summing. That is, (*) is impossible. This completes the proof of sufficiency.

REMARK. We wanted Theorem 2 as follows : Let E be a separable σ-complete Banach lattice, which is of cotype p for some $p<\infty$. A nonnegative symmetric bounded linear operator R from E^* into E is a Gaussian covariance if and only if there exists a constant K such that for all disjoint sequences $\{g_i\}$ in E and $\{g_i^*\}$ in E^* with $\langle g_i^*, g_j\rangle = \delta_{ij}$, $\|\sum_{i=1}^{\infty}\langle QRQ^* g_i^*, g_i^*\rangle^{\frac{1}{2}} g_i\| \leqq K$. But by finding the following example we know that the above statement is false.

Let us take the separable σ-complete Banach lattice E as $l_2 \oplus l_1$. Since $l_2 \oplus l_1$ is a direct sum of a cotype 2 space and a cotype 2 space, $l_2 \oplus l_1$ is of cotype 2. Let $y_j = (e_j, \delta_j)$, where $\{e_j\}$ is the unit vector basis of l_2 and $\{\delta_j\}$ is the unit vector basis of l_1, and $y_j^* = e_j^*$. Now define an operator $A: l_2 \to E$ by $A(e_i) = \beta_i e_i$ with $\{\beta_i\} \in l_2 - l_1$. Then

$$\|\sum_{j=1}^{\infty}\langle AA^* y_j^*, y_j^*\rangle^{\frac{1}{2}} y_j\|_E = \|\sum_{j=1}^{\infty}\|A^* y_j^*\| y_j\|_E = \|\sum_{j=1}^{\infty}\beta_j y_j\|_E$$

$=\|\sum_{j=1}^{\infty}\beta_j e_j\|_{l_2}+\|\sum_{j=1}^{\infty}\beta_j \delta_j\|_{l_1}=\infty$. That is, the series $\sum_{j=1}^{\infty}\langle AA^* y_j^*, y_j^*\rangle^{\frac{1}{2}}$ y_j does not converge.

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Dongguk University
Seoul 100—715, Korea