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ON KAUFFMAN POLYNOMIALS OF LINKS

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§1. Introduction

L. Kauffman concocted a polynomial invariant of links last year [K]. For an unoriented link K, a polynomial $L_K(\alpha, z)$ in commuting variables α, z and with integer coefficients is defined through axioms:

1. Regularly isotopic links have the same polynomial;
2. $L_O=1$;
3. $L_{K_+}+L_{K_-}=z(L_{K_0}+L_{K_\infty})$;
4. Removing a positive (or negative) curl as in Fig 1.2 multiplies the polynomial by α (or by α^{-1}).

Axioms need to be made clear. Links are regularly isotopic if their diagrams become identical through 2-dimensional isotopies and Reide-

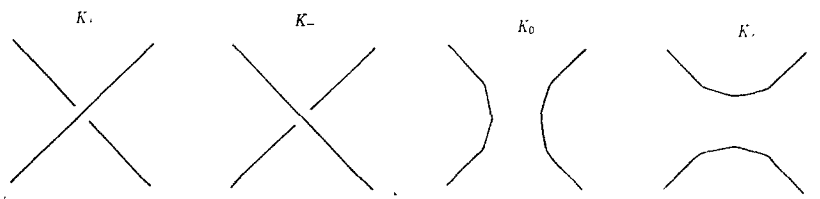

Fig. 1.1

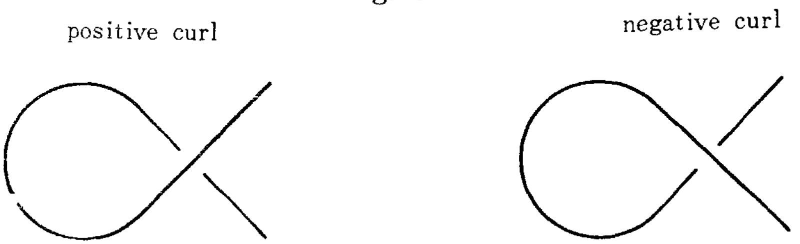

Fig. 1.2

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meister moves of types II and III. O denotes the trivial knot. K_+, K_-, K_0, and K_∞ are diagrams of four links that are exactly the same except near one crossing where they are as Fig. 1.1.

For an oriented link K, an integer $w(K)$, called the writhe number of K, is define by taking the sum of all crossing signs in the link diagram K. Then the Kauffman polynomial $F_K(\alpha, z)$ of an oriented link K is defined by

$$F_K(\alpha, z) = \alpha^{w(K)} L_K(\alpha, z).$$

Earlier Lickorish and Millet defined a polynomial $Q_K(z)$, known as Q–polynomial for an unoriented link K. The simple formula $Q_K(z) = F_K(1, z)$ relates the two polynomials.

In [4], it was proved that the Jones polynomial $V_K(t)$ is a special case of $F_K(\alpha, z)$, that is, $V_K(t) = F_K(-t^{-3/4}, t^{-1/4} + t^{1/4})$. Since Jones polynomial turned out to be useful in many aspects [6], Kauffman polynomial needs to be studied carefully.

The primary purpose of this paper is to develop the preskein theory for the Kauffman polynomials which has extensively been done for the HOMFLY polynomials in [5]. A typical use of this theory shows that the Kauffman polynomial is an invariant of mutation. However it is not an invariant of the skein equivalence. Therefore it can be utilized to detect links that are skein equivalent but not mutations each other. As another application of the theory, a general formula of computing the polynomials of pretzel links is obtained.

Before ending this section, few properties of the Kauffman polynomial is observed.

LEMMA 1.1. *Let K be an oriented link and K' the same link with the reversed orientation on each component. Then $F_K = F_{K'}$.*

Proof. Clearly $w(K) = w(K')$.

LEMMA 1.2. *Let K be an oriented link and K'' the same link with the reversed orientation on the i–th component. Let l_i be the sum of the linking numbers of the i–th component of K with all other components. Then*

$$F_K = \alpha^{4l_i} F_{K''}.$$

Proof. Let K_j be the j–th component of $K, j=1, \ldots, m$ and $l_{ik}(K)$

the linking number of K_j and K_k. Then $w(K)=\sum_{j\neq k} L_{jk}(K)+\sum_j w(K_j)$ But $l_{ik}(K)=-l_{ik}(K'')$ and $l_i=\sum_k l_{ik}$. Thus $w(K'')=w(K)-4l_i$.

LEMMA 1.3. *Let K be an (oriented) link and $\bar{K}$ its mirror image. Then*

$$L_{\bar{K}}(\alpha, z)=L_K(\alpha^{-1}, z),$$
$$F_K(\alpha, z)=F_K(\alpha^{-1}, z).$$

Proof. This follows from the fact that all crossings are reversed, hence that Axiom 3 is unaffected and the contributions via Axiom 4 are all reversed. Similarly $w(\bar{K})=-w(K)$

2. Regular preskein theory

In this section, we mainly concern the polynomial L, hence all links are assumed to be unoriented unless said otherwise. A *room* R is a 2-disk (possibly punctured) on the boundary of which a finite set of points is given. An *inhabitant* of R is a properly embedded tangle diagram in R, which meets the boundary precisely in the given set of points. The *regular preskein* of R is the set of regular isotopy classes, keeping the boundary fixed during isotopies, of all inhabitants of R. Two useful rooms that are not punctured are a *prison* with the empty subset on its boundary and a *quad* with four points on its boundary.

For a room R, let $M(R)$ be the free module over $\boldsymbol{Z}[\alpha^{\pm 1}, z^{\pm 1}]$ generated by the regular preskeins of R. Let $L(R)$ be the quotient of $M(R)$ by the submodule generated by all elements of the forms

$$s_+ + s_- - z(s_0 + s_\infty),\quad s_{c+} - \alpha s,\quad \text{and}\quad s_{c-} - \alpha^{-1} s$$

where s_+, s_-, s_0, and s_∞ are elements of the regular preskein that have representative inhabitants identical except near a point where they are ×, ×,) (, and ⌣⌢, and s_{c+} (or s_{c-}) become s by removing a positive (or negative) curl. The first example to consider is the prison P. As any link diagram can be reduced to the unknot O by regular isotopies, changing crossing and removing curls. Hence $L(P)$ is generated by O. Since the polynomial L is well-defined, any generator K of $M(R)$ is uniquely written as $L_K O$. Any inhabitant in a quad Q can be reduced to either $v_0, v_1,$ or v_{-1} as show in Fig. 2.1. Hence $L(Q)$ is generated by $v_0, v_1,$ and v_{-1}.

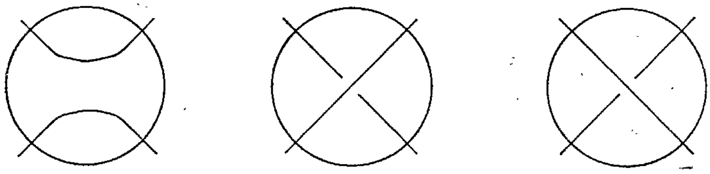


Fig. 2.1

A *house* H is a punctured room together with a specific inhabitant, the *wiring*, and specified components $R_1, R_2, \ldots, R_n$ (called the rooms of H) in the complement of H, where the point on ∂R_i are the points of $\partial R_i \cap$ (wiring). Let $\tilde{H}$ be the room H with its rooms $R_1, R_2, \ldots, R_n$ filled in.

PROPOSITION 2.1. *A house H with rooms $R_1, R_2, \ldots, R_n$ defines a multilinear map*

$$L(R_1)\times L(R_2)\times\cdots\times L(R_n) \longrightarrow L(\tilde{H}).$$

Proof [5]. Insertion of an inhabitant into each room produces an inhabitant of $\tilde{H}$. This function on generators extends by multilinearity to a multilinear function $M(R_1)\times M(R_2)\times\cdots\times M(R_n) \longrightarrow M(\tilde{H})$ which passes to quotients to give immediately the required function.

LEMMA 2.2. *If K_1 and K_2 are two (oriented) links, separated by a 2-sphere, then*

$$L_{K_1\cup K_2}=(\alpha z^{-1}+\alpha^{-1}z^{-1}-1)\,L_{K_1}L_{K_2},$$
$$F_{K_1\cup K_2}=(\alpha z^{-1}+\alpha^{-1}z^{-1}-1)\,F_{K_1}F_{K_2},$$
$$L_{K_1\# K_2}=L_{K_1}L_{K_2},$$
$$F_{K_1\# K_2}=F_{K_1}F_{K_2}.$$

where $K_1\cup K_2$ and $K_1\# K_2$ denote the distant union and any connected sum of K_1 and K_2, respectively.

Proof. Can be done by either a direct use of Axiom 4 (see [3]) or a typical use of the preskein theory we have developed (see [5]).

Let K_1 and K_2 be link diagrams, then K_2 is a *mutation* of K_1 (and vice versa), if K_2 can be obtained from K_1 by the following process:

i) remove from K_1 an inhabitant T of a copy of a quad Q;

ii) rotate T through angle π about the central axis (perpendicular to the plane of diagram) or about the E–W or the N–S axis (see Fig. 2.2);

iii) place this new inhabitant in Q.

In fact, this operation can also be performed on oriented liks.

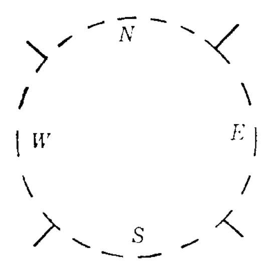


Fig. 2.2

Fig. 2.3

In Fig. 2.3, the Kinoshita–Terasaka knot on the left is a mutation of the Conway knot on the right, but they have different genus (see [G]).

Proposition 2.3. *If K_1 and K_2 are (oriented) links and K_1 is a mutation of K_2, then $L_{K_1} = L_{K_2}$ and $F_{K_1} = F_{K_2}$.*

Proof. Let ρ be one of the three involutions on the set of inhabitants T of Q described in the above definition. Then ρ induces a linear map from $L(Q)$ to $L(Q)$. Since ρ fixes the generators v_0 and $v_{\pm 1}$, ρ must be the identity on $L(Q)$. Let P be the house with Q filled in to give K_1 or K_2 depending on inhabitant T or ρT. Then the linear map $L(Q) \longrightarrow L(P)$ sends T to $L_{K_1}O$, and ρT to $L_{K_2}O$. Therefore $L_{K_1} = L_{K_2}$. Clearly the mutation operation does not change the writhe numbers and hence $F_{K_1} = F_{K_2}$

When he first invented the algorithmic method of computing the

Alexander polynomial, Conway saw the usefulness of the following equivalence among links [1]. *Skein equivalence* is the smallest equivalence relation $\sim$ on the set of all oriented links such that

i) if K and K are ambient isotopic then $K \sim K$;

ii) if K_+, K_-, and K_0 are three oriented links identical except near one point where they are , , and  respectively and K_+', K_-', and K_0 is another such triple then

a) $K_+ \sim K_+'$ and $K_0 \sim K_0'$ imply $K_- \sim K_-'$ and

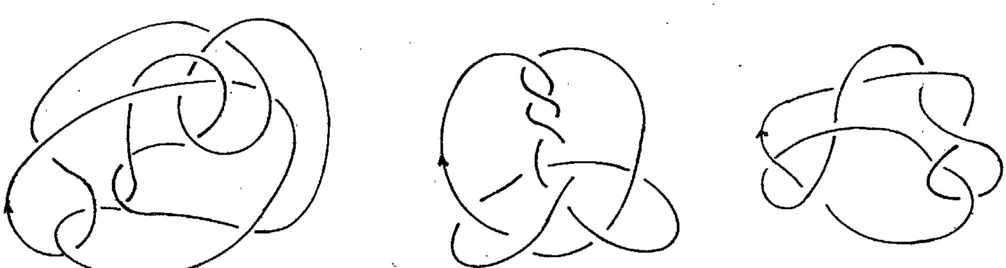
b) $K_- \sim K_-'$ and $K_0 \sim K_0'$ imply $K_+ \sim K_+'$.

Skein equivalence is by its definition the weakest equivalence relation of oriented link that make the HOMFLY polynomial well-defined [5].

LEMMA 2.4. *If K_1 and K_2 are oriented links and K_1 is a mutation of K_2, then K_1 is skein equivalent to K_2.*

Proof. Can be done by the induction on the complexity of the inhabitant rotated where the complexity is the lexicographic order of pairs that consist of the number of crossings and the number of crossing changes necessary to create an ascending tangle in the quad (see [5]).

Except iterated mutations, it is hard to generate skein equivalent knots. On a computer, one can generate links and compute their polynomials. If two links have the same HOMFLY polynomial, it is very likely that they are skein equivalent. On the other hand if the two links have the distinct Kauffman polynomial, they are not related



13_{6714} 10_{129} 8_8

Fig. 2.4

by mutations. As an example of this phenomenon, $(13_{6714}, 10_{129}, O^2)$ and $(8_8, 10_{129}, O^2)$ are the (K_+, K_-, K_0) triples (see Fig. 2.4) where O^2 is the unlink of two components. Thus 8_8 and 13_{6714} are skein equivalent.

Proposition 2.5. *$F_{13_{6714}} \neq F_{8_8}$. Therefore the knot 13_{6714} is not a mutation of the knot 8_8.*

Proof. Since Kauffman gave the table of his polynomials for all knots with less than 9-crossings, we reduce the number of crossings down to 9. The relevant $(K_+, K_-, K_0, K_\infty)$ quadruples are (see Fig. 2.5):

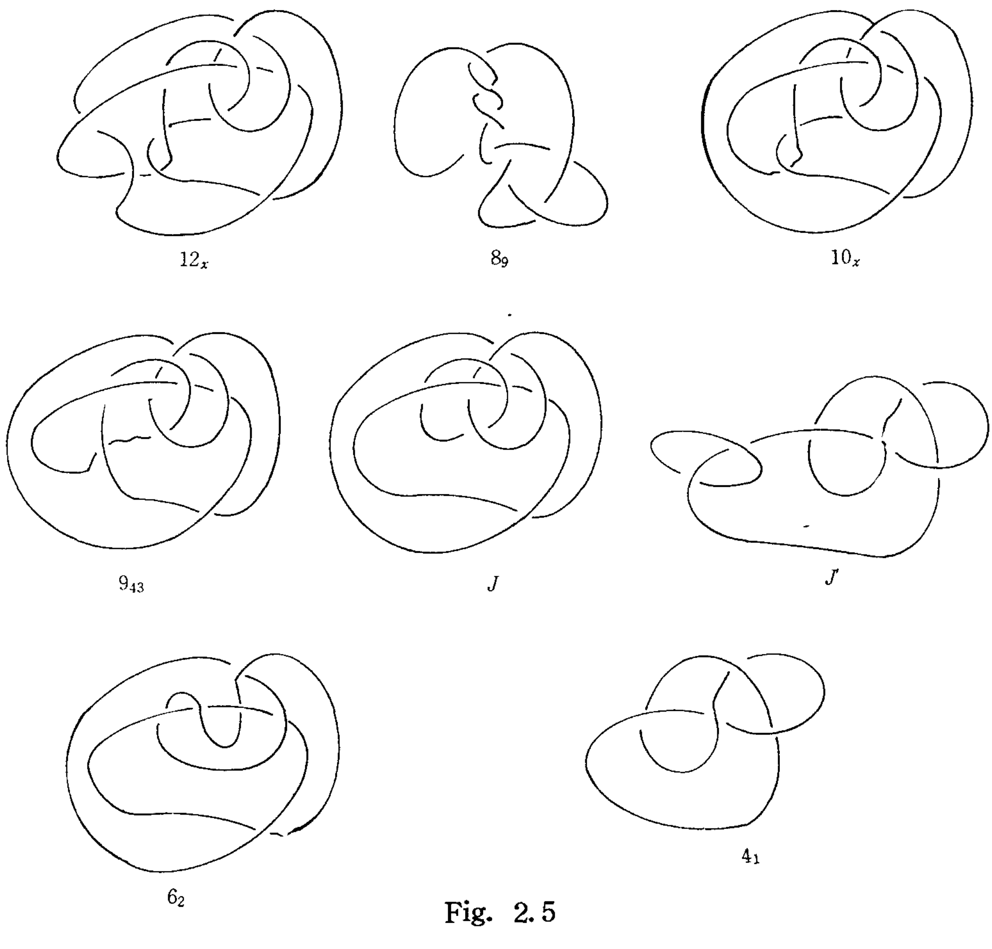


Fig. 2.5

$(13_{6714}, \alpha^{-1}10_{129}, O^2, 12_x)$, $(10_{129}, 8_8, \alpha O^2, \alpha 8_9)$,
$(12_x, 10_x, \alpha^{-1}O^2, \alpha^{-1}10_{129})$, $(10_x, \alpha^{-4}O, \alpha J, 9_{43})$,
$(J, J', \alpha^{-3}O, \alpha 6_2)$, $(J', O \cup 4_1, \alpha 4_1, \alpha^{-1}4_1)$

where $\alpha^n K$ denotes the link diagram K with $|n|$ removable $+$ or $-$ curls depending on the sign of n. We give the highest degree term in z to save space.

$$F_{12_{6714}} = (\alpha^{-2}+1)z^{10}+\cdots,$$
$$F_{8_8} = (\alpha^{-3}+\alpha^{-1})z^7+\cdots,$$

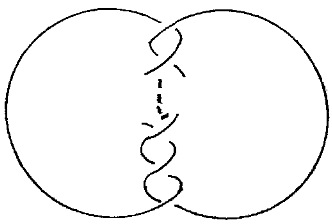
This proposition shows that the Kauffman polynomial is not an invariant under skein equivalence.

3. Computations

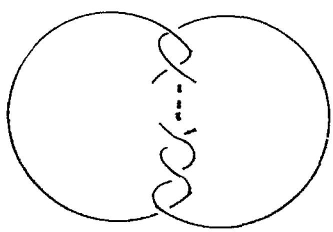
Proposition 3.1. *Let X_r be the polynomial L of the r–crossing link in Fig.* 3.1. *Then*

$$X_r = (1,0,0,)\begin{pmatrix} \alpha^{-1}+z & -1-\alpha^{-1}z & \alpha^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{r-2}\begin{pmatrix} X_2 \\ X_1 \\ X_0 \end{pmatrix} \text{ for } r \geq 3,$$

$$X_2 = (\alpha+\alpha^{-1})(z-z^{-1})+1,$$
$$X_1 = \alpha,$$
$$X_0 = (\alpha+\alpha^{-1})z^{-1}-1,$$
$$X_r(\alpha, z) = X_{-r}(\alpha^{-1}, z) \text{ for } r<0.$$



$r>0$



$r<0$

Fig. 3.1

Proof. Assume $r \geq 3$. Applying Axiom 3 to one of crossings, we have a recursive formula $X_r + X_{r-2} = z(X_{r-1} - \alpha^{-(r-1)})$. By eliminating the constant term, we have

$$X_r = (\alpha^{-1}+z)X_{r-1} - (1+\alpha^{-1}z)X_{r-2} + \alpha^{-1}X_{r-3}.$$

By solving this recursion, we have the result. Note that X_2 is the Hopf link, X_1 is of the trivial knot with a positive curl, X_0 is of the trivial link of twocomponent and X_{-r} is the mirror image of X_r.

For integers $a_1, a_2, \dots, a_n$ the pretzel link $[a_1, a_2 \dots, a_n]$ is the house of Fig 3.2 with n rooms, each of which is a quad with the inhabitant T_{a_i} of Fig. 3.3.

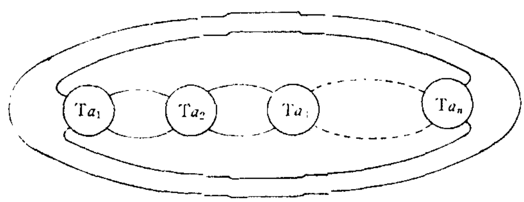


Fig. 3.2

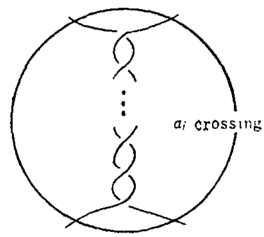


Fig. 3.3

PROPOSITION 3.2. *Let K be the pretzel link* $[a_1, a_2, \dots, a_n]$. *Then*

$$L_k = \sum_{\delta} X_{a_1}{}^{\delta(1)} X_{a_2}{}^{\delta(2)} \cdots X_a{}^{\delta(n)} X_{\Sigma(\delta)}$$

where the summation is over the 3^n *functions*

$$\delta : \{1, 2, \dots, n\} \to \{0, 1, -1\},$$

and $\sum(\delta) = -\sum_{i=1}^{n} \delta(i)$, *and finally*

$(X_a{}^0, X_a{}^1, X_a{}^{-1}) =$

$$(1, 0, 0) \begin{pmatrix} \alpha^{-1}+z & -1-\alpha^{-1}z & \alpha^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{a-2} \begin{pmatrix} 1-\alpha^{-1}z & z-z^{-1} & -z^{-1} \\ 0 & 1 & 0 \\ -1 & z^{-1} & z^{-1} \end{pmatrix} \textit{for } a \geq 2,$$

$(X_1{}^0, X_1{}^1, X_1{}^{-1}) =$

$(X_0{}^0, X_0{}^1, X_0{}^{-1}) = (-1, z^{-1}, z^{-1})$

$(X_a{}^0, X_a{}^1, X_a{}^{-1})(\alpha, z) = (X_{-a}{}^0, X_{-a}{}^1, X_{-a}{}^{-1})\ (\alpha^{-1}, z)$ *for* $a > 0$.

Proof. Let Q be a quad. Let T_a be the inhabitant in Q as in Fig 3.3. Recall the generators v_0, v_1, and v_{-1} of $L(Q)$ from Fig. 2.1. Using the recursive formula:

$$T_a + T_{a-2} = z(T_{a-1} - \alpha^{-(r-1)} v_0),$$

we have $T_a = (1, 0, 0) \begin{pmatrix} \alpha^{-1}+z & -1-\alpha^{-1}z & \alpha^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{a-2} \begin{pmatrix} T_2 \\ v_1 \\ T_0 \end{pmatrix}$ for $a \geq 2$. But

$T_2+T_0=z(v_1-\alpha^{-1}v_0)$ and $T_0=z^{-1}(v_1+v_{-1})-v_0$, hence

$$\begin{pmatrix} T_2 \\ v_1 \\ T_0 \end{pmatrix} = \begin{pmatrix} 1-\alpha^{-1}z & z-z^{-1} & -z^{-1} \\ 0 & 1 & 0 \\ -1 & z^{-1} & z^{-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_{-1} \end{pmatrix}$$

Thus $$T_a=X_a{}^0v_0+X_a{}^1v_1+X_a{}^{-1}v_{-1} \text{ for } a\geq 2.$$

The house P with n rooms of Fig. 3.2 defines an multilinear map

$$\varphi : L(Q)\times L(Q)\times\cdots\times L(Q) \longrightarrow L(P).$$

Then,

$L_KO=\varphi(T_{a_1}, T_{a_2}, \cdots T_{a_n})$

$=\varphi((X_{a_1}{}^0v_0+X_{a_1}{}^1v_1+X_{a_1}{}^{-1}v_{-1}, X_{a_2}{}^0v_0+X_{a_2}{}^1v_1+X_{a_2}{}^{-1}v_{-1}, \ldots, X_{a_n}{}^0v_0$

$+X_{a_n}{}^1v_1+X_{a_n}{}^{-1}v_{-1})$.

$\varphi(y_1, y_2, \ldots, y_n)$ where r of the y_i are v_1, s of the y_i are v_{-1} and the remaining $(n-r-s)$ are v_0 is the $(s-r)$-crossing link and hence

$$\varphi(y_1, y_2, \ldots, y_n)=x_{s-r}O.$$

Then the result follows from the multilinearity of φ.

References

1. J. Conway *An enumeration of knots and links,* In Computational Problems in Abstract Algebra (Edited by J. Leech) pp. 329–358. Pergamon Press (1969).
2. D. Gabai, *Foliations and genera or links,* Topology **23** (1984), 381–394.
3. L. Kauffman, *An invariant of Regular isotopy,* preprint.
4. _______, *State models and the Jones polynomial,* Topology **26** (1987), 395–407.
5. W. B. R. Lickorish and K. Millett, *A polynomial invariant of oriented links,* Topology **26** (1987), 107–104.
6. V. Jones, *Hecke algebra representations of braid groups and link polynomials,* Ann. of Math. **126** (1987), 355–388.

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