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COMPACT GROUP ACTIONS ON ADMISSIBLE MANIFOLDS

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Conner and Raymond [CR] proved that the only connected, compact Lie groups that act effectively on a closed aspherical manifold are tori. Subsequently, many generalizations have been obtained by enlarging the class of spaces which enjoy some of the features of compact groups acting on aspherical manifolds in several ways. A list of classes of manifolds satisfying the aforementioned property can be found in [LR].

To describe the main results, we need some notations and terminologies. Throughout this paper, M will be connected, oriented, closed manifold, whose universal covering is denoted by $\tilde{M}$. Let G be a group acting on M. We define a subset $F(G)$ of G as the set of all elements which have some lift to $\tilde{M}$ of finite order. It will be shown that this set is in fact a normal subgroup of G if the action of G on M is inner (see below for a definition). An action of a compact Lie group G on M is called inner if the induced homomorphism $\phi : G \to \mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ is trivial, $\pi = \pi_1(M)$. Recall that M is said to be *admissible* if all periodic self homeomorphisms of the universal covering $\tilde{M}$ of M commuting with $\pi = \pi_1(M)$ are elements of the center $Z(\pi)$ of π.

Lee and Raymond proved the following implications hold: For M a connected, oriented, closed manifold,

$$\text{Admissible}$$
$$\Downarrow$$
$$\text{Any effective finite inner action is abelian.}$$
$$\Downarrow$$
$$\text{Any compact connected effective Lie group acting on } M \text{ is a torus.}$$

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In this paper, we formulate the admissibility in terms of $F(G)$ for a finite group G acting effectively and innerly. More importantly, we found that $F(G)$ being central in G for every finite inner G implies that M must satisfy the last property. Of course this is weaker than asking that any effective finite inner action be abelian. More precisely, we prove

THEOREM 1. *M is admissible if and only if $F(G)$ is trivial for every finite group G acting effectively and innerly on M.*

THEOREM 2. *Suppose, for every finite group G acting effectively and innerly on M, $F(G)$ is central. Then any compact connected effective Lie group acting on M is a torus.*

Let M be a connected manifold and let G be a group acting on M. If $p : \tilde{M} \to M$ is a universal covering, we shall denote by E the group of all homeomorphisms $h : \tilde{M} \to \tilde{M}$ covering those in G, i.e., $ph = gp$ for some $g \in G$. The group E fits in the exact sequence

$$1 \to \pi_1(M) \to E \to G \to 1.$$

Let $\psi : G \to \mathrm{Out}(\pi)$ be the abstract kernel for this extension. Then we obtain an exact sequence

$$1 \to Z(\pi) \to C_E(\pi) \to \mathrm{Ker}\psi \to 1$$

where $\pi = \pi_1(M)$, $Z(\pi)$ is the center and $C_E(\pi)$ is the centralizer of π in E [GLO ; 2.1].

PROPOSITION 1. *Let G be a finite group acting effectively on M. Suppose the G-action is inner. Then*

(i) *$F(G)$ is a normal subgroup of G,*

(ii) *$G/F(G)$ is abelian.*

Proof. Since the G-action is inner, G maps trivially into $\mathrm{Out}(\pi)$. Therefore, $1 \to Z(\pi) \to C_E(\pi) \xrightarrow{\rho} G \to 1$ is exact. This extension is central. Let m be the order of G. Then there is a homomorphism $\phi : C_E(\pi) \to Z(\pi)$, defined by $\phi(x) = x^m$, [GLO ; 1.1]. Let $t(C_E(\pi))$ be the set of all elements of finite order in $C_E(\pi)$. Then $t(C_E(\pi)) = \phi^{-1}t(Z(\pi))$ is a normal subgroup of $C_E(\pi)$, and $C_E(\pi)/t(C_E(\pi))$ is torsion-free. Furthermore, $t(C_E(\pi))$ maps onto $F(G) \subset G$ by ρ, sho wing that $F(G)$ is normal in G. This proves (i).

For (ii), observe that $1 \to t(Z(\pi)) \to t(C_E(\pi)) \to F(G) \to 1$ is exact, since $\rho(t(C_E(\pi)))=F(G)$. Hence, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & t(Z(\pi)) & \rightarrow & t(C_E(\pi)) & \rightarrow & F(G) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z(\pi) & \rightarrow & C_E(\pi) & \rightarrow & G & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z(\pi)/t(Z(\pi)) & \rightarrow & C_E(\pi)/t(C_E(\pi)) & \rightarrow & G/F(G) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

From the last row, $C_E(\pi)/t(C_E(\pi))$ is a torsion-free central extension of an abelian group $Z(\pi)/t(Z(\pi))$ by the finite group $G/F(G)$. Applying [GLO ; 1. 3], we conclude that $C_E(\pi)/t(C_E(\pi))$ and hence $G/F(G)$ is abelian.

Proof of Theorem 1. If M is admissible, then clearly $F(G)$ is trivial for every G. For the converse, suppose that M is not admissible. Then there exists a homeomorphism h of $\tilde{M}$ so that

(a) h commutes with π,

(b) $h^k=\mathrm{id}$ for some $k>1$, and

(c) $h\notin Z(\pi)$.

Let p be the smallest integer so that $h^p\in Z(\pi)$, $1\leq p\leq k$. Let $k=dp$. We may assume p is a prime by choosing a power of h if necessary. Then $\boldsymbol{Z}_k=\{h, h^2, \ldots, h^k\}$ is a subgroup of $H(\tilde{M})$ commuting with π. Now $\boldsymbol{Z}_k \cap \pi=\{h^p, h^{2p}, \ldots, h^{dp}\}\cong \boldsymbol{Z}_d$.

The action of $\boldsymbol{Z}_k$ on $\tilde{M}$ induces an effective action of $\boldsymbol{Z}_k/\boldsymbol{Z}_k\cap\pi\cong\boldsymbol{Z}_p$ on M. Let E be the lifting of the $\boldsymbol{Z}_p$ action on M so that $1\rightarrow\pi\rightarrow E\rightarrow\boldsymbol{Z}_p\rightarrow 1$ is exact. By construction, E is generated by π and $\boldsymbol{Z}_k$. Furthermore, the abstract kernel $\boldsymbol{Z}_p\rightarrow\mathrm{Out}(\pi)$ of this exact sequence is trivial, becaue $\boldsymbol{Z}_k$ commutes with π. We conclude that the $\boldsymbol{Z}_p$ action on M is inner. Clearly, $F(\boldsymbol{Z}_p)$ is $\boldsymbol{Z}_p$ itself, which is not a trivial group.

By Theorem 1, $F(G)$ is trivial, and so G is abelian from (ii) of Proposition 1. This gives an another proof of the result of Lee and Raymond [LR].

Corollary. *If a finite group acts on an admissible manifold effectively* *and innerly, then it is abelian.*

The following generalizes [GLO ; 1.4] from the abelian to the nilpotent case. It is crucial for a proof of Theorem 2.

PROPOSITION 2. *If every finite subgroup of a compact connected Lie group L is m-step nilpotent for a fixed $m>0$, then L is abelian.*

Proof. Let T^r be a maximal torus (of rank r), $N_L(T^r)$ be its normalizer in L and $W=N_L(T^r)/T^r$ be the Weyl group with $n=|W|$. For any natural number k, $(\boldsymbol{Z}_{kn})^r=\{t\in T^r \mid t^{kn}=1\}$ is a characteristic subgroup of T^r. Hence the standard action of W on T^r restricts to an action on $(\boldsymbol{Z}_{kn})^r$, giving the exact sequence of W-modules

$$0\to(\boldsymbol{Z}_{kn})^r\xrightarrow{i}T^r\xrightarrow{kn}{}'T^r\to 0.$$

In cohomology, we have the exact sequence

$$H^2(W;\boldsymbol{Z}_{kn})^r\}\xrightarrow{i_*}H^2(W;T^r)\xrightarrow{(kn)_*}H^2(W;{}'T^r).$$

Since $|W|=n$, $(kn)_*$ is a trivial map so that i_* is onto. There is $[W_k]\in H^2(W;(\boldsymbol{Z}_{kn})^r)$ such that $i_*[W_k]=[N_L(T^r)]$. Then

$$\begin{array}{ccccccccc} 0 & \to & (\boldsymbol{Z}_{kn})^r & \to & W_k & \to & W & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & T^r & \to & N_L(T^r) & \to & W & \to & 1 \end{array}$$

is commutative, and we have

$$W_1\subset W_2\subset W_3\subset\ldots\subset N_L(T^r).$$

Since $\bigcup_k(\boldsymbol{Z}_{kn})^r$ is dense in T^r, $\bigcup_k W_k$ is also dense in $N_L(T^r)$.

We give an argument for $m=2$. The general case is similar. Let $a, b, c\in N_L(T^r)$. Then $a=\lim a_i$, $b=\lim b_j$, and $c=\lim c_k$, where $a_i\in W_i$, $b_j\in W_j$ and $c_k\in W_k$ for each i, j and k. Since $[[-,-],-] : L\times L\times L\to L$ is a continuous map, we have $[[a,b],c]=[[\lim a_i, \lim b_j], \lim c_k]=\lim[[a_i,b_j],c_k]=1$. Hence $N_L(T^r)$ is nilpotent. The following Lemma shows that G is abelian, finishing the proof.

LEMMA. *Let L be a compact, connected Lie group. If L is not a torus, then the normalizer of a maximal torus is not nilpotent.*

Proof. Any connected compact Lie group L is of the form

$$L=(T_0\times L_1\times L_2\times\cdots\times L_n)/K$$

where T_0 is a torus, L_i's are simple, and K is a finite subgroup of the center of L. Let $\tilde{L}$ be the finite covering of L so that $\tilde{L}=T_0\times L_1\times L_2\times\cdots\times L_n$. For a subgroup H of the maximal torus which is closed and normal in L, it is known that the Weyl groups $W(L/H)$ and $W(L)$ are isomorphic. See [Ad ; 5.6]. Applying this fact twice, we may assume that $L=L_1\times L_2\times\cdots\times L_n$. This implies that $W(L)=W(L_1)\times W(L_2)\times\cdots\times W(L_n)$. However, all the simple groups are known to have nontrivial Weyl groups. We conclude that $W(L)$ is nontrivial. In fact, it is generated by "reflections".

Choose an element α of $W(L)$ of order 2. It acts on the Lie algebra of the maximal torus T of L as reflections along a linear subspace, called a wall. Therefore, it has an eigen-value -1. Let $\alpha_*(v)=-v$. We pick a lift $\tilde{\alpha}\in N_L(T)$ of α. Consider the subgroup S of $N_L(T)$ generated by $\{\exp(tv)\,|\,t\in\boldsymbol{R}\}$ and $\tilde{\alpha}$. Since $\tilde{\alpha}_*$ leaves the line $\{tv\,|\,t\in\boldsymbol{R}\}$ invariant and $\tilde{\alpha}^2\in T^r$, S is an extension of either $\boldsymbol{R}^1$ or S^1 by $\boldsymbol{Z}_2$ or $\boldsymbol{Z}$. Clearly, $[v,\tilde{\alpha}]=2v$, which shows that S is not nilpotent.

Proof of Theorem 2. Let L be a compact, connected Lie group acting on M, and let G be any finite subgroup of L. Since L is connected, the action of L on M is inner, hence so is the action of G. By the hypothesis, $F(G)$ is central in G. Now consider the exact sequence

$$1\rightarrow F(G)\rightarrow G\rightarrow G/F(G)\rightarrow 1.$$

By (ii) of Proposition 1, $G/F(G)$ is abelian. We conclude that every finite subgroup G of L is 2-step nilpotent. Now apply Proposition 2 to see that L is abelian.

Example. The example [LR ; 4.8] shows that the converse of Theorem 2 does not hold: $M=(S^2\times S^1)\#(S^2\times S^1)$ admits no compact connected group action other than the circle action with fixed points. Further, M admits a dihedral inner action, which is, of course, not nilpotent.

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