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EXPONENTIALLY EQUI-CONTINUOUS C-SEMIGROUPS IN LOCALLY CONVEX SPACE

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1. Introduction

In this paper we are concerned with exponentially equi-continuous C-semigroups for calibration. A calibration Γ for a topological vector space X is a family of seminorms which induces the topology of X. Let X be sequentially complete locally convex space and let C ; $X \to X$ be an injective bounded linear operator with dense range. A family $\{S(t)\ ;\ t \geqq 0\}$ of bounded linear operators from X into itself is called an exponentially equi-continuous C-semigroup if

(1) $S(t+s)C = S(t)S(s)$ for $t, s \geqq 0$ and $S(0) = C$,

(2) there exists $a \geqq 0$ *such that* $\{e^{-at}S(t)x\ ;\ t \geqq 0\}$ is equi-continuous in X.

(3) for every $x \in X$, $S(t)x$ is continuous in $t \geqq 0$.

For every $t \geqq 0$, let $T(t)$ be the closed linear operator defined by $T(t)x = C^{-1}S(t)x$ for $x \in D(T(t)) = \{x \in X\ ;\ S(t)x \in R(C)\}$.

We define the operator G by

$$D(G) = \left\{x \in R(C)\ ;\ \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\right\}$$

and

$$Gx = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \quad \text{for } x \in D(G). \tag{1.1}$$

$\bar{G}$ is called the C-c. i. g. (C-complete infinitesimal generator) of $\{S(t)\ ;\ t \geqq 0\}$, where $\bar{G}$ denotes the closure of G.

In Section 2, we deal with the C-c. i. g. and representation of exponentially equi-continuous C-semigroups, Section 3 treats the generation of exponentially equi-continuous C-semigroups and Section 4 investigate a Cauchy problem in locally convex space.

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2. C-complete infinitesimal generator and representation of exponentially equi-continuous C-semigroups

Let x be any fixed element in X and let us put

$$L_\lambda^w x=\int_0^w e^{-\lambda t}S(t)x\ dt$$

for each $w>0$ and $\lambda>0$ (We can define the integral of Riemann type since $e^{-\lambda t}S(t)x$ is continuous on $[0,\infty)$ with values in X). Then we have for any seminorm $p\in\Gamma$,

$$p(L_\lambda^w x-L_\lambda^{w'}x)\leqq\int_{w'}^w e^{-\lambda t}p(S(t)x)\,dt$$

for any $0<w'<w$. By the assumption (2) there exist $a\geqq0$ and $q\in\Gamma$ such that $p(S(t)x)\leqq e^{at}q(x)$ for all $t\geqq0$.
Hence if $\lambda>a$, then

$$p(L_\lambda^w x-L_\lambda^{w'}x)\leqq q(x)\int_{w'}^w e^{-(\lambda-a)t}dt\to0$$

as $w,w'\to\infty$. Thus the limit $\lim_{w\to\infty}L_\lambda^w x$ exist. For every $\lambda>a$, define the bounded linear operator $L_\lambda\,;\,X\to X$ by

(2. 1) $$L_\lambda x=\int_0^\infty e^{-\lambda t}S(t)x\ dt \text{ for } x\in X.$$

Similarity as in Banach space [1, 3], we have

(a) G is densely defined and closable,

(b) for every $\lambda>a$,

(2. 2) $$(\lambda-\overline{G})L_\lambda x=Cx \text{ for } x\in X$$
$$L_\lambda(\lambda-\overline{G})x=Cx \text{ for } x\in D(\overline{G}).$$

In fact, let $\lambda>a$. For $x\in R(C)$, $L_\lambda x\in R(C)$ and

$$\frac{1}{h}(T(h)-I)L_\lambda x=\frac{1}{h}\left(\int_0^\infty e^{-\lambda t}S(t+h)x\,dt-\int_0^\infty e^{-\lambda t}S(t)x\,dt\right)$$
$$=\frac{1}{h}(e^{\lambda h}-1)\int_h^\infty e^{-\lambda t}S(t)x\,dt-\frac{1}{h}\int_0^\infty e^{-\lambda t}S(t)x\,dt$$
$$\to\lambda L_\lambda x-Cx$$

as $h\to0^+$, and hence $L_\lambda x\in D(G)$ and $GL_\lambda x=\lambda L_\lambda x-Cx$.
Next, for $x\in D(G)$, $L_\lambda x\in R(C)$ and

$$\frac{1}{h}(T(h)-I)L_\lambda x=L_\lambda\left[\frac{1}{h}(T(h)-I)x\right]\to L_\lambda Gx$$

as $h\to0^+$, i.e., $GL_\lambda x=L_\lambda Gx$. Therefore

(2. 3) $\quad(\lambda-G)L_\lambda x=Cx$ for $x\in R(C)$,

$$L_\lambda(\lambda-G)x=Cx \text{ for } x\in D(G).$$

Note that

$$(2.4) \qquad \lim_{\lambda\to\infty}\lambda L_\lambda x=\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda t}S(t)xdt=Cx \text{ for } x\in X.$$

Since $\lambda L_\lambda Cx\in D(G)$ and $\lim_{\lambda\to\infty}\lambda L_\lambda Cx=C^2x$ for $x\in X$ by (2.3) and (2.4), we have that $D(G)$ is dense in X by noting $R(C^2)$ is dense in X. Now the closability of G follows from $L_\lambda(\lambda-G)x=Cx$ for $x\in D(G)$ and (2.4), then (2.3) implies (2.2). The proof is complete.

We also see that

$$(2.5) \qquad \frac{d}{ds}S(s)x=S(s)\overline{G}x=\overline{G}S(s)x$$

for $x\in D(\overline{G})$ and $s\geqq 0$. The proof is similarly as in [1, Lemma 8].

THEOREM 2.1. *The family of operators*

$$(2.6) \quad \{[(\lambda-a)(\lambda-\overline{G})^{-1}]^nCx\ ;\ x\in D((\lambda-\overline{G})^{-1}),\ \lambda>a,\ n=1,2,3,\dots\}$$

is equi-continuous.

Proof. From the resolvent equation

$$(\mu-\overline{G})^{-1}Cx-(\lambda-\overline{G})^{-1}Cx=(\lambda-\mu)(\lambda-\overline{G})^{-1}(\mu-\overline{G})^{-1}Cx$$

for λ, $\mu>a$, we obtain

$$\begin{aligned}\frac{d}{d\lambda}(\lambda-\overline{G})^{-1}Cx&=\lim_{\mu\to\lambda}(\mu-\lambda)^{-1}[(\mu-\overline{G})^{-1}C-(\lambda-\overline{G})^{-1}C]x\\&=-\lim_{\mu\to\lambda}(\lambda-\mu)^{-1}[(\mu-\overline{G})^{-1}C-(\lambda-\overline{G})^{-1}C]x\\&=-\lim_{\mu\to\lambda}(\lambda-\overline{G})^{-1}(\mu-\overline{G})^{-1}Cx\\&=-(\lambda-\overline{G})^{-2}Cx.\end{aligned}$$

Moreover, since

$$(\mu-\overline{G})^{-n}C-(\lambda-\overline{G})^{-n}C=(\lambda-\mu)\sum_{k=0}^{n-1}(\mu-\overline{G})^{-k-1}(\lambda-\overline{G})^{-n+k}C$$

for $\lambda,\mu>a$, we have $\frac{d}{d\lambda}(\lambda-\overline{G})^{-n}Cx=-(\lambda-\overline{G})^{-n-1}Cx$.

Therefore $(\lambda-\overline{G})^{-1}Cx$ is infinitely differentiable with respect to $\lambda>a$ and

$$\frac{d^n}{d\lambda^n}(\lambda-\overline{G})^{-1}Cx=(-1)^nn!(\lambda-\overline{G})^{-n-1}Cx$$

for $\lambda>a$ and $x\in D((\lambda-\overline{G})^{-1})$.

On the other hand, we have, by (2.2) and differenting (2.1)

n-times with respect to,

$$\frac{d^n}{d\lambda^n}(\lambda-\overline{G})^{-1}Cx=\int_0^\infty e^{-\lambda t}(-t)^n S(t)\,xdt.$$

Hence $[(\lambda-a)(\lambda-\overline{G})^{-1}]^{n+1}Cx=\frac{(\lambda-a)^{n+1}}{n!}\int_0^\infty e^{-\lambda t}t^n S(t)\,x\,dt$ for $x\in D((\lambda-\overline{G})^{-1})$, and so, for any $p\in\Gamma$ on X and $\lambda>a$, $n>0$,

$$p([(\lambda-a)(\lambda-\overline{G})^{-1}]^{n+1}Cx)\leqq\frac{(\lambda-a)^{n+1}}{n!}\int_0^\infty e^{-(\lambda-a)t}\,t^n dt\ \sup_{t\geqq 0} p(e^{-at}S(t)x)$$
$$=\sup_{t\geqq 0} p(e^{-at}S(t)x).$$

This proves theorem by the equi-continuity of $\{e^{-at}S(t)x\,;\,t\geqq 0\}$.

We now define

$$S_\lambda^n(t)x=e^{-\lambda t}\sum_{k=0}^{n}\frac{t^k\lambda^{2k}}{k!}(\lambda-\overline{G})^{-k}Cx \text{ for } x\in X.$$

For each fixed $\lambda>a$, $t\geqq 0$ and $x\in X$, the sequence $\{S_\lambda^n(t)x\,;\,n=0,1,2,\ldots\}$ is Cauchy sequence.

Indeed, for any continuous seminorm $p\in\Gamma$ on X,

$$p(S_\lambda^n(t)x-S_\lambda^m(t)x)\leqq e^{-\lambda t}\sum_{k=m+1}^{n}\frac{t^k\lambda^{2k}}{k!}p((\lambda-\overline{G})^{-k}Cx)$$

and there exist $q\in\Gamma$ by Theorem 2.1., $p((\lambda-\overline{G})^{-k}Cx)\leqq\frac{q(x)}{(\lambda-a)^k}$ for all k, and so that

$$p(S_\lambda^n(t)x-S_\lambda^m(t)x)\leqq q(x)e^{-\lambda t}\sum_{k=m+1}^{n}\frac{t^k\lambda^{2k}}{k!(\lambda-a)^k}$$
$$\to 0 \text{ as } m,n\to\infty.$$

Then, for each $\lambda>a$, $t\geqq 0$ and $x\in X$, the limit

$$S_\lambda(t)x=\lim_{n\to\infty}S_\lambda^n(t)x=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^{2k}}{k!}(\lambda-\overline{G})^{-k}Cx \tag{2.7}$$

exists. By $S_\lambda^n(t)$ is continuous and $S_\lambda(t)$ is uniformly converges in t, we also see that $S_\lambda(t)$ is continuous in $t\geqq 0$.

THEOREM 2.2. *Let $\{S(t)\,;\,t\geqq 0\}$ be an exponentially equi-continuous C-semigroup. If $\overline{G}$ is the C-c. i. g. of $\{S(t)\,;\,t\geqq 0\}$ and*

$$S_\lambda(t)x=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^{2k}}{k!}(\lambda-\overline{G})^{-k}Cx$$

for $x\in X$ and $t\geqq 0$, then we have

$$\frac{d}{ds}(S_\lambda(t-s)S(s)x)=S_\lambda(t-s)S(s)(\overline{G}x-\lambda\overline{G}(\lambda-\overline{G})^{-1}x) \tag{2.8}$$

for all $x \in CD(\overline{G})$.

Proof. From the definition of $S_\lambda(t)$

$$\frac{d}{ds}S_\lambda(s)Cx = S_\lambda(s)(-\lambda + \lambda^2(\lambda-\overline{G})^{-1})Cx$$
$$= S_\lambda(s)\lambda\overline{G}(\lambda-\overline{G})^{-1}Cx$$

for $x \in X$, and we see that

$$\frac{d}{ds}S(s)x = S(s)\overline{G}x$$

for $x \in CD(\overline{G})$ by (2.5). By (2.2),

$$S(s)(\lambda-\overline{G})^{-1}Cx = S(s)L_\lambda x = L_\lambda S(s)x = (\lambda-\overline{G})^{-1}CS(S)x$$

for $x \in CD(\overline{G})$ i.e., $\overline{G}(\lambda-\overline{G})^{-1}C(=\lambda(\lambda-\overline{G})^{-1}C - C)$ commutes with $S(s)$. Now, let $x \in CD(G)$ and $x = Cy$, $y \in D(\overline{G})$. Then we have

$$\frac{d}{ds}(S_\lambda(t-s)S(s)x) = S_\lambda(t-s)S(s)(\overline{G}x - \lambda\overline{G}(\lambda-\overline{G})^{-1}x)$$

for $x \in CD(\overline{G})$.

LEMMA 2.1. *For each* $x \in X$,

(2.9) $$\lim_{\lambda\to\infty} \overline{G}(\lambda-\overline{G})^{-1}Cx = 0.$$

Proof. For all $x \in D(\overline{G})$ and for any $p \in \Gamma$, there exists $q \in \Gamma$ such that

$$p(\overline{G}(\lambda-\overline{G})^{-1}Cx) = p((\lambda-\overline{G})^{-1}C\overline{G}x) \leqq \frac{q(x)}{(\lambda-a)} \to 0$$

as $\lambda \to \infty$, $\lambda > a$, and by (2.6).

Since $D(\overline{G})$ is dence in X,

$$\lim_{\lambda\to\infty} \overline{G}(\lambda-\overline{G})^{-1}Cx = 0 \text{ for any } x \in X.$$

LEMMA 2.2. *For each* $x \in D(\overline{G})$,

(2.10) $$\lim_{\lambda\to\infty} \lambda\overline{G}(\lambda-\overline{G})^{-1}Cx = \overline{G}Cx.$$

Proof. Let $\mu > a$, $x \in D(\overline{G})$ and $x = (\mu-\overline{G})^{-1}y$ for some $y \in R(\mu-\overline{G})$. Then

$$\lambda\overline{G}(\lambda-\overline{G})^{-1}C^2x - \overline{G}C^2x = \lambda\overline{G}(\lambda-\overline{G})^{-1}C(\mu-\overline{G})^{-1}Cy - \overline{G}C^2(\mu-\overline{G})^{-1}y$$
$$= \lambda\overline{G}(\lambda-\overline{G})^{-1}C(\mu-\overline{G})^{-1}Cy - \overline{G}(\mu-\overline{G})^{-1}C^2y$$
$$= \frac{\lambda\overline{G}}{\mu-\lambda}((\lambda-\overline{G})^{-1}C^2y - (\mu-\overline{G})^{-1}C^2y)$$

$$-\overline{G}(\mu-\overline{G})^{-1}C^2y$$
$$=\frac{\lambda}{\mu-\lambda}\overline{G}(\lambda-\overline{G})^{-1}C^2y-\frac{\mu}{\mu-\lambda}\overline{G}(\mu-\overline{G})^{-1}C^2y.$$

Thus for any seminorm $p\in\Gamma$

$$p(\lambda\overline{G}(\lambda-\overline{G})^{-1}C^2x-\overline{G}C^2x)\leqq\frac{\lambda}{\mu-\lambda}p(\overline{G}(\lambda-\overline{G})^{-1}C^2y)+\frac{\mu}{\mu-\lambda}(\overline{G}(\mu-\overline{G})^{-1}C^2y)\to 0$$

as $\lambda\to\infty$ by (2.9). Since $D(\overline{G})$ is dense in X, we have

$$\lim_{\lambda\to\infty}\lambda\overline{G}(\lambda-\overline{G})^{-1}Cx=\overline{G}Cx \text{ for any } x\in X.$$

THEOREM 2.3. *Let $\{S(t)\ ;\ t\geqq 0\}$ be an exponentially equ-continuous C-semigroup. If $\overline{G}$ is the C-c.i.g. of $\{S(t)\ ;\ t\geqq 0\}$, then*

$$S(t)x=\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^{2k}}{k!}(\lambda-\overline{G})^{-k}Cx$$

for $x\in X$ and $t\geqq 0$.

Proof. since

$$S_\lambda(t)x=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^k}{k!}(1-\frac{a}{\lambda})^{-k}[I-\frac{1}{\lambda-a}(\overline{G}-a)]^{-k}Cx$$

$$=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^k}{k!}(1+\frac{a}{\lambda-a})^k[(\lambda-a)(\lambda-\overline{G})^{-1}]^kCx \tag{2.11}$$

for $\lambda>a$ and $x\in X$. there exists $q\in\Gamma$ such that

$$\begin{aligned}p(S_\lambda(t)x)&\leqq e^{-\lambda t}\sum_{k=0}^{\infty}\frac{t^k\lambda^k}{k!}(1+\frac{a}{\lambda-a})^kq(x)\\&=e^{(a+a^2/(\lambda-a))t}q(x)\\&\leqq e^{2at}q(x)\end{aligned} \tag{2.12}$$

for any $p\in\Gamma$, $x\in X$ and $\lambda>2a$ by (2.11) and (2.6).
From (2.8), we have

$$S(t)C^2x-S_\lambda(t)C^2x=\int_0^t\frac{d}{ds}S_\lambda(t-s)S(s)Cx\ ds$$
$$=\int_0^t S_\lambda(t-s)S(s)(\overline{G}Cx-\lambda\overline{G}(\lambda-\overline{G})^{-1}Cx)\,ds$$

for $x\in D(\overline{G})$ and $0\leqq s\leqq t$. By property (2) of $\{S(t)\ ;\ t\geqq 0\}$, there exist $q, \tilde{q}\in\Gamma$ such that

$$p(S(t)C^2x-S_\lambda(t)C^2x)\leqq\int_0^t p(S_\lambda(t-s)S(s)(\overline{G}Cx-\lambda\overline{G}(\lambda-\overline{G})^{-1}Cx)\ ds$$
$$\leqq\int_0^t e^{2a(t-s)}q(S(s)(\overline{G}Cx-\lambda\overline{G}(\lambda-\overline{G})^{-1}Cx))\,ds$$

$$\leqq \int_0^t e^{2a(t-s)} e^{as} \tilde{q}(\bar{G}Cx - \lambda \bar{G}(\lambda - \bar{G})^{-1}Cx)\, ds$$

$$= -\frac{1}{a}(e^{-at}-1) e^{2at} \tilde{q}(\bar{G}Cx - \lambda \bar{G}(\lambda - \bar{G})^{-1}Cx)$$

for any $p \in \Gamma$, $\lambda > 2a$ and $0 \leqq s \leqq t$, and so, for any $p \in \Gamma$ and $t \geqq 0$, $p(S(t)C^2x - S_\lambda(t)C^2x) \to 0$ as $\lambda \to \infty$, because (2. 10).
Hence $\lim_{\lambda\to\infty} S_\lambda(t)x = S(t)x$ for each $t \geqq 0$ and $x \in C^2D(\bar{G})$.
Since $C^2D(\bar{G})$ is dense in X, for any $x \in X$, we have $\lim_{\lambda\to\infty} S_\lambda(t)x = S(t)x$ for $t \geqq 0$.

3. Generation of exponentially equi-continuous C-semigroup

We consider the following conditions:
(i) $D(A)$ is dense in X,
(ii) for each $x \in D((\lambda - A)^{-1})$, $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$.
(iii) for each $x \in X$, $\{[(\lambda - a)(\lambda - A)^{-1}]^k Cx\ ;\ \lambda > a, k = 0, 1, 2, \ldots\}$ is equi-continuous,
(iv) $CD(A)$ is a core of A.

LEMMA 3. 1. *Let* $S_\lambda(t)x = e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k \lambda^{2k}}{k!}(\lambda - A)^{-k}Cx$ *for* $x \in X$, $t \geqq 0$ *and* A *satisfying* (i)——(iv). *Then for every bounded set* B, *the operator* $\{e^{-2at}S_\lambda(t)x\ ;\ \lambda \geqq 2a,\ t \geqq 0\}$ *is equi-continuous for any* $x \in B$.

Proof. We already proved in Theorem 2. 3.

LEMMA 3. 2. *Let* $S_\lambda(t)x = e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k \lambda^{2k}}{k!}(\lambda - A)^{-k}Cx$ *for* $x \in X$, $t \geqq 0$ *and* A *satisfying* (i)——(iv). *Then the limit* $\lim_{\lambda\to\infty} S_\lambda(t)x$ *exists uniformly with respect to t in any bounded interval.*

Proof. By definition of $S_\lambda(t)$, we have

$$\frac{d}{ds} S_\lambda(t)x = \lambda A(\lambda - A)^{-1} S_\lambda(t)x \text{ for } x \in CD(A).$$

Thus if $x \in CD(A)$ and $x = Cy$, $y \in D(A)$, then

$$\frac{d}{ds} S_\lambda(t-s) S_\mu(s) Cx = S_\lambda(t-s) \mu A(\mu - A)^{-1} S_\mu(s) Cx$$

$$-\lambda A(\lambda-A)^{-1}\cdot S_\lambda(t-s)S_\mu(s)Cx$$
$$=S_\lambda(t-s)S_\mu(s)\mu A(\mu-A)^{-12}Cy$$
$$-S_\lambda(t-s)\cdot S_\mu(s)\lambda A(\lambda-A)^{-1}C^2y$$

Hence

$$S_\mu(t)C^2x-S_\lambda(t)C^2x=\int_0^t\frac{d}{ds}S_\lambda(t-s)S_\mu(s)Cx\ ds$$
$$=\int_0^t[S_\lambda(t-s)S_\mu(s)\mu A(\mu-A)^{-1}C^2y-S_\lambda(t-s)$$
$$\cdot S_\mu(s)\lambda A(\lambda-A)^{-1}C^2y]ds$$
$$=\int_0^t S_\lambda(t-s)S_\mu(s)(\mu A(\mu-A)^{-1}C^2y$$
$$-\lambda A(\lambda-A)^{-1}C^2y)ds$$

for every $x\in C^3D(A)$. By Lemma 3.1., for any $p\in\Gamma$, there exists $q\in\Gamma$ such that $p(S_\lambda(t-s)S_\mu(s)x)\leqq e^{2at}q(x)$ for all $\mu,\lambda>2a$ and $0\leqq s\leqq t$, and (2.10) show that $p(\mu A(\mu-A)^{-1}Cx-\lambda A(\lambda-A)^{-1}Cx)\to 0$ as $\lambda,\mu\to\infty$. Then, for each $t\geqq 0$ and $p\in\Gamma$, there exist $q\in\Gamma$ such that

$$p(S_\mu(t)C^2x-S_\lambda(t)C^2x)\leqq\int_0^t e^{2at}q(\mu A(\mu-A)^{-1}Cx-\lambda A(\lambda-A)^{-1}Cx)ds$$
$$\leqq te^{2at}q(\mu A(\mu-A)^{-1}Cx-\lambda A(\lambda-A)^{-1}Cx)$$
$$\to 0$$

as $\lambda,\mu\to\infty$. Hence for each $x\in C^3D(A)$ and $t\in[0,\infty)$, the limit $\lim_{\mu\to\infty}S_\mu(t)x$ exists unformly with respect to t in any bounded interval, and from the uniform convergence in t, this limit is a continuous linear operator from X into itself.

THEOREM 3.1. *If* $\{S(t)\ ;\ t\geqq 0\}$ *is an exponentially equi-continuous C-semigroup and* A *is the C-c.i.g. of* $\{S(t)\ ;\ t\geqq 0\}$, *then* A *satisfy the conditions* (i)——(iv).

Proof. We already proved (i)——(iii). We must show that (iv). Let $z\in X$ and $t\geqq 0$. Since $D(A)$ is dense in X, there exists $z_n\in D(A)$ such that $\lim_{n\to\infty}z_n=z$. Noting $S(\tau)z\in D(A)$ and

$\frac{d}{d\tau}S(\tau)z_n=AS(\tau)z_n=S(\tau)Az$, we obtain

$$S(t)z_n-Cz_n=\int_0^t S(\tau)Az_n d\tau$$

$$=\int_0^t AS(\tau)z_n d\tau = A\int_0^t S(\tau)z_n d\tau.$$

Since $\lim_{n\to\infty}\int_0^t S(\tau)z_n d\tau=\int_0^t S(\tau)zd\tau$, $\lim_{n\to\infty} A\int_0^t S(\tau)z_n d\tau = S(t)z-Cz$ and the closedness of A,

(3.1) $\quad \int_0^t S(\tau)zd\tau\in D(A)$ and $A\int_0^t S(\tau)zd\tau = S(t)z-Cz$

for $z\in X$ and $t\geqq 0$. It suffices to show

(3.2) $\quad \overline{A|_{CD(A)}}\supset A.$

To this end, let $x\in D(A)$ and $\varepsilon>0$. Using (3.1) with $z=C^{-1}x$, we have

$$\frac{1}{t}\int_0^t S(\tau)C^{-1}x\, d\tau \to x \text{ and}$$

$$A\left(\frac{1}{t}\int_0^t S(\tau)C^{-1}x\, d\tau\right)=\frac{1}{t}(S(t)C^{-1}x-x) \tag{3.3}$$

$$=\frac{1}{t}(T(t)x-x)\to Ax$$

as $t\to 0^+$. Hence there is a $t_0>0$ such that

$$p\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)C^{-1}xd\tau - x\right)+p\left(A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)C^{-1}x\, d\tau\right)-Ax\right)<\frac{\varepsilon}{2}$$

for any $p\in\Gamma$. Since $CD(A)$ is dense in X, we can choose $x_n\in CD(A)$ such that $x_n\to C^{-1}x$ as $n\to\infty$. By (3.1) again, $\frac{1}{t_0}\int_0^{t_0} S(\tau)x_n d\tau\in CD(A)$ and

$$A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)x_n\ d\tau\right)=\frac{1}{t_0}(S(t_0)x_n-Cx_n)$$

$$\to \frac{1}{t_0}(S(t_0)C^{-1}x-C\cdot C^{-1}x) \tag{3.4}$$

$$=A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)C^{-1}x\ d\tau\right)$$

as $n\to\infty$. By (3.4), there is a $n_0\geqq 0$ such that

$$p\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)x_{n_0}\ d\tau-\frac{1}{t_0}\int_0^{t_0} S(\tau)C^{-1}xd\tau\right)+p\left(A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)x_{n_0}d\tau\right)\right.$$

$$\left.-A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)C^{-1}x\ d\tau\right)\right)<\frac{\varepsilon}{2}$$

for any $p\in\Gamma$. Then we have $\frac{1}{t_0}\int_0^{t_0} S(\tau)x_{n_0}d\tau\in CD(A)$ and

$$p\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)x_{n_0}\, d\tau - x\right) + p\left(A\left(\frac{1}{t_0}\int_0^{t_0} S(\tau)x_{n_0} d\tau - Ax\right) - Ax\right) < \varepsilon$$

for any $p \in \Gamma$. Thus (3.2) is satisfied and the proof is complete.

THEOREM 3.2. *If A is a closed linear operator satisfying* (i)——(iv), *then A is the C-c.i.g. of an exponentially equi-continuous C-semigroup* $\{S(t)\ ;\ t \geqq 0\}$.
Further

$$S(t)x = \lim_{\lambda\to\infty} e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k\lambda^{2k}}{k!}(\lambda - A)^{-k}Cx \tag{3.5}$$

for all $x \in X$ and $t \geqq 0$.

Proof. Put $S_\lambda(t) = e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k\lambda^{2k}}{k!}(\lambda - A)^{-k}Cx$ for $x \in X$ and $t \geqq 0$. By virtue of Lemma 3.2., we may $S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x$.

We first prove that $\{S(t)\ ;\ t \geqq 0\}$ satisfies an exponentially equi-continuous C-semigroup property. Clearly $S(0) = C$.

Since $S_\lambda(t+s)C = S_\lambda(t)S_\lambda(s)$ [1, Theorem 11] and $\{e^{-2at}S_\lambda(t)\ ;\ t \geqq 0, \lambda \geqq 2a\}$ is equi-continuous, for any $p \in \Gamma$, there exists $q \in \Gamma$ such that

$$\begin{aligned}
&p(S(t+s)Cx - S(t)S(s)x) \\
&\leqq p(S(t+s)Cx - S_\lambda(t+s)Cx) + p(S_\lambda(t+s)Cx - S_\lambda(t)S_\lambda(s)x) \\
&\quad + p(S_\lambda(t)S_\lambda(s)x - S_\lambda(t)S(s)x) + p(S_\lambda(t)S(s)x - S(t)S(s)x) \\
&\leqq p(S(t+s)Cx - S_\lambda(t+s)Cx) + e^{2at}q(S_\lambda(s)x - S(s)x) \\
&\quad + p((S_\lambda(t) - S(t))S(s)x) \\
&\to 0
\end{aligned}$$

as $\lambda \to \infty$, which proves $S(t+s)Cx = S(t)S(s)x$ for $t, s \geqq 0$.
From (2.11) and (iii), for any $p \in \Gamma$, $\lambda > a$ and $t \geqq 0$, there exists $q \in \Gamma$ such that

$$\begin{aligned}
p(S_\lambda(t)x) &\leqq e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k\lambda^k}{k!}\left(1 + \frac{a}{\lambda - a}\right)^k p([(\lambda - a)(\lambda - A)^{-1}]^k Cx) \\
&\leqq e^{-\lambda t}\sum_{k=0}^{\infty} \frac{t^k\lambda^k}{k!}\left(1 + \frac{a}{\lambda - a}\right)^k q(x) \\
&= e^{(a + a^2/(\lambda - a))t}q(x),
\end{aligned}$$

this implies $\lim_{\lambda\to\infty} p(S_\lambda(t)x) = p(S(t)x) \leqq e^{at}q(x)$ for all $x \in X$.

Thus the operators $\{e^{-at}S(t)x\ ;\ t \geqq 0, \lambda > a\}$ are equi-continuous in X, and from the uniform convergence in t, $S(t)x$ is continuous in

$t \geqq 0$.

Next we shall prove that A is the C-c. i. g. of an exponentially equi-continuous C-semigroup $\{S(t)\ ;\ t \geqq 0\}$.

Since $\frac{d}{dt} S_\lambda(t) Cx = S_\lambda(t) \lambda A(\lambda - A)^{-1} Cx$ for $x \in D(A)$,

$$\frac{1}{t}(S_\lambda(t)Cx - C^2x) = \frac{1}{t}\int_0^t S_\lambda(s)\lambda A(\lambda-A)^{-1}Cx\,ds \tag{3.6}$$

for each $x \in D(A)$. By (2.10) and Lemma 3.1., for any $p \in \Gamma$, there is $q \in \Gamma$ and $\lambda_0 > 0$ such that

$$p(S_\lambda(s)(\lambda A(\lambda-A)^{-1}Cx - ACx)) \leqq e^{2as} q(\lambda A(\lambda-A)^{-1}Cx - ACx) \leqq \frac{\varepsilon}{2}$$

for all $\lambda > \max\ (\lambda_0, 2a)$ and $0 \leqq s \leqq t$. By Lemma 3.2., there exists λ_0' > 0 such that $p((S_\lambda(s) - S(s))ACx) < \frac{\varepsilon}{2}$ for all $\lambda > \lambda_0'$ and $0 \leqq s \leqq t$. Thus if $\lambda > \max(\lambda_0, \lambda_0', 2a)$, then

$$\begin{aligned} & p\left(\frac{1}{t}\left(\int_0^t S_\lambda(s)\lambda A(\lambda-A)^{-1}Cx\,ds - \int_0^t S(s)ACx\,ds\right)\right) \\ & \leqq p\left(\frac{1}{t}\int_0^t S_\lambda(s)(\lambda A(\lambda-A)^{-1}Cx - ACx)\,ds\right) + p\left(\frac{1}{t}\int_0^t (S_\lambda(s) - S(s))ACx\,ds\right) \\ & \leqq \frac{1}{t}\int_0^t p(S_\lambda(s)(\lambda A(\lambda-A)^{-1}Cx - ACx))\,ds + \frac{1}{t}\int_0^t p((S_\lambda(s) - S(s))ACx)\,ds \\ & < \varepsilon \end{aligned}$$

for any $p \in \Gamma$. Them passing to the limit λ in (3.6) we have

$$\frac{1}{t}(S(t)Cx - C^2x) = \frac{1}{t}\int_0^t S(s)ACx\,ds$$

for $x \in D(A)$ and thus we see that

$$\frac{1}{t}(C^{-1}S(t)x - x) = \frac{1}{t}\int_0^t C^{-1}S(s)Ax\,ds$$

for $x \in CD(A)$. Hence

$$\lim_{t \to 0^+}\frac{1}{t}(T(t)x - x) = \lim_{t \to 0^+}\frac{1}{t}\int_0^t T(s)Ax\,ds = Ax$$

for $x \in CD(A)$. Let A' be the C-c. i. g. of an exponentially equi-continuous C-semigroup $\{S(t)\ ;\ t \geqq 0\}$ and $D(A')$ be its domain. Since $D(A') \supset CD(A)$ and $ACx = A'Cx$ for $x \in D(A)$, thus $D(A') \supset CD(A)$ and $A|_{CD(A)} = A'|_{CD(A)}$. Hence $A|_{CD(A)} \subset A'$. Since $CD(A)$ is a core of A, we have $A \subset A'$. On the other hand, by [1]

(3.7) $$(\lambda-A)^{-1}Cx=\int_0^\infty e^{-\lambda t}S(t)xdt$$

and define

$$L_\lambda x=\int_0^\infty e^{-\lambda t}S(t)xdt$$

for $\lambda>a$. From (3.7) and (ii), we have $L_\lambda(\lambda-A)x=Cx$ for $x\in D(A)$ and for $x\in X$, $(\lambda-A')L_\lambda x=Cx$ by (2.2). Thus $A'L_\lambda x=L_\lambda Ax$ for $x\in D(A)$. Letting $\lambda\to\infty$, we see that $A'Cx=CAx=ACx$ for $x\in D(A)$ that is $A'\supset A'|_{CD(A)}=A|_{CD(A)}$ Since $CD(A)$ is core of $A, A'\supset A$. Thus the proof is complete.

4. The abstract Cauchy problem

In this section we consider the following abstract Cauchy problem

(4.1) $$\frac{d}{dt}u(t)=Au(t) \text{ for } t\geqq 0 \text{ and } u(0)=x.$$

By a solution $u(t)$ of the (4.1) we mean that $u(t)$ is continuously differentiable in $t\geqq 0$, $u(0)=x, u(t)\in D(A)$ and $\frac{d}{dt}u(t)=Au(t)$ for every $t\geqq 0$.

THEOREM 4.1. *Let A be a densely defined closed linear operator which commutes with C. Then A is the C–c.i.g. of an exqonentially equi–continuous C–semigroup $\{S(t)\ ;\ t\geqq 0\}$ if and only if A satisfies the following conditions*:

(α) *The (4.1) has unique solution $u(t)$ for all $x\in CD(A)$,*

(β) *for every $x\in CD(A)$ and $p\in\Gamma$, there exists $q\in\Gamma$ such that*
$p(u(t))\leqq e^{at}q(C^{-1}x)$ *for* $t\geqq 0$,

(γ) *$CD(A)$ is a core of A.*

proof. Let A be the C–c.i.g. of $\{S(t)\ ;\ t\geqq 0\}$ and let $x\in CD(A)$, $x=Cy, y\in D(A)$. Put $u(t)=T(t)x$ for $t\geqq 0$. Then $u(0)=x$ and $u(t)=S(t)y\in D(A)$. By [1, Lemma 8], we have

$$T(t)x-x=S(t)y-Cy=\int_0^t S(\tau)Ayd\tau$$

and so

$$\frac{d}{ds}S(s)y=S(s)Ay=AS(s)y.$$

Thus $\frac{d}{dt}u(t)=Au(t)$ and $u(t)$ is continuously differentiable in $t\geqq 0$. To prove uniqueness, we suppose $v(t)\in D(A)$, $\frac{d}{dt}v(t)=Av(t)$ and $v(0)=x\in CD(A)$. For $s, t\geqq 0$

$$\frac{d}{ds}[S(t-s)v(s)]=S(t-s)\frac{d}{ds}v(s)+\left[\frac{d}{ds}S(t-s)\right]v(s)$$
$$=0,$$

whence $S(t-s)v(s)$ is independent of s. Setting $s=0$, $s=t$ yields $v(t)=C^{-1}S(t)x=T(t)x=u(t)$. Clearly, the condition (β) and (γ) satisfy by the definition of exponentially equi-continuous C-semigroup and Theorem 3. 1.

Conversely, let for every $x\in CD(A)$, (4. 1) has a unique continuously differentiable solution on $[0, \infty)$ and let define the operator $\tilde{T}(t)$: $CD(A)\rightarrow D(A)$ by $\tilde{T}(t)x=u(t\ ;\ x)$. From the uniqueness of the solution $\tilde{T}(t)$ is a linear operator defined on all of $CD(A)$. First we show that

(4. 2) $\qquad \tilde{T}(t)Ax=A\tilde{T}(t)x$

for $x\in CD(A^2)$. Let $x\in CD(A^2)$. Since $Ax\in CD(A)$, there exists a solution $u(t\ ;\ Ax)$ of (4. 1) such that

(4. 3) $\qquad \frac{d}{ds}u(s\ ;\ Ax)=Au(s\ ;\ Ax).$

Integrating (4. 3) from 0 to t, we obtain

$$u(t\ ;\ Ax)=Ax+\int_0^t Au(s\ ;\ Ax)ds$$
$$=A(x+\int_0^t u(s\ ;\ Ax)ds).$$

put $z(t)=x+\int_0^t u(s\ ;\ Ax)ds$. Then $z(t)$ is continuously differentiable, $\frac{d}{dt}z(t)=u(t\ ;\ Ax)=Az(t)$ for $t\geqq 0$ and $z(0)=x$.

It follows that $z(t)$ is a solution of (4. 1) with initial value x. From the uniqueness of solution, $z(t)=u(t\ ;\ x)$.

Hence $u(t\ ;\ Ax)=Au(t\ ;\ x)$ for $t\geqq 0$. Therefore $\tilde{T}(t)Ax=A\tilde{T}(t)x$ by the definition of $\tilde{T}(t)$.

Since $C\tilde{T}(s)x\in CD(A)$, for $0\leqq s\leqq t$ and $x\in CD(A)$, $\tilde{T}(t-s)C\tilde{T}(s)x$ is a solution of (4. 1). Using (4. 2) and commutating A with C, we have

$$\frac{d}{ds}\tilde{T}(t-s)C\tilde{T}(s)x=\tilde{T}(t-s)CA\tilde{T}(s)x-A\tilde{T}(t-s)C\tilde{T}(s)x$$

whence $\tilde{T}(t-s)C\tilde{T}(s)x$ is independent of s. Setting $s=0$, $s=t$ yields

(4.4) $\qquad \tilde{T}(t)Cx=C\tilde{T}(t)x$

for $x\in CD(A)$.

We define w ; $[0, \infty) \to X$ by

$$w(t)=\begin{cases} u(t\ ;\ x) & (0\leqq t\leqq s) \\ u(t-s\ ;\ u(s\ ;\ x)) & (t>s) \end{cases}$$

for $x\in CD(A)$. Since $Cu(s\ ;\ x)\in CD(A)$ we note that the existance of solution $u(t\ ;\ u(s\ ;\ x))$ of (4.1). Clearly, $w(t)$ is a solution of (4.1) with initial deta x. By the uniqueness of solution, we may $w(t)=u(t\ ;\ x)$ for $t\in[0, \infty)$. From the uniqueness of the solution (4.1), $w(t+s)=u(t+s\ ;\ x)=u(t\ ;\ u(s\ ;\ x))$. It follows that

(4.5) $\qquad \tilde{T}(t+s)x=\tilde{T}(t)\tilde{T}(s)x$

for $x\in CD(A)$. From (β) and (4.4), for any $p\in\Gamma$ and $x\in CD(A)$, there exists $q\in\Gamma$ such that

(4.6) $\qquad p(C\tilde{T}(t)x)\leqq e^{at}q(x)$.

We define the continuous linear operator by $S(t)x=C\tilde{T}(t)x$ for $x\in CD(A)$. Clearly, for $x\in CD(A)$, $S(t)x$ is equi-continuous by (4.6), $S(0)x=Cx$ and $S(t)x$ is continuous in $t\geqq 0$.

From (4.4) and (4.5), we obtain

$$S(t+s)Cx=C\tilde{T}(t+s)Cx=\tilde{T}(t)C\tilde{T}(s)Cx=C\tilde{T}(t)C\tilde{T}(s)x=S(t)S(s)x$$

for $x\in X$ and $t, s\geqq 0$. Since $S(t)x$ is continuous on $CD(A)$ and $CD(A)$ is dense in X, $S(t)x$ can be extended to all of X. Consequentely $\{S(t)\ ;\ t\geqq 0\}$ becomes an exponentially equi-continuous C-semigroup on X.

Let G be the operator defined by (1.1) and let $x\in CD(A)$.
Since $\tilde{T}(t)x=C^{-1}S(t)x$ and

we have $\quad \dfrac{d}{dt}\tilde{T}(t)x=\lim\limits_{t\to 0^+}\dfrac{\tilde{T}(t)x-x}{t}=\lim\limits_{t\to 0^+}\dfrac{C^{-1}S(t)x-x}{t}=Gx,$

(4.7) $\quad CD(A)\subset D(G)$ and $G\big|_{CD(A)}=A\big|_{CD(A)}$.

So that (γ) implies that $A\subset\overline{G}$. To conclude the proof, we shall show that $\overline{G}\subset A$. Since $CD(A)$ is dense in X, there exists $x_n\in CD(A)$ such that $\lim\limits_{n\to\infty} x_n=x$ for all $x\in X$. From $S(t)x_n=C\tilde{T}(t)x_n\in CD(A)$ and (4.7), we obtain $AS(t)x_n=GS(t)x_n=S(t)\overline{G}x_n$.

Using the closedness of A,

$$A\int_0^\infty e^{-\lambda t}S(t)x_n dt=\int_0^\infty e^{-\lambda t}AS(t)x_n dt=\int_0^\infty e^{-\lambda t}S(t)\overline{G}x_n\ dt,$$ i.e., $AL_\lambda x_n = L_\lambda\overline{G}x_n$. Combing this with (2.2), $AL_\lambda x_n=\lambda L_\lambda x_n - Cx_n$. Since A is closed, $L_\lambda x_n \to L_\lambda x$ and $AL_\lambda x_n=\lambda L_\lambda x_n - Cx_n \to \lambda L_\lambda x - Cx$, we have

$$L_\lambda x \in D(A) \text{ and } AL_\lambda x=\lambda L_\lambda x - Cx \text{ for } x\in X. \tag{4.8}$$

Now (4.8) and (2.2), $A(\lambda L_\lambda x)=\lambda L_\lambda \overline{G}x$ for $x\in D(\overline{G})$. By closedness of $A, \lambda L_\lambda x \to Cx$ and $A(\lambda L_\lambda x)=\lambda L_\lambda\overline{G}x \to C\overline{G}x$ as $\lambda\to\infty$, we obtain

$$Cx\in D(A) \text{ and } ACx=C\overline{G}x=\overline{G}Cx \text{ for } x\in D(\overline{G}).$$

Thus $CD(\overline{G})\subset D(A)$ and $\overline{G}|_{CD(\overline{G})}=A|_{CD(\overline{G})}\subset A$. Since $CD(\overline{G})$ is core of $\overline{G}$, we see that $\overline{G}=\overline{G}|_{CD(\overline{G})}\subset A$. Therefore $A=\overline{G}$, i.e., A is the C-c.i.g. of the equi-continuous C-semigroup $\{S(t)\ ;\ t\geqq 0\}$. Thus the proof is complete.

COROLLARY 4.1. *If A is a closed linear operator satisfying* (i)——(iv) *in Section* 3, *then A has the condition* (α)——(γ).

Proof. The consequence of Theorem 3.2. and Theorem 4.1.

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