J. Korean Math. Soc. 26(1989), No. 2, pp. 311-322

REAL HYPERSURFACES WITH η-HARMONIC WEYL TENSOR OF A COMPLEX SPACE FORM

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Introduction

The study of real hypersurfaces of a complex projective space P_nC was initiated by Takagi[15], who proved that all homogeneous hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC have first been investigated by Montiel [13] and they are completely classified by Berndt [2]. Real hypersurfaces of P_nC and H_nC have been studied by many authors from various different points of view and, in particular, there are recently some studies [1], [3], [4], [8], [9], [10], [11] and so on.

Now, a real hypersurface M of a complex space form $M_n(c)$ is said to be *harmonic Weyl tensor* if the Ricci tensor S and the scalar curvature r satisfy

$$(*) \qquad \nabla_X S(Y) - \nabla_Y S(X) = \{dr(X)Y - dr(Y)X\}/4(n-1)$$

for any vector fields X and Y, where ∇ denotes the Riemannian connection. The induced almost contact metric structure of the real hypersurface M is denoted by (ϕ, ξ, η, g). In particular, M is said to have *η-harmonic Weyl tensor*, if it satisfies

$$(**) \quad g(\nabla_X S(Y) - \nabla_Y S(X), Z) = \{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}/(4(n-1)$$

for any vector fields X, Y and Z in the orthogonal complement $\xi^\perp$ of ξ. It is seen in [3] that there exist no real hypersurfaces with harmonic Weyl tensor of P_nC and H_nC, but a real hypersurface of type A_1, A_2 or B (resp. A_0, A_1, A_2, B) of P_nC (resp. H_nC) has η-harmonic Weyl tensor.

For the shape operator A, we denote by A' the restriction of A

Received March 13, 1989.

to the orthogonal complement $\xi^{\perp}$ of ξ. The purpose of this note is to investigate the converse of the above property and then to prove the following

THEOREM. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structure vector ξ is principal. Assume that the restricted shape operator A' has no simple roots. Then the hypersurface has η-harmonic Weyl tensor if and only if M is locally congruent to one of the real hypersurfaces of type $A_1\sim B$ of $P_n\boldsymbol{C}$ or of type $A_0\sim B$ of $H_n\boldsymbol{C}$.*

1. Preliminaries

Let M be a real hypersurface of an $n(\geqq 2)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformations of X and C under J can be represented as

$$JX=\phi X+\eta(X)\xi,\quad JC=-\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X)=\eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies then

(1.1) $$\phi^2=-I+\eta\otimes\xi,\quad \phi\xi=0,\quad \eta(\phi X)=0,\quad \eta(\xi)=1,$$

where I denotes the identity transformation. Accordingly, the set becomes an almost contact metric structure. Moreover the covariant derivatives of the structure tensors are given by

(1.2) $$\nabla_X\phi(Y)=\eta(Y)AX-g(AX, Y)\xi,\quad \nabla_X\xi=\phi AX,$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows:

(1.3) $$R(X, Y)Z=c\{g(Y, Z)X-g(X, Z)Y+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z\}/4+g(AY, Z)AX-g(AX, Z)AY,$$

(1.4) $$\nabla_XA(Y)-\nabla_YA(X)=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is a tensor of type $(0, 2)$ given by $S'(X, Y) = tr\{Z \to R(Z, X)Y\}$. But it may be also regarded as the tensor of type $(1, 1)$ and denoted by $S : TM \to TM$; it satisfies $S'(X, Y) = g(SX, Y)$. By the Gauss equation, (1.1) and (1.2) the Ricci tensor S is given by

$$(1.5) \qquad S = c\{(2n+1)I - 3\eta\otimes\xi\}/4 + hA - A^2,$$

where h is a trace of the shape operator A. The covariant derivative of S is also given by

$$(1.6) \qquad \nabla_X S(Y) = -3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}/4 + dh(X)AY + (hI - A)\nabla_X A(Y) - \nabla_X A(AY).$$

Now, some fundamental properties about the structure vector ξ are stated here for later use. First of all, we have the following fact, which is proved by Maeda [12] and Ki and Suh [4], according as $c > 0$ and $c < 0$.

PROPOSITION A. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If the structure vector ξ is principal, then the corresponding principal curvature α is locally constant.*

In the sequel, that the structure vector ξ is principal, that is, $A\xi = \alpha\xi$ is assumed. It follows from (1.4) that we have

$$(1.7) \qquad 2A\phi A = c\phi/2 + \alpha(A\phi + \phi A)$$

and therefore, if $AX = \lambda X$ for any vector field X, then we have

$$(1.8) \qquad (2\lambda - \alpha)A\phi X = (\alpha\lambda + c/2)\phi X.$$

Accordingly, it turns out that in the case where $\alpha^2 + c \neq 0$, ϕX is also a principal vector with principal curvature $\mu = (\alpha\lambda + c/2)/(2\lambda - \alpha)$, namely, we have

$$2\lambda - \alpha \neq 0,$$

$$(1.9) \qquad A\phi X = \mu\phi X, \quad \mu = (\alpha\lambda + c/2)/(2\lambda - \alpha).$$

On the other hand, for any principal curvature λ we find

$$(1.10) \qquad d\lambda(\xi) = 0$$

by the Codazzi equation (1.4) and Proposition A. In fact, the Codazzi equation gives $\nabla_X A(\xi) - \nabla_\xi A(X) = -c\phi X/4$ for any X orthogonal to ξ. Accordingly, for any principal vector X in $\xi^\perp$ with principal curvature λ, we have $g(\nabla_X A(\xi) - \nabla_\xi A(X), X) = (\alpha - \lambda)$

$g(\nabla_X \xi, X) + d\lambda(\xi) g(X, X)$, which implies that $d\lambda(\xi)=0$, because of (1.2). This is due to Kimura and Maeda [8].

Now, let $A(\lambda)$ be an eigenspace of A associated with the eigenvalue λ. Then the subspace $\xi_x{}^\perp$ of the tangent space T_xM at x can be decomposed as

(1.11) $$\xi_x{}^\perp = A(\lambda_1) + A(\lambda_2) + \cdots + A(\lambda_p).$$

By P the operator defined by $A^2 - hA$ is denoted. Then, for any vector fields X, Y and Z in $\xi^\perp$ we have

(1.12) $$g(\nabla_X S(Y), Z) = -g(\nabla_X P(Y), Z),$$

where

(1.13) $$g(\nabla_X P(Y), Z) = g(\nabla_X A(AY), Z) + g(\nabla_X A(Y), AZ) - dh(X) g(AY, Z) - hg(\nabla_X A(Y), Z),$$

because the function h is smooth on M. In particular, for any $X \in A(\lambda)$, $Y \in A(\mu)$ and $Z \in A(\sigma)$ we get

(1.14) $$g(\nabla_X P(Y), Z) = (\mu + \sigma - h) g(\nabla_X A(Y), Z) - dh(X) g(AY, Z).$$

2. Proof of Theorem

A Riemannian manifold N is said to have harmonic curvature, if the Ricci tensor S_N is a Codazzi tensor, that is, it satisfies $dS_N=0$, where S_N is regarded as a 1-form with values in the tangent bundle TN. In particular, the Riemannian manifold N is said to be ***harmonic Weyl tensor***, if the tensor $S_N - r_N g_N/2(n-1)$ is a Codazzi tensor, that is, if it satisfies $d(S_N - r_N g_N/2(n-1)) = 0$, where $n = \dim N$ and r_N denotes the scalar curvature of N.

Now, let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$, on which the structure vector ξ is principal. Then the hypersurface M is said to have *η-harmonic Weyl tensor*, if the Ricci tensor S satisfies

(2.1) $$g(\nabla_X S(Y) - \nabla_Y S(X), Z) = \{dr(X) g(Y, Z) - dr(Y) g(X, Z)\}/4(n-1)$$

for any vector fields in $\xi^\perp$. Assume that the hypersurface M has η-harmonic Weyl tensor and furthermore assume that the restriction A' of the shape operator A to the orthogonal complement $\xi^\perp$ of ξ has no simple roots, that is, the multiplicity of each eigenvalue of A' at any point is greater than or equal to 2. Then (2.1) is equivalent to

(2.2) $$g(\nabla_X P(Y) - \nabla_Y P(X), Z) = -\{dr(X) g(Y, Z)$$

$$-dr(Y)g(X,Z)\}/4(n-1).$$

On the other hand, combining (1.13) and the Codazzi equation (1.4) together with (2.2), we have

$$(2.3)\quad g(\nabla_X A(AY)-\nabla_Y A(AX)-dh(X)AY+dh(Y)AX, Z) + \{dr(X)g(Y,Z)-dr(Y)g(X,Z)\}/4(n-1)=0.$$

By M_A we denote a subset of M consisting of points x at which there exists a neighborhood $U(x)$ so that the multiplicity of each principal curvature is constant on $U(x)$. The restricted shape operator A' is a Codazzi tensor of type (1, 1) by means of the Codazzi equation (1.4) and moreover it is seen that M_A is open and dense and each principal curvature λ is smooth on each connected component of M_A. Given a point x in M and an eigenvalue λ of A'_x, let $A_x(\lambda)$ in $\xi^\perp{}_x\subset T_xM$ be the corresponding eigenspace of A'. On every connected component of the open and dense set M_A the principal curvatures of A' form mutually distinct smooth eigenvalue functions and for such a function λ, the assignment $A(\lambda):\lambda\to A_x(\lambda)$ defines a smooth eigenspace distribution.

Now, on any connected component M_0 of the set M_A, we denote by $A(\lambda)$, $A(\mu)$ and $A(\sigma)$ the eigenspace distributions corresponding to λ, μ and σ, respectively. For any unit vector fields $X\in A(\lambda)$, $Y\in A(\mu)$ and $Z\in A(\sigma)$, it follows from (1.4) and (2.3) that we have

$$(2.4)\quad (\mu-\lambda)g(\nabla_X A(Y),Z)+\{dr(X)/4(n-1)-\mu dh(X)\}g(Y,Z) - \{dr(Y)/4(n-1)-\lambda dh(Y)\}g(X,Z)=0.$$

On the other hand, it is easily seen that

$$(2.5)\quad g(\nabla_X A(Y),Z)=d\mu(X)g(Y,Z)+(\mu-\sigma)g(\nabla_X Y,Z).$$

Then, putting again $Y=Z$ in (2.4) and supposing that X and Y are orthonormal, and making use of (2.5), we get

$$(2.6)\quad (\mu-\lambda)d\mu(X)+dr(X)/4(n-1)-\mu dh(X)=0.$$

By the assumption that the restricted shape operator A' has no simple roots, there are orthonormal vector fields X and Y in $A(\lambda)$. For the pair (X, Y), the last equation is reformed as

$$(2.7)\quad dr(X)-4(n-1)\lambda dh(X)=0 \text{ for any } X\in A(\lambda).$$

By means of (2.6) and (2.7) we have

$$(2.8)\quad (\mu-\lambda)d(\mu-h)(X)=0 \text{ for any } X\in A(\lambda).$$

First of all, the following property is verified.

LEMMA 2.1. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$,* *on which ξ is principal. If A' has no simple roots and if M has η-harmonic Weyl tensor, then the mean curvature of M is constant.*

Proof. In order to prove this lemma, it suffices to show that the function h is constant on each connected component M_0 of the open and dense set M_A. Since α is constant by Proposition A, the proof is divided into two parts.

Suppose first that $\alpha^2+c=0$. Then, without loss of generality, we may suppose that there are at least one principal curvature, say λ, different from $\alpha/2$. For any X in $A(\lambda)$, it is seen that ϕX belongs to $A(\lambda')$, where $\lambda'=\alpha/2$. The equation (2.8) can be applied to this situation (λ, λ') because of $\lambda\neq\lambda'$, and hence we have $dh(X)=d\lambda'(X)=0$ for any X in $A(\lambda)$ and any fixed λ. On the other hand, since h is constant along the ξ-direction, the mean curvature h/n is constant on M_0.

The case where $\alpha^2+c\neq 0$ is next considered. For any fixed principal curvature λ and for any vector X in $A(\lambda)$, ϕX is also principal and the corresponding principal curvature λ' is given by $(\alpha\lambda+c/2)/(2\lambda-\alpha)$. Let $\lambda_1, \ldots, \lambda_p$ be distinct principal curvatures except for α on M_0 and let $n_1, \ldots, n_p$ be multiplicities of $\lambda_1, \ldots, \lambda_p$, respectively, such that $\lambda_1=\lambda$. Then we have $d\lambda_r(X)=dh(X)$, $r=2, \ldots, p$, for any $X\in A(\lambda)$. Since h is given by $\alpha+\sum_{r=1}^{p} n_r\lambda_r$, we have $dh(X)=\Sigma n_r d\lambda_r(X)=n_1 d\lambda_1(X)+\sum_{r=2}^{p} n_r d\lambda_r(X)$ for any $X\in A(\lambda)$. Hence, combining above two equations, we have

$$d\lambda_1(X)=(1-\sum_{r=2}^{p} n_r)\,dh(X)/n_1=a\,dh(X), \tag{2.9}$$

where $a=(1-\sum_{r=2}^{p} n_r)/n_1\in \boldsymbol{R}$. Here the constant a is non-zero. In fact, if $a=0$, then $\sum_{r=2}^{p} n_r=1$, from which it turns out that $r=2$ and $n_2=1$, a contradiction.

Suppose that $\lambda=\lambda'$. Then we have $2\lambda^2-2\alpha\lambda-c/2=0$ and hence the principal curvature λ is constant on M_0. By means of (2.9), it means that we get $dh(X)=0$ for any X in $A(\lambda)$.

Suppose next that $\lambda\neq\lambda'$. The equation (2.8) yields that $d\lambda'(X)=dh(X)$. Because of $d\lambda'=-(\alpha^2+c)d\lambda/(2\lambda-\alpha)^2$, it follows from (2.9) that we have

$$\{(2\lambda-\alpha)^2+a(\alpha^2+c)\}dh(X)=0$$

for any $X\in A(\lambda)$. For any fixed X in $A(\lambda)$, a connected component of a subset of M consisting of points y at which $dh(X)(y)\neq 0$ is denoted by $M(X)$. Suppose that $M(X)$ is not empty. Then, as is easily seen, λ is constant on $M(X)$ and hence so is λ', which implies that $dh(X)=0$, a contradiction. Thus $M(X)$ is empty and hence we have $dh(X)=0$ on M_0 for any X in $A(\lambda)$.

This means that $dh(X)=0$ on M_0 for any principal curvature λ and any vector field X in $A(\lambda)$, which yields that the function h is constant on M_0.

Proof of Thoerem. Assume that the hypersurface M has η-harmonic Weyl tensor. First of all, we shall prove that all principal curvatures are constant under this assumption. Let M_0 be a connected component of the subset M_A of M. From Lemma 2.1 and (2.7) it follows that the scalar curvature r is constant on M. We denote by p the number of distinct principal curvatures different from α on M_0. If $p=1$, then we have $h=\alpha+2(n-1)\lambda$, from which together with Lemma 2.1 it follows that λ is costant on M_0. Next, suppose that $p\geqq 2$. For any distinct principal curvatures λ and μ, the equation (2.6) is reformed as $(\lambda-\mu)d\mu(X)=0$ for any $X\in A(\lambda)$. Therefore we have $d\mu(X)=0$ on M_0 for any $X\in A(\lambda)$. Let $\lambda_1, \dots, \lambda_p$ be mutually distinct principal curvatures on M_0 for the restricted shape operator A' such that $\lambda_1=\lambda$ and $n_1, \dots, n_p$ be their multiplicities, respectively. Since h and α are both constant, the last equation $d\mu(X)=0$ and the relationship $h=\alpha+\sum_{a=1}^{p} n_a\lambda_a$ then give rise to $n_1 d\lambda_1(X)=0$ for any principal curvature λ and for any vector field X, i.e., any λ is constant along the $\xi^\perp$-direction. While it is already seen that λ is constant along the ξ-direction, so is any principal curvature λ on M_0 and on M. Thus, by the classification theorem due to Takagi [15], Kimura [6] and Berndt [2], M is locally congruent to one of the real hypersurfaces of type $A_1\sim E$ of $P_n\boldsymbol{C}$ or $A_0\sim B$ of $H_n\boldsymbol{C}$.

Let M be a real hypersurface of type $A_1\sim B$ of P_nC or of type $A_0\sim B$ of H_nC. By the characterization theorems of the η-parallel shape operator due to Kimura and Maeda [8] and Suh [14], the shape operator A is η-parallel. Since the subspace $\xi^\perp$ is A-invariant, it follows from (1.11) and (1.12) that the operator P is also η-parallel and hence so is the Ricci tensor S. Accordingly, M has η-harmonic Weyl tensor, because the scalar curvature is constant.

In order to prove the theorem, it suffices to show that the real hypersurface of type C, D or E can not occur. Let M be a real hypersurface of type C, D or E. Suppose that the hypersurface M has η-harmonic Weyl tensor. The shape operator A and the structure tensor ϕ satisfy the condition

$$(A\phi-\phi A)(A\phi+\phi A-k\phi)=0,$$

where $k=-c/\alpha$. On the orthogonal complement $\xi^{\perp}$, it follows from (1.7) that we have $Q\phi-\phi Q=0$, where Q denotes the symmetric linear transformation defined by A^2-kA. Accordingly we have

$$\nabla_X Q(\phi Y)+Q\nabla_X\phi(Y)-\nabla_X\phi(QY)-\phi\nabla_X Q(Y)=0. \tag{2.10}$$

Since ξ is principal, $\xi^{\perp}$ is Q-invariant and while ϕ is η-parallel by (1.2). By these properties (2.10) can be deformed as $g(\nabla_X Q(Y)-\phi\nabla_X Q(Y), Z)=0$. Since the operator $\nabla_X Q$ is also symmetric and ϕ is skew-symmetric, it is equivalent to

$$g(\nabla_X Q(Y), \phi Z)=-g(\nabla_X Q(Z), \phi Y) \tag{2.11}$$

for any vector fields X, Y and Z in $\xi^{\perp}$. On the other hand, taking account of the fact that k is constant and the Codazzi equation (1.4), we get

$$g(\nabla_X Q(Y)-\nabla_Y Q(X), \phi Z)=g(\nabla_X A(AY)-\nabla_Y A(AX), \phi Z),$$

from which together with (2.4) it follows that

$$g(\nabla_X Q(Y), \phi Z)=g(\nabla_Y Q(X), \phi Z) \tag{2.12}$$

for any vector fields X, Y and Z in $\xi^{\perp}$. The equations (2.11) and (2.12) show that $g(\nabla_X Q(Y), \phi Z)$ is symmetric with respect to X and Y and also skew-symmetric with respect to Y and Z. This yields that $g(\nabla_X Q(Y), \phi Z)=0$ for any vector fields in $\xi^{\perp}$. Since the transformation $\phi: \xi^{\perp}\rightarrow\xi^{\perp}$ is isomorphic, the above equation means that Q is η-parallel. By means of the theorem due to Aiyama, Suh and one of the authors [1] the transformation Q is η-parallel and ξ is principal if and only if M is locally congruent to one of the real hypersurfaces of type $A_1\sim B$ of $P_n\boldsymbol{C}$ or $A_0\sim B$ of $H_n\boldsymbol{C}$. This is a contradiction.

This completes the proof.

Now, a real hypersurface M of $M_n(c)$, $c\neq 0$, is said to have η-harmonic curvature if the Ricci tensor S satisfies the condition that dS is proportional to ξ, which is equivalent to

$$g(dS(X, Y), Z)=0$$

for any vector fields X, Y and Z orthogonal to ξ. The following property holds as a direct consequence of the main theorem.

COROLLARY 2.2. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which ξ is principal. Assume that the restricted shape operator A' has no simple roots. Then M has η-harmonic curvature if and only if M is locally congruent to one of the real hypersurfaces of type $A_1\sim B$ of P_nC and of type $A_0\sim B$ of H_nC.*

REMARK 2.1. Let M be a real hypersurface with harmonic Weyl tensor of $M_n(c)$, $c\neq 0$, $n\geqq 3$. Then it is already shown in [3] that the structure vector ξ becomes principal. In the case where M has η-harmonic Weyl tensor, we do not know whether or not ξ is principal.

REMARK 2.2. Can the assumption that the restricted shape operator A' has no simple roots be rejected?

REMARK 2.3. For a Riemannian manifold N, $R_N(X, Y)$ operates as a derivation on the algebra consisting of tensor fields on N, where R_N denotes the Riemannian curvature tensor of N. Then N is said to be *cyclic Ryan* if the Ricci tensor S_N satisfies

(2.13) $$\mathfrak{S}(R_N(X, Y)\cdot S_N)(Z)=0$$

for any vector fields, where $\mathfrak{S}$ denotes the cyclic sum and it is also said to be *cyclic pseudo-Ryan* if it satisfies

(2.14) $$g(\mathfrak{S}(R_N(X, Y)\cdot S_N)(Z), W)=0$$

for any vector fields in $\xi^{\perp}$. In [3], two of the authors and Suh showed that a Riemannian manifold with harmonic Weyl tensor is cyclic Ryan and it is pseudo-Einstein. But one should here notice that a real hypersurface with η-harmonic Weyl tensor of $M_n(c)$ is not necessarily cyclic pseudo-Ryan.

3. Parallel harmonic curvature

In this section we consider whether or not a condition that M has η-harmonic curvature is weakened. A Riemannian manifold N is said to have *parallel harmonic curvature*, if it satisfies $\nabla^N dS_N=0$, where ∇^N denotes the induced connection of the vector bundle $\Lambda^2T^*N\otimes TN$. Let M be a real hypersurface of $M_n(c)$. In particular, M is said to have *η-parallel harmonic curvature* if ∇dS is proportional to ξ.

Now, let M be a real hypersurface with η-parallel harmonic curvature of $M_n(c)$, $c\neq 0$, $n\geqq 3$. Then the Ricci tensor S on M satisfies

$$(3.1)\qquad g(\nabla_X\nabla_Y S(Z)-\nabla_X\nabla_Z S(Y),\ W)=0$$

for any vector fields X, Y, Z and W orthogonal to ξ. In fact, the covariant derivative of the 2-form dS with values in TM is given by

$$\nabla dS(X, Y, Z)=\nabla_X(dS(Y, Z))-dS(\nabla_X Y, Z)-dS(Y, \nabla_X Z).$$

Consequently the condition $g(\nabla dS(X, Y, Z), W)=0$ is equivalent to (3.1) because of

$$\nabla_X\nabla_Y S(Z)=\nabla_X(\nabla_Y S(Z))-\nabla_{\nabla_X Y}S(Z)-\nabla_Y S(\nabla_X Z).$$

THEOREM 3.1. *There are no real hypersurfaces with η-parallel harmonic curvature of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structure vector ξ is principal.*

Since the curvature operator $R(X, Y)$ satisfies $(R(X, Y)\cdot S)(Z)=R(X, Y)(SZ)-S(R(X, Y)Z)$ for any vector fields X, Y and Z, the first Bianchi formula gives rise to

$$(3.2)\qquad \mathfrak{S}(R(X, Y)\cdot S)(Z)=\mathfrak{S}R(X, Y)(SZ).$$

The relation between the condition (3.1) and the right hand side of (3.2) is first investigated.

LEMMA 3.2. *Let M be a real hypersurface with η-parallel harmonic curvature of $M_n(c)$, $c\neq 0$, $n\geqq 3$. Then M is cyclic pseudo-Ryan.*

Proof. By means of the assumption it satisfies that

$$\{\nabla_X\nabla_Y S(Z)-\nabla_X\nabla_Z S(Y)\}+\{\nabla_Y\nabla_Z S(X)-\nabla_Y\nabla_X S(Z)\}$$
$$+\{\nabla_Z\nabla_X S(Y)-\nabla_Z\nabla_Y S(X)\}$$

is proportional to ξ for any vector fields orthogonal to ξ. From the Ricci formula for the Ricci tensor it follows that the property is equivalent to $g(\mathfrak{S}(R(X, Y)\cdot S)(Z), W)=0$, which means that M is cyclic pseudo-Ryan.

Proof of Theorem 3.1. As a direct consequence of the above lemma and a theorem due to Ki and Suh [4], M is pseudo-Einstein, i.e. $S=aI+b\eta\otimes\xi$, where the coefficients a and b are both non-zero constants. Accordingly we have

$$\begin{aligned}&\nabla_X\nabla_Y S(Z)\\(3.3)\qquad &=b[\nabla_X\{\nabla_Y\eta(Z)\xi+\eta(Z)\nabla_Y\xi\}-\nabla_{\nabla_X Y}\eta(Z)\xi+\eta(Z)\nabla_{\nabla_X Y}\xi\}\\&\qquad -\{\nabla_Y\eta(\nabla_X Z)\xi+\eta(\nabla_X Z)\nabla_Y\xi\}],\end{aligned}$$

which implies that we have by means of the assumption and (3.4)

$$b[\{g(\nabla_X\nabla_Y\xi-\nabla_{\nabla_X Y}\xi, Z)-g(\nabla_X\nabla_Z\xi-\nabla_{\nabla_X Z}\xi, Y)\}\xi$$
$$+\{g(\nabla_Y\xi, Z)-g(\nabla_Z\xi, Y)\}\nabla_X\xi+g(\nabla_X\xi, Z)\nabla_Y\xi-g(\nabla_X\xi, Y)\nabla_Z\xi$$
$$+\eta(Z)(\nabla_X\nabla_Y\xi-\nabla_{\nabla_X Y}\xi)-\eta(Y)(\nabla_X\nabla_Z\xi-\nabla_{\nabla_X Z}\xi)]=f\xi,$$

for any vector fields X, Y and Z orthogonal to ξ, where f is a smooth function on M. The inner product of this equation and the vector W orthogonal to ξ gives us the following

$$ab\{2g(\phi X, W)g(\phi Y, Z)+g(\phi X, Z)g(\phi Y, W)$$
$$-g(\phi X, Y)g(\phi Z, W)\}=0$$

because of (1.2). If we put $X=Y$ and $Z=W$ in this equation, then we have $abg(\phi X, Y)^2=0$, a contradiction.

REMARK 3.1. In [5] and [9] it is proved that there are no real hypersurfaces with harmonic curvature of $M_n(c)$, $c\neq 0$, $n\geqq 3$, on which the structure vector ξ is principal. Theorem 3.1 is a slight generalization of these results.

REMARK 3.2. If a Riemannian manifold N has parallel harmonic curvature, then it has harmonic curvature. In fact, assume that $\nabla^N dS_N=0$. Then the Ricci formula for dS_N gives $R_N(X, Y)\cdot dS_N=0$ for any vector fields $X$and Y, which is equivalent to

$$g_N(\nabla^N_{R(X,Y)Z}S_N(U)-\nabla^N_U S(R_N(X, Y)Z, V)$$
$$+g_N(\nabla^Z_N S(R_N(X, Y)U-\nabla^N_{R(X,Y)U}S_N(Z), V)$$
$$+g_N(\nabla^N_Z S_N(U)-\nabla^N_U S_N(Z), R_N(X, Y)V)=0$$

for any vector fields. Putting $X=Z=E_k$ and $Y=U=E_j$ in the equation for any orthonormal frame $\{E_j\}$ and summing up j and k, we have

$$\Sigma g_N(\nabla^N_{E_j}S_N(E_k)-\nabla^N_{E_k}S_N(E_j), R_N(E_j, E_k)U)=0.$$

Differentiating it covariantly in the direction of $U=E_i$, one gets

$$\Sigma g_N(\nabla^N_{E_j}S_N(E_k)-\nabla^N_{E_k}S_N(E_j), E_i)^2=0,$$

which gives the conclusion.

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